Numerical Methods
By Dr. Ameeya Kumar Nayak
Department of Mathematics
Indian Institute of Technology, Roorkee
Lecture 34
Numerical Integration Part 4

Welcome to the lecture series on numerical methods, last class we have discussed this numerical integration based on trapezoidal rule, composite trapezoidal rule and Simpon's rule. So today we will just go for the discussion of this integration method based on composite Simpson's 1 by 3 rule and some of the examples based on composite Simpson's 1 by 3 rule and then Simpson's 3 by 8 rule and then we will just go for some examples of 3 by 8 Simpson's rule.

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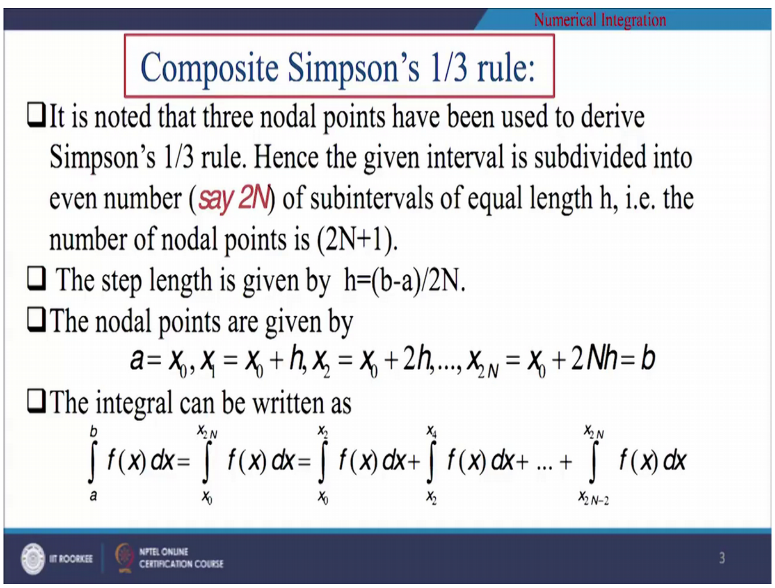

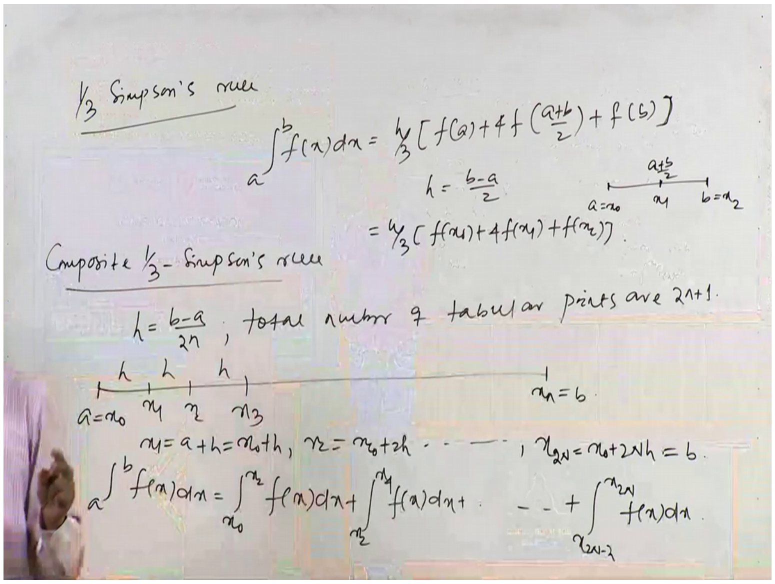

So last class we have discussed about this 1 by 3 Simpson's rule there itself I have just written this formulae this 1 by 3 Simpson's rule as integration a to b f of x dx this can be written in form of like h by 3 f of a plus 4 f of a plus b by 2 plus f of b where h can be defined as b minus a by 2 since we are just dividing this total domain into 2 sub parts here if our starting point is a equals to x 0 here then middle point we are just assuming that a plus b by 2 here and the last point b as x 2 here so middle point is x 1.

Then we can just rewrite this formula as in the form of like h by 3 f of x 0 plus 4 f of x 1 plus f of x 2 here. So if you just go for composite formula for this 1 by 3 Simpson's rule then since we are just considering only 2 sub intervals here then the total interval should be divided by into 2 N plus 1 number of nodal points then we can just use this Simpson's 1 by 3 rule there and if you just go for this step size here then for composite Simpson's rule here composite 1 by 3 Simpson's rule we have to divide this total interval that b minus a by n sub intervals here this means that sorry this is 2 N we have to consider since total number of tabular points is 2 n plus 1.

And the nodal points it can be chosen either in the form of like a equals to x 0 then x 1, then x 2 since all are of equi spaced here then we can just write x 3 so likewise we can just write the last point here that is in the form of like x 0 plus 2 n h that is x n here this equals to b here. So that is why it is just considered it of like 2 n plus 1 points here. So that is why we can just consider a equals to x 0, then x 1 equals to a plus h or x 0 plus h, then x 2 equals to x 0 plus 2 h so likewise just we can just write the last point it can be written in the form of x of 2 N if you just write since

2 n plus 1 points here, so if you just write x of 2 N so then it can be written as x 0 plus 2 n h here this equals to b here.

So if you just write this 1 by 3 Simpson's rule in a composite form then we can just write this integration that as in the form of a to b f of x dx since we are just starting this point from x 0 here, so x 0 to x 2 f of x dx plus x 2 to x 4 f of x dx plus likewise we can just write the last point as x of 2 n minus 1 x of 2 N f of x dx here, sorry we have to write this one as x of 2 N minus 2 since twice points it is just required there that is a 1 sub interval since we are just dividing if this point is written from x 0 to x n here then x n can be replaced as 2 n here then equally we are just subdividing this domain into 2 n parts there.

So that is why we can just write since the starting point is x 0 to x 2 here then x 2 to x 4 so last point it will go from x of 2 n minus 2 to x of 2 n here and in each of the intervals if you just use this 1 by 3 Simpson's rule then especially it is called composite Simpson's rule here.

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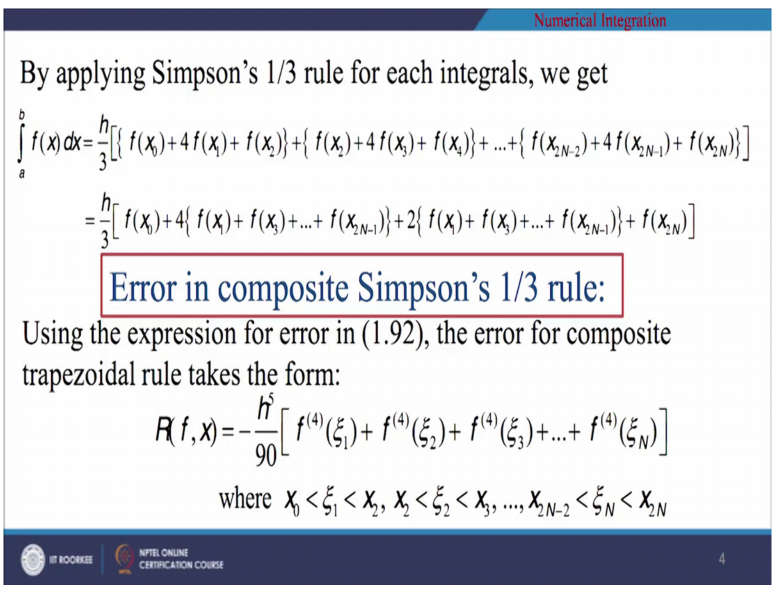

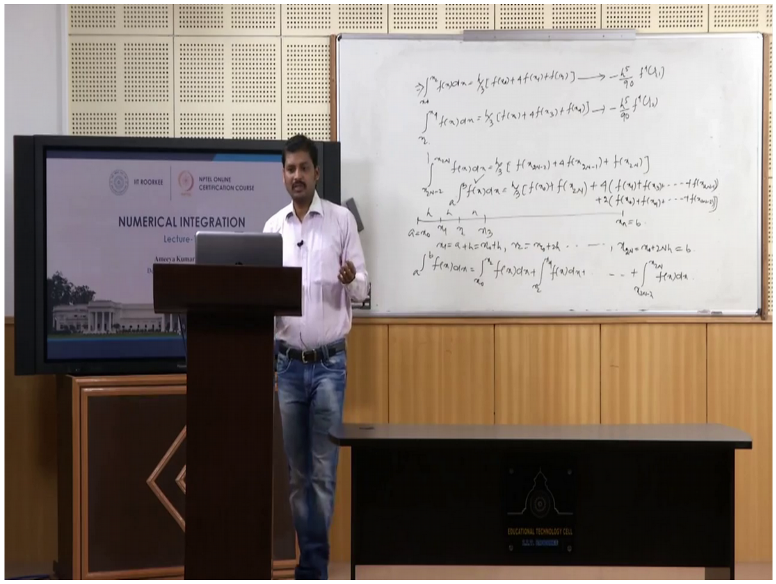

So then if you just apply this 1 by 3 Simpson's rule in each of this intervals here then this formula can be written as, in the first interval x 0 to x 2 f of x dx this can be written as like h by 3 f of x 0 4 f of x 1 plus f of x 2 here. Similarly in the second interval if you just write x 2 to x 4 f of x dx this can be written as h by 3 f of x to 4 f of x 3 plus f of x 4 here. So likewise if you just write the last interval it can be written in the form of x of 2 N minus 2 x of 2 N f of x dx here this can be written as h by 3 f of x of 2 N minus 2 plus 4 f of x of 2 N minus 1 plus f of x of 2 N here.

So if you just R all this terms, then we can just obtain this composite trapezoidal rule that as in the form of like integration a to b f of x dx this equals to h by 3 f of first point that is x not plus last point f of x 2 N plus the 4 coefficient terms that as f of x 1, f of x 3 up to f of x 2 minus 1 here. So that can be written as like f of x 1 plus f of x 3 so up to f of x of 2 N minus 1 plus twice of the repeated terms if you just see here like f of x 2 is appearing here, f of x 2 is appearing here so then again f of x 4 will appear here then f of x 4 is appearing here.

So that is why we can just write twice of this even number terms that is f of x 2 plus f of x 4 plus up to f of x of 2 N minus 2. So this is basically called the composite 1 by 3 Simpson's formula here and if you just go for this error competition of this 1 by 3 Simpson's rule here then last class we have just derived this error for this Simpson's 1 by 3 rule and if you will just consider like our earlier trapezoidal rule, so in each of this intervals we will have a maximized error that as in the form of like minus h by 90.

So first error if you just see for this interval here that is minus h to the power 5 by 90 f fourth of zeta 1, second one if you just consider here that is h to the power 5 by 90 f to the power 4 of zeta 2 here. So likewise if we in each of these intervals if you will just take this error terms and if you just add all these error terms then we can just get this one as in the form of like minus h by h to the power 5 by 90 f to the power 4 of zeta 1 plus f to the power 4 of zeta 2 up to f to the power 4 of zeta n since we are just dividing this 2 n number of points or 2 n plus one number of points into 2 n number of intervals here and then in twice of this interval or sub intervals like 2 points within 2 points we are just getting one error term so that why this total number of error terms it will be like n number of terms there.

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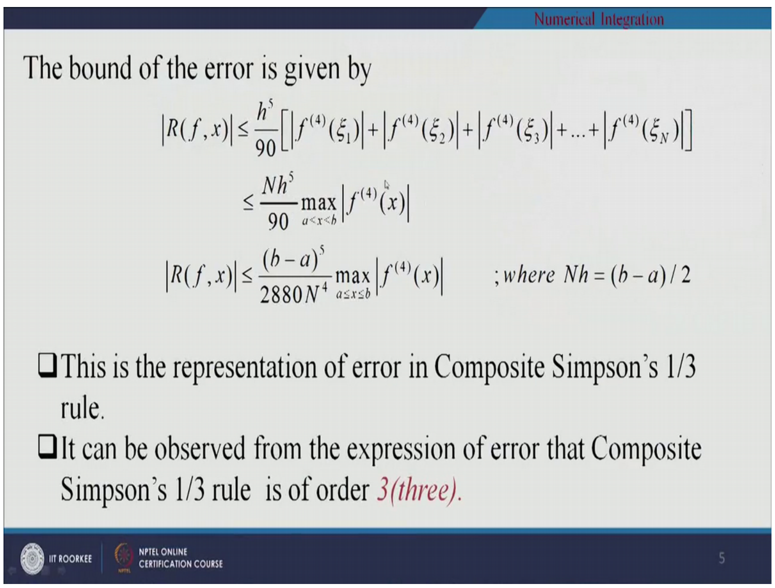

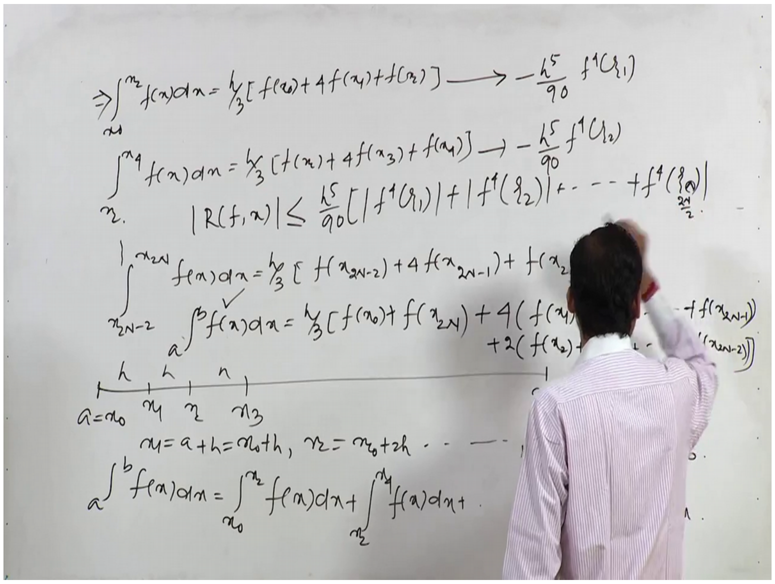

So that is why the last point of this error term it is just coming as f to the power 4 of zeta n here. And in a composite form if you just write that is the maximum bound of error that is absolute modulus of R of f of x if you will just write here maximum modulus that is R of f of x here. So you can just write this should be less or equal to first term here that is h to the power 5 by 90 and if you just consider all other terms f to the power 4 of zeta 1, f to the power 4 of zeta 2, plus up to f to the power 4 of zeta n here or we can just write this way n as like 2 n by 2.

So that is why if you just consider all these terms at a time so it can be written in a form of like a this should be less or equal to h to the power or n h to the power 5 by 90 so this can be written as n h to the power 5 by 90 and this maximum error that is occurring between a to b for this f 4 of x here, this one we can just write maximum of f to the power 4 of x where x should be lies between a to b this one.

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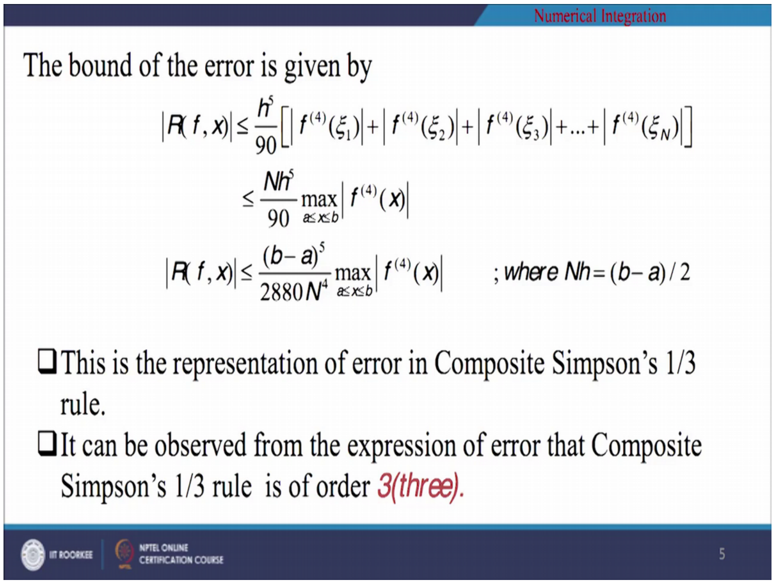

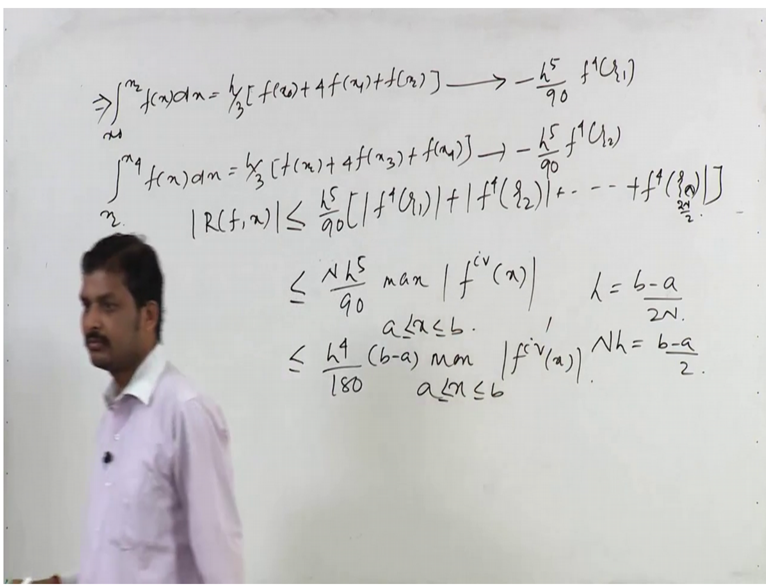

And obviously sometimes we can just write that one n h equals to b minus a by 2 if you just see here that is your h size that is defined as b minus a by 2 N here, so N h if you just replace here like N h this equals to b minus a by 2 so we can just write this one as like h to the power 4 by 180 and b minus a maximum of x lies between a to b f to the power 4 of x here. So this one is the representation of error in composite Simpson's 1 by 3 rule here and it can be observed from the expression of error that this composite 1 by 3 Simpson's rule is of order 3 here since fourth order term it is just giving the error term here.

So since we have already explained that one usually if this error term is written in form of like integration a to b f of x dx minus summation k equals to 0 to n lambda k f of x k this equals to c by p plus 1 factorial into f to the power p plus 1 zeta there. So that is why p should be the order of this polynomial which will give you the exact solution, so that is why for degree 3 we are just exactly getting 0 values at this integration level so that why this order of this error terms can be considered as 3 here since p plus 1 equals to 4 here.

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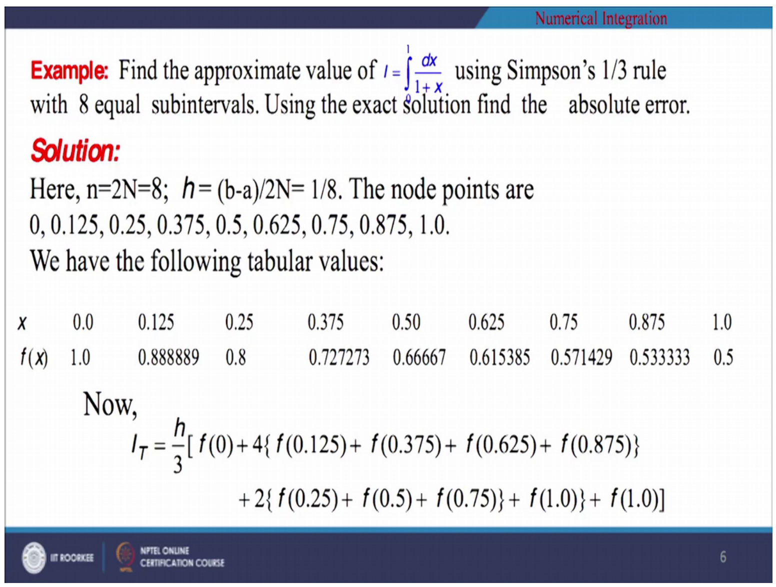

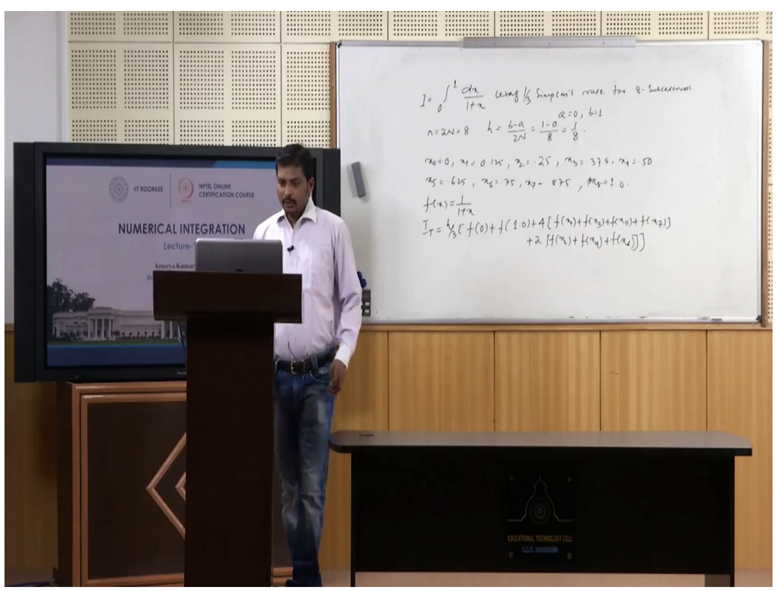

So next we will just go for the example of this 1 by 3 Simpson's rule and this composite 1 by 3 Simpson's rule here. So if you just go for this Simpson's 1 by 3 rule, then suppose the example is given as, like find the approximate value of this integration that is I equals to integration 0 to 1 dx by 1 plus x using Simpson's 1 by 3 rule with 8 equals sub intervals 1 by 3 Simpson's rule for 8 sub intervals. So if you just use these 8 subintervals here then we can just consider that N equals to 2 N that equals to 8 here.

And then we can just define this h that in the form of like b minus a by 2 N here and then we can just write this one as 1 by 8 obviously since here integration range a is given as 0 here, b as 1 so that is why it can be written as 1 minus 0 by 8 that as 1 by 8 and the nodal points if you just write in this form here then it can be written in the form of like x 0 equals to 0, x 1 equals to 1 by 8 means 0.125, x 2 equals to like 0.25, x 3 equals to 0.375, x 4 equals to 0.5, then x 5 this equals to 0.625, x6 equals to 0.75, x 7 equals to 0.875, x 8 equals to 1.0.

And then if you just use this following tabular values for this function f of x equals to 1 by x here 1 by 1 plus x then this functional values that will just give like the values as 1.0 for 0 here since if you just see x equals to 0 means this is f of x equals to 1, then 0.125 it will just give you 0.888889, then 0.25 it will just give you 0.8 here, 0.375 this will just give you 0.727273, 0.5 it will just give you 0.66667 here, 0.625 this value will just give as 0.615385, 0.75 the value will just give as 0.571429, 0.875 this is given as 0.533333 and 1.0 obviously it is 1 by 1 plus 1 so that is why it will just give you 1 by 2 means 0.5 here.

So if you just use this formula here then this integration can be written in the form of like h by 3 f of x 0 plus f of x n that as like 1.0 here plus 4 times the even terms here that is in the form of like if you will just write x 1 plus f of x 3, f of x 5 or terms here, sorry and then last point is f of x 7 here then plus 2 into f of x 2 plus f of x 4 plus f of x 6.

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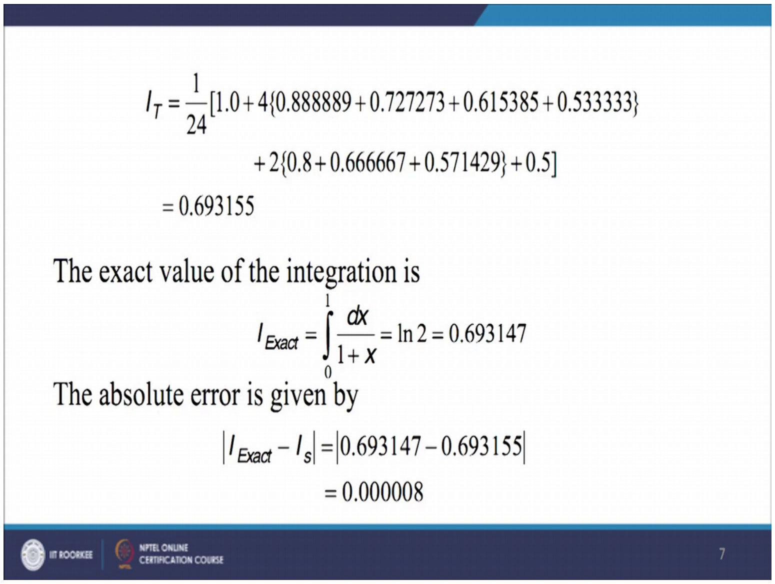

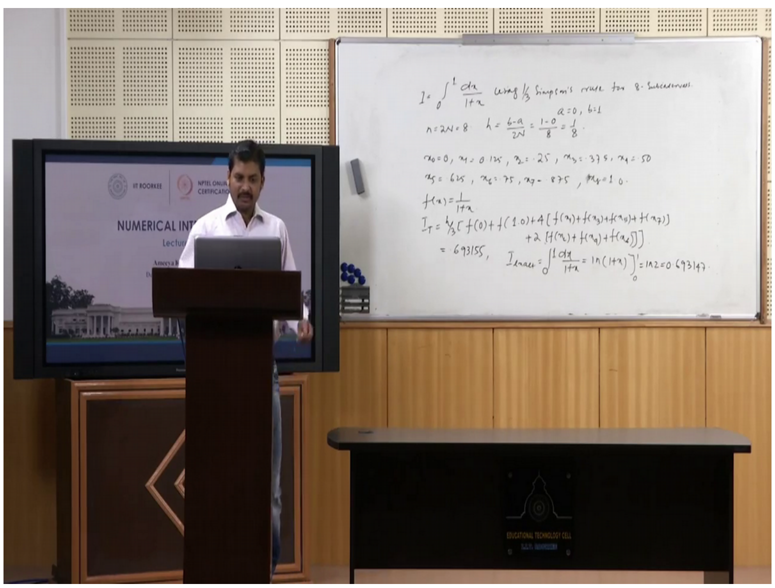

So if you just put all these values here the corresponding nodal points whatever it is just define, so correspondly this functional values if you just put in this formulation here then you can just obtain this value as 0.10615 here.

And if you just see this exact value so this value is just giving you like 0.10615 here and if you just see I exact value so this is nothing but 0 to 1 dx y 1 plus x here, so it can be represented in the form of ln of 1 plus x this is 0 to 1 here and this is nothing but ln of 2 this one since ln of 1 is 0 and this value is coming as 0.693147 and first value if you just see this one so this integral value just coming as like a so this value is coming as like a 0.693155 here and if you just take the difference of I exact minus I s here then we can just obtain this value as 0.693147 minus 0.693155 here.

So this will just give you like 0.000008 so error is very less here for this case. So next we will just go for the Simpson's 3 by 8 rule here, in this method so f of x is approximated by a cubic polynomial since in the beginning of the first case that is trapezoidal rule we have written that f of x is approximated with a linear polynomial, then in the second case like 1 by 3 Simpson's rule we have just taken that f of x is approximated by a quadratic polynomial, then for Simpson's 3 by 8 rule we will just consider that f of x will be approximated by a cubic polynomial here.

To construct this cubic polynomial so 4 nodal points are equated based we can just say like x 0, x 1, x 2 and x 3 here, hence this interval will be subdivided into 3 equal to parts to obtain 4 nodal points.

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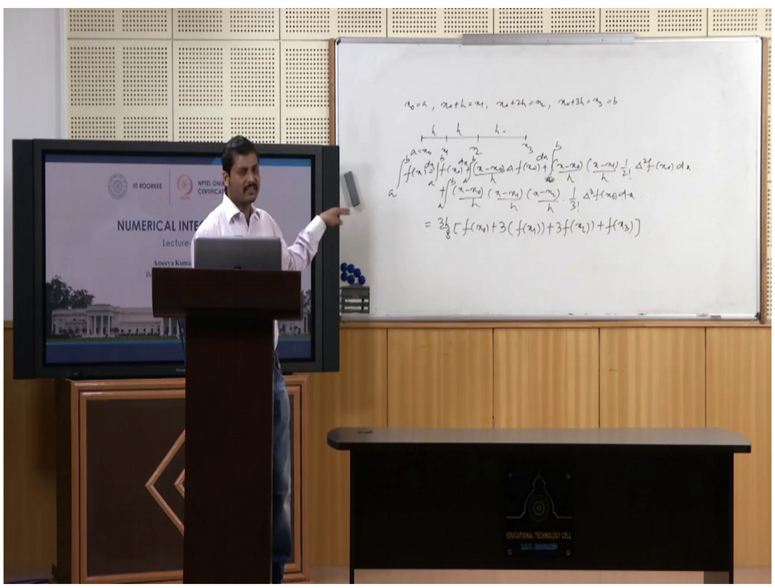

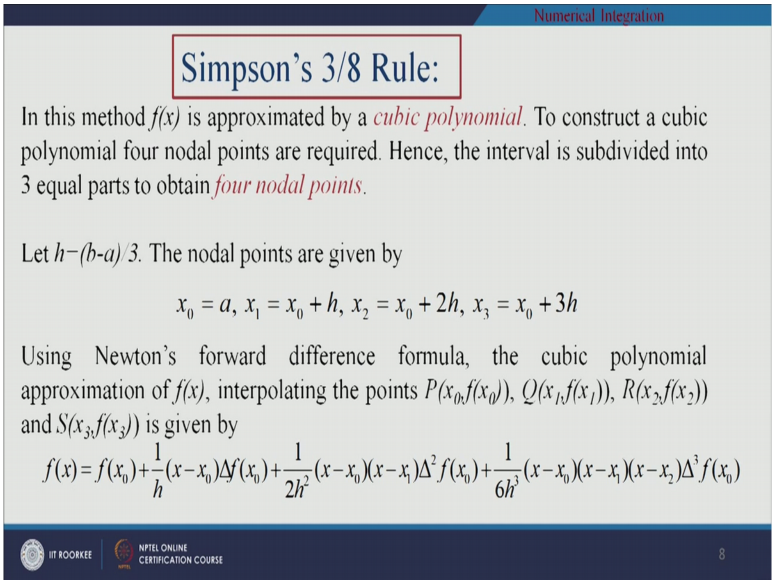

So for that we have to consider h equals to b minus a by 3 here and then nodal points can be written as nodal points are like a x 0 equals to a, then x 0 plus h equals to x 1, x 0 plus 2 h as x 2 here, then x 0 plus 3 h this equals to x 3 here. So usually we are just defining these subintervals 3 subintervals so that is why we can just write this one a equal to x 0 1, 2, then 3 subintervals x 1, x 2, x 3 here h, h, h here.

Now if you just go for this Newton's forward difference formula like the earlier cases, so the cubic polynomial approximation for f of x interpolating these points like p of x 0 f of x 0 then Q of x 1 f of x 1, then R of x 2 f of x 2 and S of x 3 f of x 3 then we can just write this formula that is in the form of like f of x 0 if you just write these 3 different terms like f of x equals to f of x 0 so first term can be written as x minus x 0 so p delta of f of x 0 plus x minus x 0, sorry by h it will be there x minus x 1 by h 1 by 2 factorial del square of f of x 0 plus x minus x 0 by h x minus x 1 by h x minus x2 by h 1 by 3 factorial delta cube of f of x 0.

Then we can just obtain this formula for this 3 by 8 Simpson's rule here. So just then we will just take this integration here that is in the range of from a to b here, a to b, then a to b, dx here, dx here integration x 0 to x 3 or you can just write a to b dx here then a to b dx here. If you will just integrate this one, then you can just obtain this formula that is in the form of like 3 h by 8 f of x 0 plus 3 into f of x 1 plus 3 into f of x 2 plus f of x 3 here. So this expression is basically called the 3 by 8 Simpson's rule, so all of these values it can be expressed in the terms of x 0 and h since x

1 can be written in the form of like x 0 plus h, x 2 can be written in the form of x 0 plus 2 h and x 3 can be written in the form of x 0 plus 3 h here.

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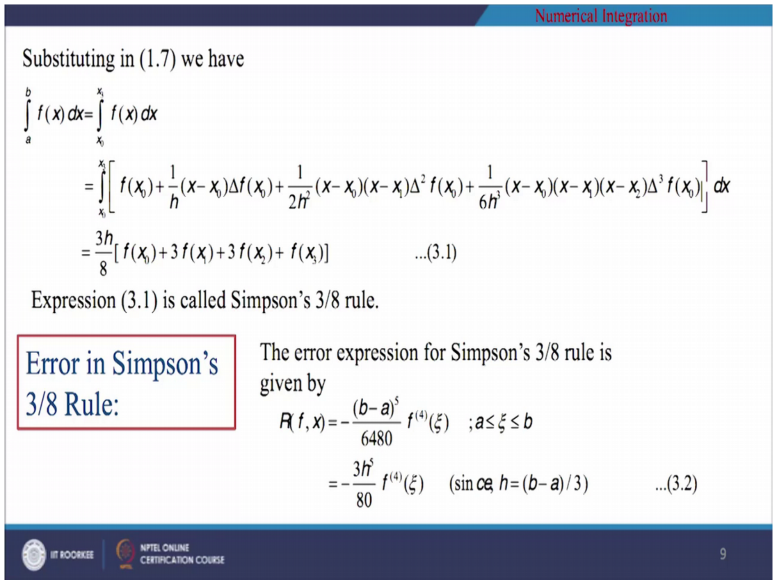

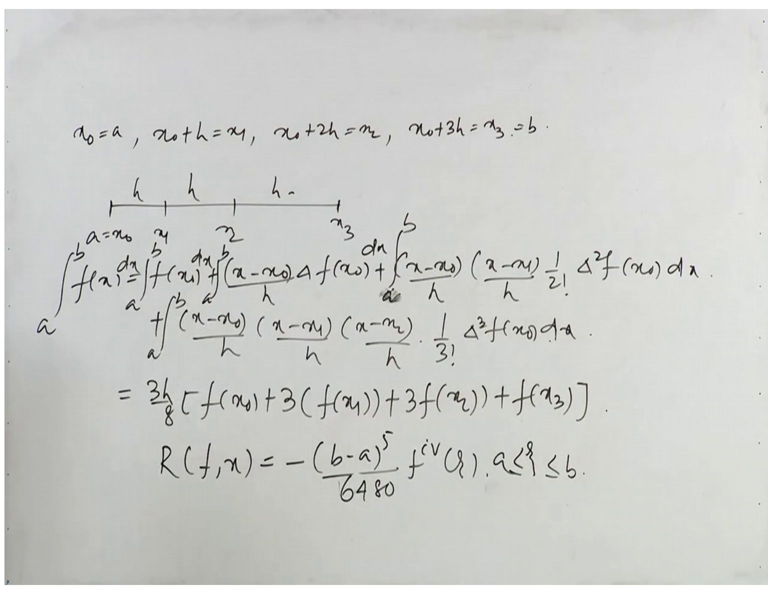

And the error approximation for this Simpson's 3 by 8 rule if you just go like our earlier computation, then we can just obtain this one as a R of f of x here this equals to minus of b minus a whole to the power 5 by 28, sorry this is 6480 f to the power 4 of zeta here, where zeta should be lies between a and b and in terms of h if you just write this can be written in the form of like a minus 3 h to the power 5 by 80 f to the power 4 of zeta here and h can be written in the form of like b minus a by 3 here.

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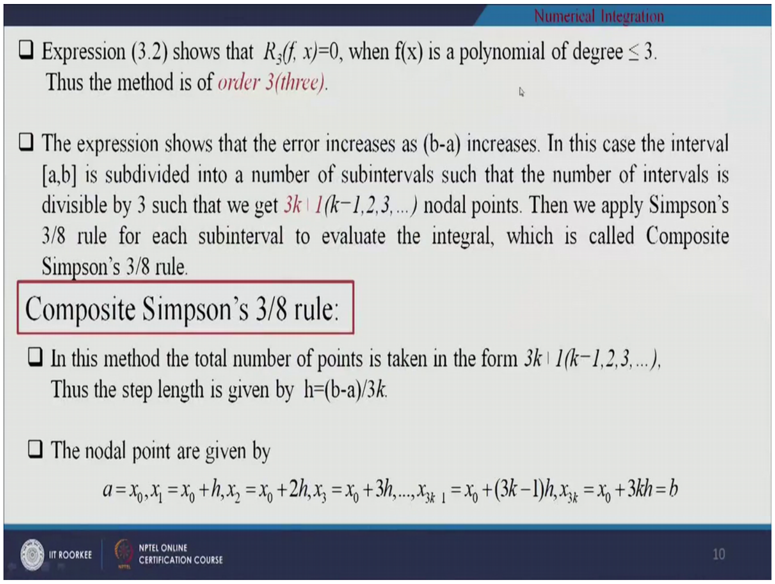

And from this expression it is also shown that this order of approximation for the error is third order here also since when f of x is a polynomial for degree less or equal to 3 then R 3 of f of x it is just giving the value here.

And this shows that the error increases as b minus a values are increasing here also and in this case this interval a, b is subdivided into a number of subintervals that can be represented in terms of 3 k plus 1 here so earlier it was 2 N plus 1 since 2 subintervals we have just considered. So here 3 subintervals so that is why it can be considered as 3 k plus 1 total number of points small n equals to 3 k plus 1 here than simple we can just use 3 by 8 Simpson's rule.

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- Expression (3.2) shows that $R_3(f, x)=0$, when f(x) is a polynomial of degree ≤ 3. Thus the method is of *order 3(three)*.

- The expression shows that the error increases as (b-a) increases. In this case the interval [a,b] is subdivided into a number of subintervals such that the number of intervals is divisible by 3 such that we get $3k+1 (k=1,2,3,\ldots)$ nodal points. Then we apply Simpson's 3/8 rule for each subinterval to evaluate the integral, which is called Composite Simpson's 3/8 rule.

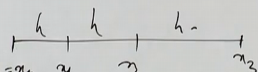
Composite Simpson's 3/8 rule:

- In this method the total number of points is taken in the form $3k+1 (k=1,2,3,\ldots)$, Thus the step length is given by $h=(b-a)/3k$.

- The nodal point are given by

$$a=x_0, x_1=x_0+h, x_2=x_0+2h, x_3=x_0+3h, \ldots, x_{3k-1}=x_0+(3k-1)h, x_{3k}=x_0+3kh=b$$

$x_0=a$, $x_0+h=x_1$, $x_0+2h=x_2$, $x_0+3h=x_3=b$.

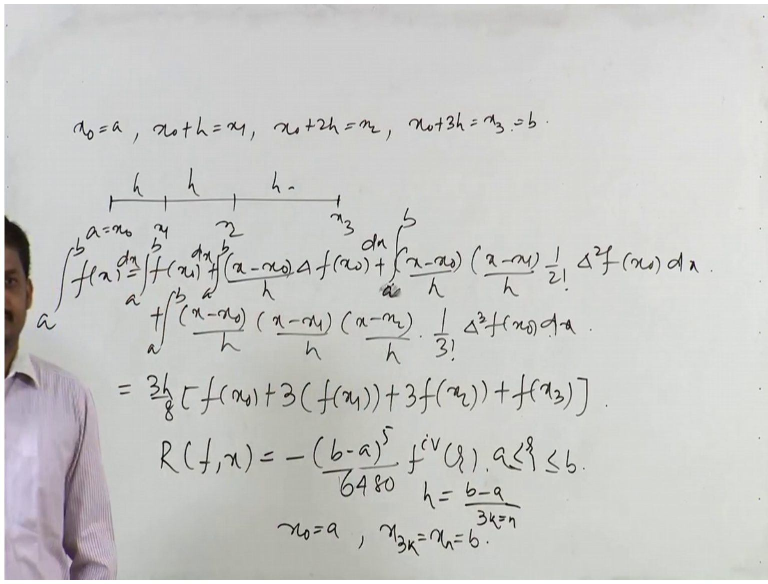


$$\int_a^b f(x)\,dx = \int_a^b f(x_0)\,dx + \int_a^b \frac{(x-x_0)}{h}\Delta f(x_0)\,dx + \int_a^b \frac{(x-x_0)}{h}\frac{(x-x_1)}{h}\frac{1}{2!}\Delta^2 f(x_0)\,dx + \int_a^b \frac{(x-x_0)}{h}\frac{(x-x_1)}{h}\frac{(x-x_2)}{h}\cdot\frac{1}{3!}\Delta^3 f(x_0)\,dx$$

$$= \frac{3h}{8}\left[f(x_0)+3(f(x_1))+3f(x_2))+f(x_3)\right]$$

$$R(f,x) = -\frac{(b-a)^5}{6480} f^{iv}(\xi),\ a<\xi<b.$$

$$h=\frac{b-a}{3k=n}$$



Then if you will just go for like composite Simpson's rule here then the total number of points can be taken as 3 k plus 1 where your h can be divided in the form of like b minus a divided by 3 k or we can just consider this 3 k equals to n is the total number of points there and all of these nodal points it can be represented in the form of a equals to x 0, then b equals to x 0 plus 3 k h that is x n usually it can be written. So if you just write here x 0 equals to a then x of 3 k this can be written as x n equals to b here and usually it can be represented in the form of x 0 plus 3 k h the last point.

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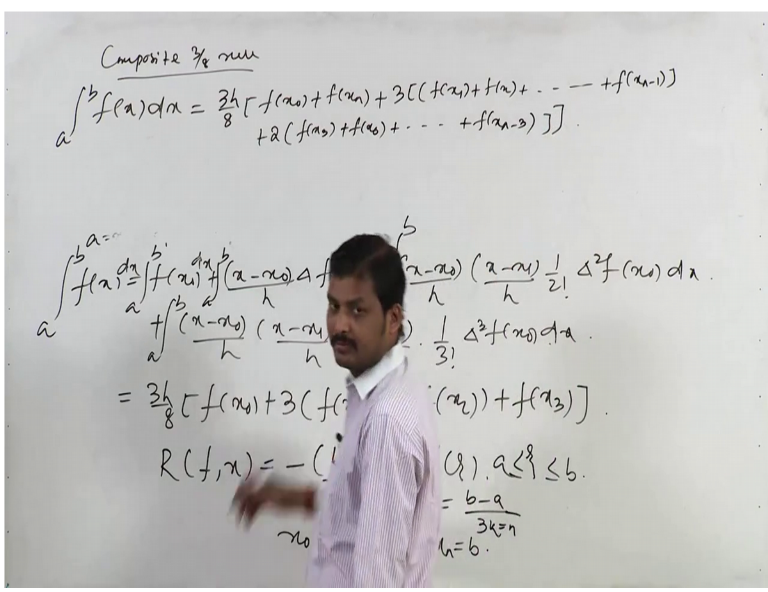

And in composite form if you just write this 3 by 8 Simpson's rule then it can be written in the form as integration a to b f of x dx this equals to this is composite 3 by 8 rule especially it can be written in the form of like a 3 h by 8 so first point f of x 0 plus f of x n is the last point plus 3 into f of x 1 f of x 2 if you just see here so 2 coefficients that is x 1 and x 2 takes the 3 coefficients and the last point f of x 3 is considered as the single coefficient again if you just apply this one the starting point will be f of x 3 there.

So that is why these points will goes up to f of x of n minus 1 points. So plus 2 into f of x 3 plus f of x 6 so likewise it will just go up to f of x n minus 3 here. This is basically called the composite Simpson's 3 by 8 rule.

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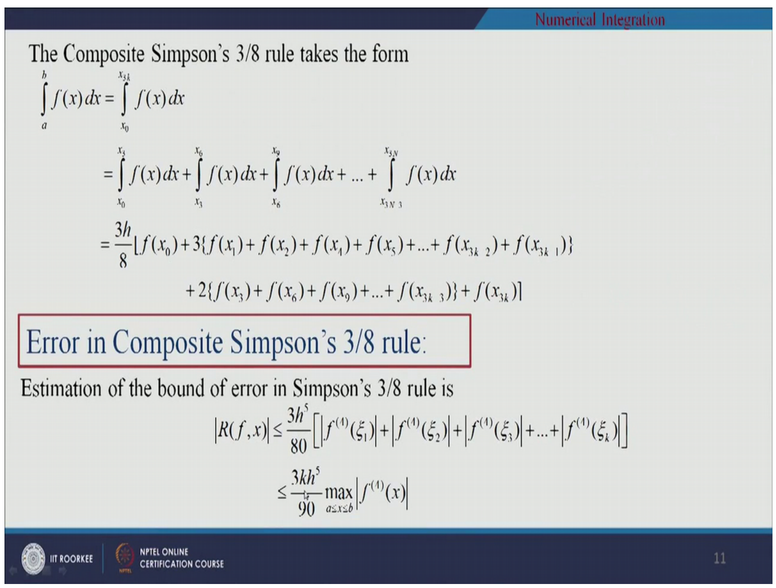

Similarly the error can be computed by considering since we are just considering this 3 subintervals at a time so that is why it can be divided by 3 means we can just finally considered as k intervals here or k maximized values within this subintervals. So that is why this final error term can be written in the form of 3 by 90 k h to the power 5 maximum of f to the power 4 x, x lies between a and b here.

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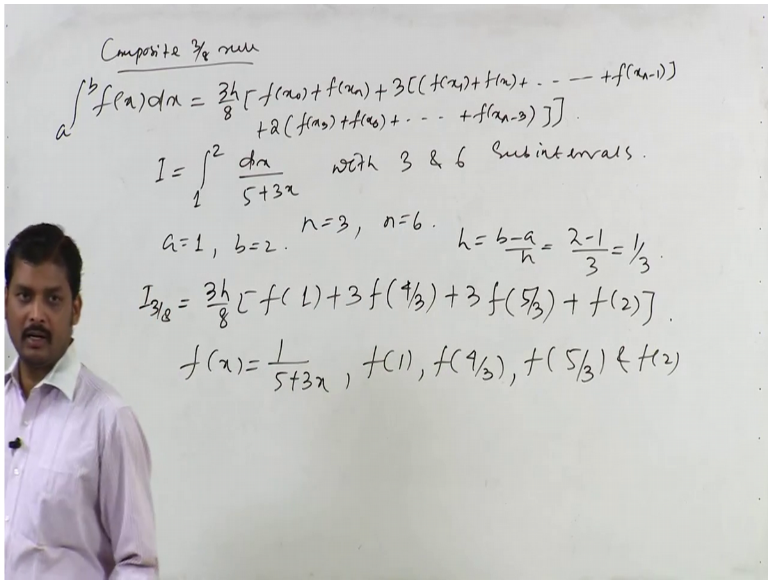

So using this 3 by 8 rule if you just try to solve one problem basically the problem statement it can be written in the form of like suppose I equals to 1 to 2 dx by 5 plus 3 x with 3 and 6 subintervals, then we can just define this problem as a given states that I equals to 1 to 2 dx by 5 plus 3 x here and we have to consider 3 and 6 subintervals.

So here we can just consider n equals to 3 first then n equals to 6 here, so a equals to 1 here, b equals to 2 then h can be written as like b minus a by n here so that is why we can just write that as like 2 minus 1 by 3 here so 1 by 3. So then we can just use this formula that as I as 3 by 8 rule for this 3 point we can just write h by 3 h by 8 usually we were just writing, so 1 by 3 we can just write this one as so directly I can just this formula that will be easy to understand so 3 h by 8 f of x 0 that is in the form of like f of 1 here so plus 3 f of x 1, x 1 can be considered as 1 plus 0.3 here so we can just consider that one as like 4 by 3 here plus 3 f of like 5 by 3 plus f of 2.

From this you can just obtain all of these values so like functional values once f of x you just write here in the form of like 1 by 5 plus 3 x from there itself you can just get f of 1, f of 4 by 3, f of 5 by 3 and f of 2 and put this values and you can just obtain these values here.

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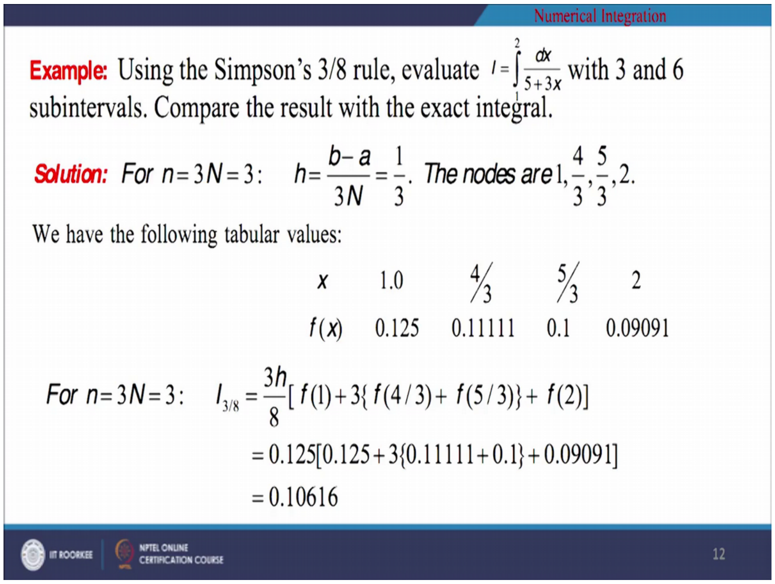

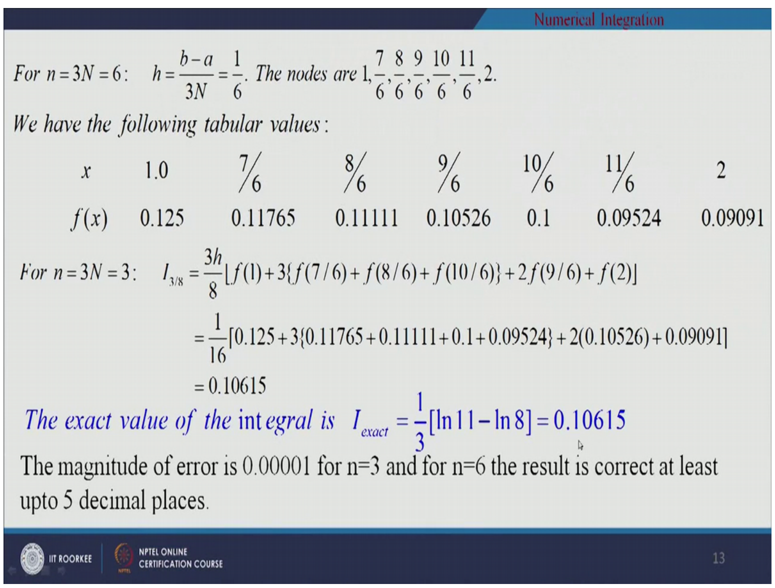

And if you just go for the calculation of 6 points here, then repeatedly you have to use all of these 2 sequences once more sequence you have to add it up to get this final answer here and if you just go for this exact value, so the exact value computation is just giving this value as a 0.10615 but in 3 point form we are just also getting 0.10616 here but in a like a 2 subintervals if at a time we are just considering like 6 points then we are just obtaining this value as 0.10615 here.

And this magnitude of error if you will just compute so for n equals to 3 we are just getting this value as 0.00001 for n equals to 6 so this correct value is giving at least up to 5 decimal places here. So the conclusion is that if we are just using like more number of subintervals then this value is just giving the accurate values in both this like exact solution and this like computed solution or numerical solution in the same form but if you are just considering less number of intervals then this error is just getting increasing. Thank you for listening this lecture.