Numerical Methods

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Lecture 33

Numerical Integration Part 3

Welcome to the lecture series on numerical methods, in the last lecture I have discussed about a numerical integration based on trapezoidal rule. So this lecture I will just continue this error approximation for this trapezoidal rule here.

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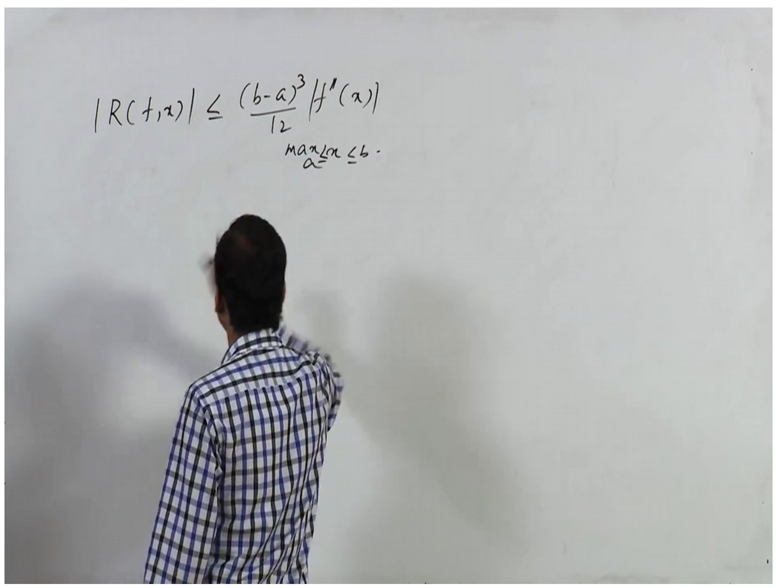

So in the trapezoidal rule if you will just go for this error calculation then we can just write this error as in the form of like as I have told you that if you will have like a 2 points only that is B and A and we will have a single interval there then we usually we are just expressing this error term as in the form like a R of f, x this should be less or equal to b minus a whole cube by 12 f double dash of x absolute values that is a maximum of x lies between a to b.

We have just consider this positive values since we have just taken here absolute value in the left hand side here.

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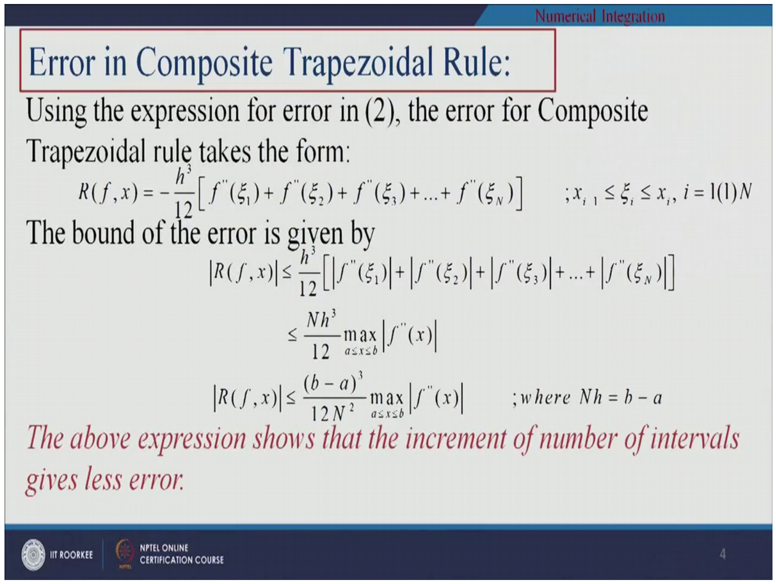

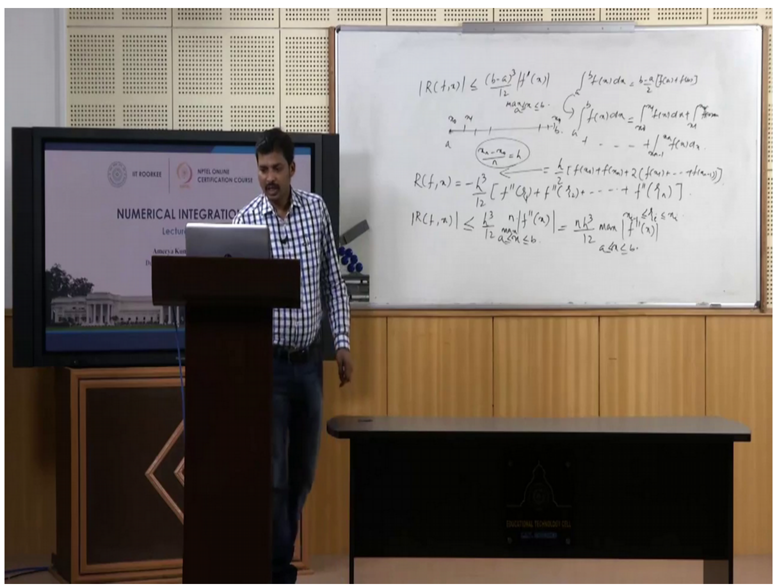

So if you will just go for this expression for a composite formula this is for like a single formula if you will just consider this formula in the form of a to b f of x and dx and h is a suppose only one interval we are just using then we are just using this formula that is in the form of b minus a by 2 f of a plus f of b there but in a composite form if we are just expressing this formula then n intervals if this interval ab is a subdivided a to b x 0, x 1 to x n here each is of like equal spacing this points are situated.

So usually we are just defining this x n minus x 0 by n this as h we are just defining this one. So that is why we will have this intervals like x 0 to x 1 f of x dx plus x 1 to x 2 f of x dx plus up to x n minus 1 to x n f of x dx here. And then they say formula is directly can be written as h by 2 f of x 0 plus f of x n plus 2 into f of x 1 to f of x n minus 1. So in each of this intervals if you will just see here we will have a error term, we will have a error term, we will have a error term since whenever we are just approximating this curve bounded area there so each of this region that is approximated by trapezium there.

So that is why in each of this trapezium we will have a error section either in the lower side or in the upper side. So that is why if you will just consider this total area for this composite formula here that can be written as R of f of x this equals to obviously this formula is written in the form of minus h cube by 12 and each of this error like a first term if you will just the error is f double dash zeta second we will have this error like zeta one f double dash of zeta 2 up to the last section we can just this write this error as f double dash of zeta n suppose where we can just write this zeta i is the error which is lying between x i minus 1 to x i here i equals to like 1, 2 up to n there.

So if you will just go for this error bound then we can just write R of f of x it should be less or equal to h cube by 12 and if you will just find a maximized term or the maximized error term from this one or we can just consider each terms of the error are of equal value or we can just consider this maximized error term there itself then we can just take the total sum as n here into f double dash of x here which can take a positive value here since we are just taking the absolute value on the left hand side so we can just take this absolute value that as a f double dash of x here also where x should be lies between a to b here that can be taken as the maximized value of f double dash x within that range here.

So we can just write that one as a maximum of x which should lies between a and b and can be taken by the this second order derivative there and in a complete form if you will just write this formula this can be written as N h cube by 2 l maximum of f double dash of x where x should have lies between a to b here and sometimes also if you just replace N h equals to b minus a then you can just replace this formula that is h square into b minus a by 12 into maximum of f double dash of x there.

So that I have just written here also that is N h cube n h equals to b minus a so that is why this can be written as b minus a whole cube by 12 N square also sometimes we can just write and maximum of f double dash x and this above expression shows that the increment of number of intervals gives less error if you will just see in comparison to the earlier one if you will just see total interval if you will just take a single step of composite rule then we will have a larger error if you will just sub divide each of this sections then we can have less error there itself since in each of this intervals if you will just see this error will be reduced to there.

And that is why we can just use this composite formula to get a better result.

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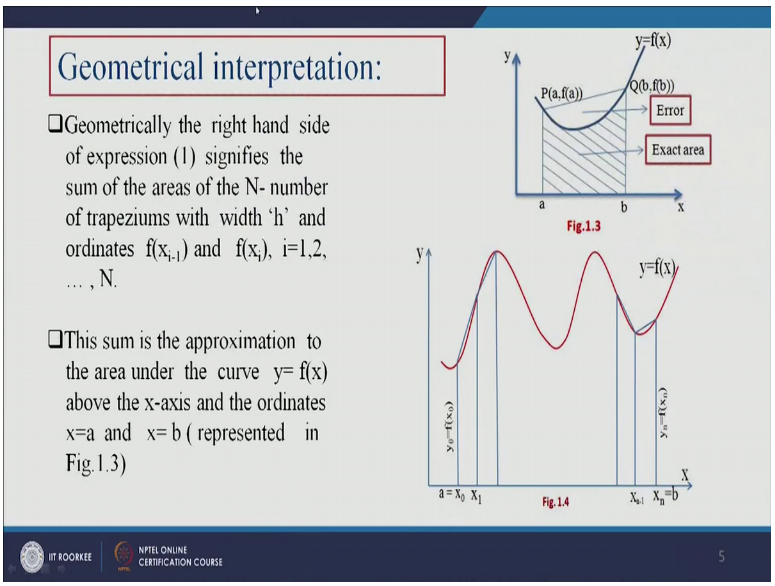

In geometrical interpretation if you will just see here in each section if you will just see this error is getting reduced here but in total section if you will just see this error is very large compare to this lower graph here that is N number of trapeziums with width is with ordinates f x i minus 1 and f of x i if you will just consider in a geometrical way, then the sum of approximation of the area for this curve y equals to f of x above the x axis and ordinates x equals a and b gives the maximized error cross sectional under this curve there.

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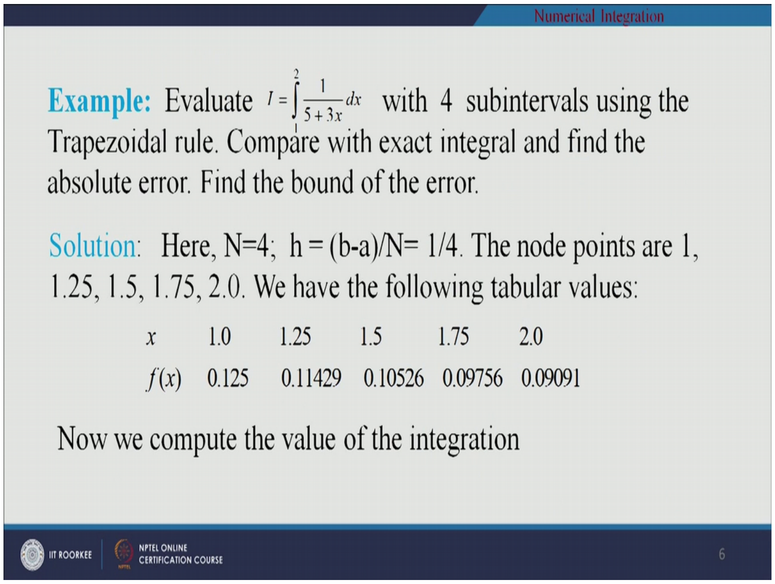

Then if suppose a question is asked like a evaluate integration i equals to 1 to 2 1 by 5 plus 3 x into dx with four subintervals using trapezoidal rule and compare with exact integral and find the absolute error and find the bound for the error. If you will just see here your A is given as 1 here and B is 2 here four subintervals means N equals to 4 here.

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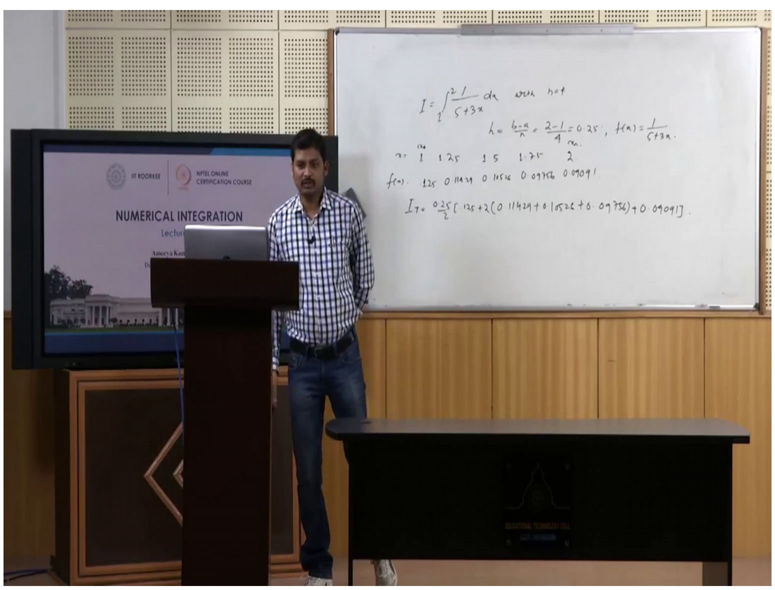

So this integration is given as I equals to 1 to 2 1 by 5 plus 3 x dx here with four subintervals this means that with n equals to 4 here and if n is equals to 4 here so then we can just define h equals to b minus a by n here so that is 2 minus 1 divided by 4 that is 0.25 here.

And if we have just defined h equals to 0.25 here then nodal points it is starting from 1, 1.25, then 1.5, then 1.75, then last point is 2 here and corresponding f values you can just define at one as like f of x is given as here 1 by 5 plus 3 x here. Corresponding values for x equals to 1 we can just get f of x as 0.125, for 1.25 we can just get 0.11429, 1.5 we can just get 0.10526, 0.09756 and 0.09091. Now we will just go for this integration so this trapezoidal integration It can be written as h by 2 so we can just write 0.25 by 2 f of x 0, x0 is given as 1 here and x n is a 2 here. So f of x 0 it is written as 0.125 plus 2 into 0.11429 plus 0.10526 plus 0.09756 plus 0.09091.

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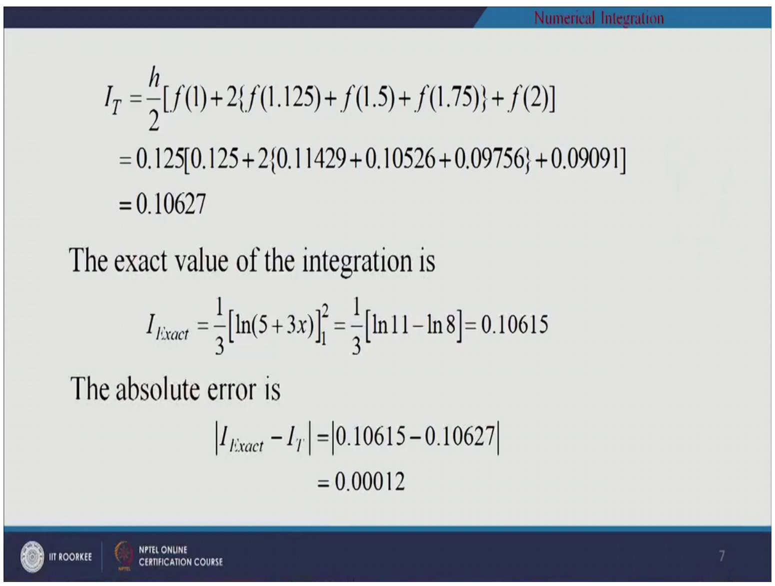

So this will just give the values as a 0.10627 and exact value of this integration if you just go for this competition we can just get 1 by 3 into l n of 5 plus 3 x if you just see and if you just put this range 1 to 2 here we can just obtain that one as 0.10615 and the absolute error is that is I exact minus I t here in absolute form if you just write 0.10615 minus 0.10627 this is 0.00012 it is very less here. And if you just go for this bound of this error so absolute value is if you just consider that can be written as error should be less or equal to b minus a h square by 12 maximum of f double dash x if you just go for this error term here so this error can be written as this should be less or equal to b minus a h square by 12 maximum of f double dash of x x should be lies between 1 and 2 here.

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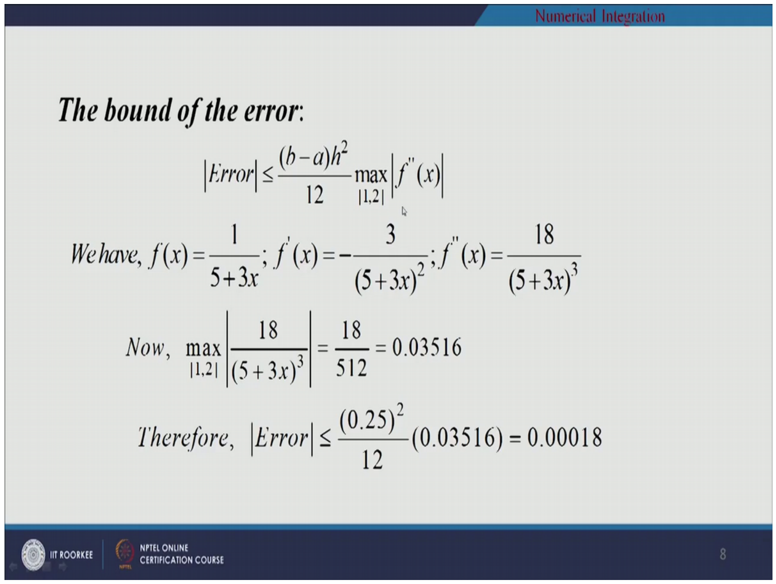

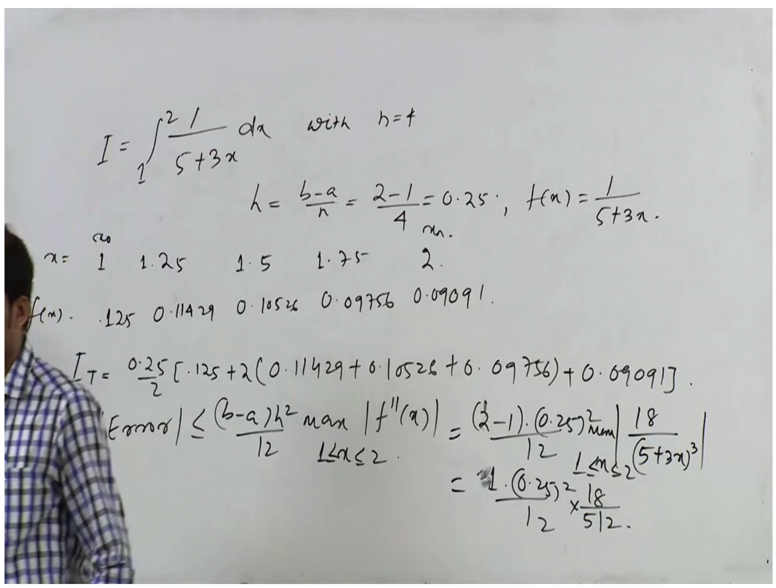

And if you will just calculate this f double dash of x we can just say this one as this equals to 1 sorry 2 minus 1 into h square is 0.25 square here by 2 1 and f double dash x if you will just compute that will just give you 18 by 5 plus 3 x whole cube here and this should be maximum value where x should be lies between 1 and 2. So if you just see here this function will achieve this maximum value whenever x equals to 1 there itself.

So if x equals to 1 there then it will be 8 to the power 3 there and its value is like 512 there. So that is why we can just write this (())(12:26) value as 2 minus 1 is 1 here into 0.25 square by 12 into 18 by 512 this will just give you this maximum error term of this composite trapezoidal rule that is just coming as a 0.00018 here.

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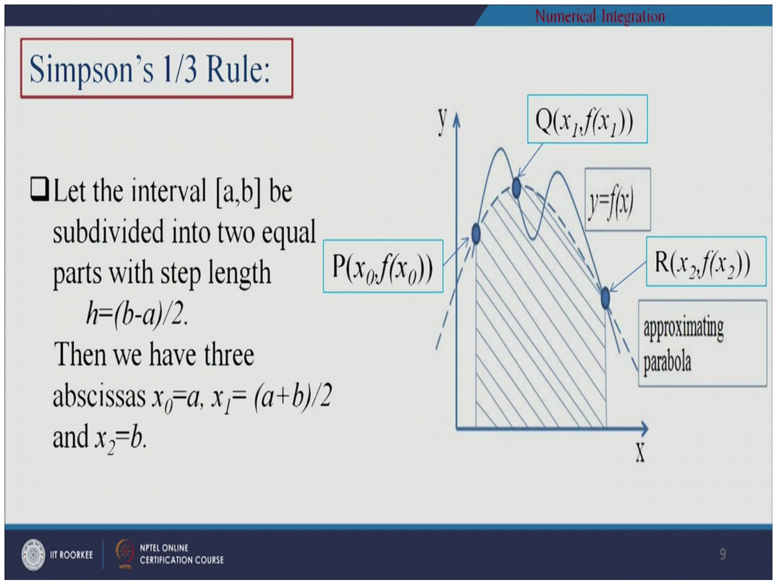

So then we will just go for 1 by 3 Simpson's rule here so whenever we will calculate this area of this curve bounded by this x axis and if this area is approximated by this parabola here then we can just obtain this 1 by 3 Simpson's rule here.

Specially we are just dividing this interval like in the earlier case we have just divided into 2 points there only single interval we have just considered, here we will just subdivide this interval into 2 parts here considering this middle part as a plus b by 2 there.

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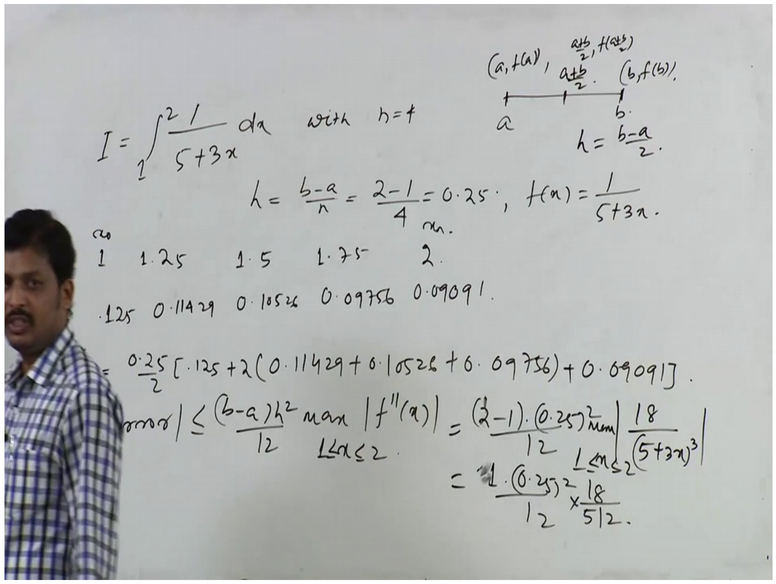

So if you will just like divide any interval like a to b here like 2 intervals 2 subintervals here, then we can just write this midpoint as a plus b by 2.

And we will have 3 points like a, f of a and like a plus b by 2 f of a plus b by 2 and then the third point b f of b here and we can just write this space size h equals to b minus a by 2 here with x 0 equals to suppose a there x 1 as a plus b by 2 and x 2 equals to b there. So if you will just see in this slide here you can just find that this parabolic section this is just approximating this aerial cross section that is bounded by this x axis and they curve there.

So if you just consider these points like x 0 and f of x 0 is the first point at x 2 f of x 2 is the last point here then the middle point is denoted as x 1 f of x 1 here.

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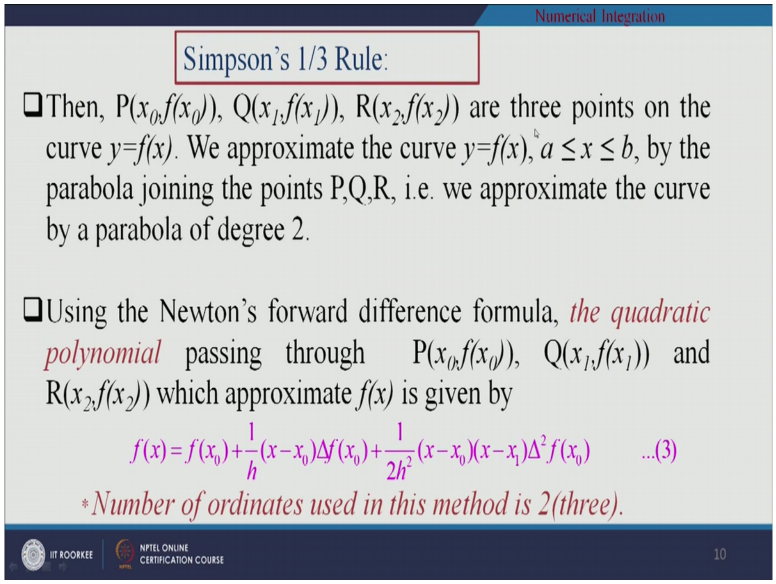

So this means that we are just approximating the 3 points of this curve y equals to f of x by parabola joining this points P, Q, R there and this parabola is approximated with this curve is of degree 2 there. So if this is approximated by a polynomial of degree 2, then we can use Newton's forward difference formula for a quadratic polynomial by considering these points x 0 f of x 0, x 1 f of x 1 and x 2 f of x 2.

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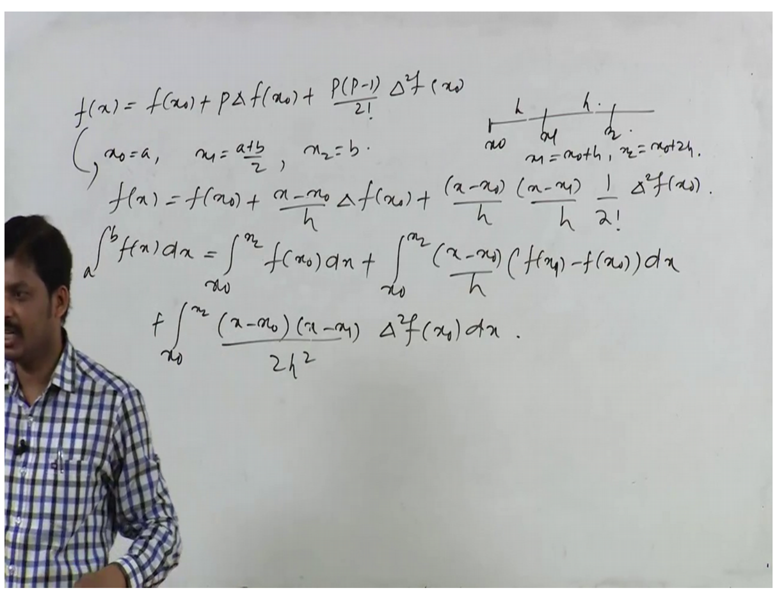

If you will just write this Newton's forward difference formula for a degree 2, so f of x can be written as f of x 0 plus p delta of f of x 0 plus p into p minus 1 by factorial 2 del square of f of x

0 here. Since, this function id approximated by a polynomial of degree 2 the higher powers will just give you the 0 values there. So in terms of x if you will just write since we have here 3 points like x 0 equals to a, x 1 equals to a plus b by 2 and x 2 equals to b here, if you will just rewrite this formula here then this can be written as f of x equals to f of x 0 plus x minus x 0 by h delta f of x 0 plus x minus x 0 by h x minus x 1 by h 1 by 2 factorial del square f of x 0.

And if you will just use this integration like a to b f of x dx here for Newton's quadrature formula a to b f of x dx this can be written as x 0 to x 2 f of x 0 dx plus x 0 to x 2 x minus x 0 by h f of x 0 minus, sorry this is f of x 1 minus f of x 0 into dx plus x 0 to x 2 x minus x 0 x minus x 1 divided by 2 h square del square of f of x 0 into dx here.

So first point if you will just integrate then you can just get x 2 minus x 0 f of x 0 here, second point if you just integrate here x minus x 0 whole square by factorial 2 h will come out so f of x 1 minus f of x 0 so that is why it can be cancel it out like x 2 minus x 0 if you will just take that will just give you 2 h there since we are just approximating here x 0, x 1, x 2 here so each is of space h here equi space points we are just considering.

So that is why we can just write x 1 equals to x 0 plus h and x 2 as x 0 plus 2 h here.

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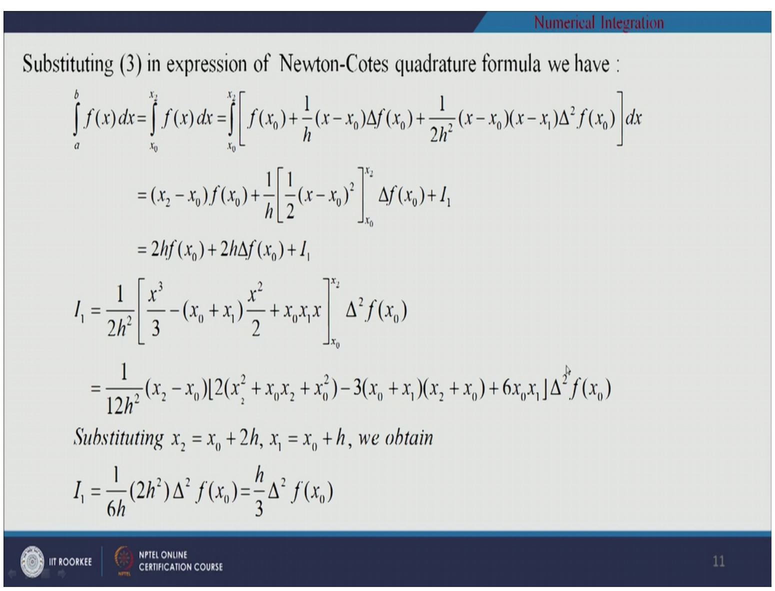

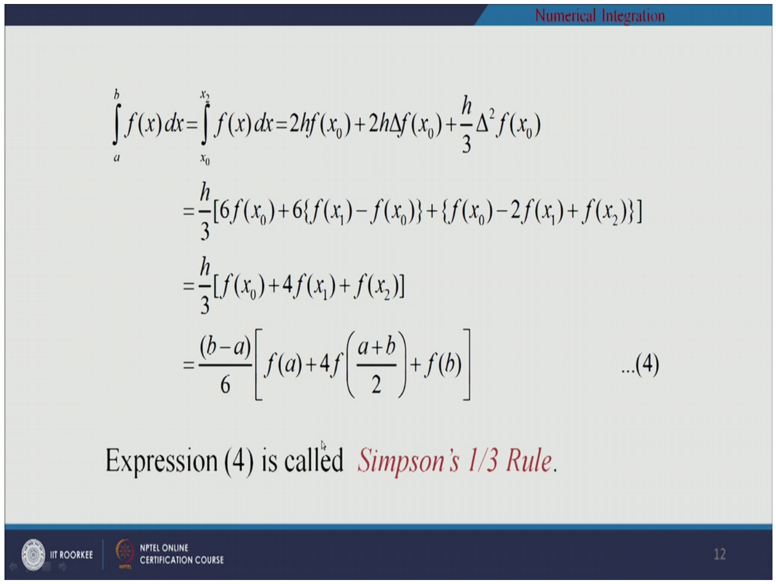

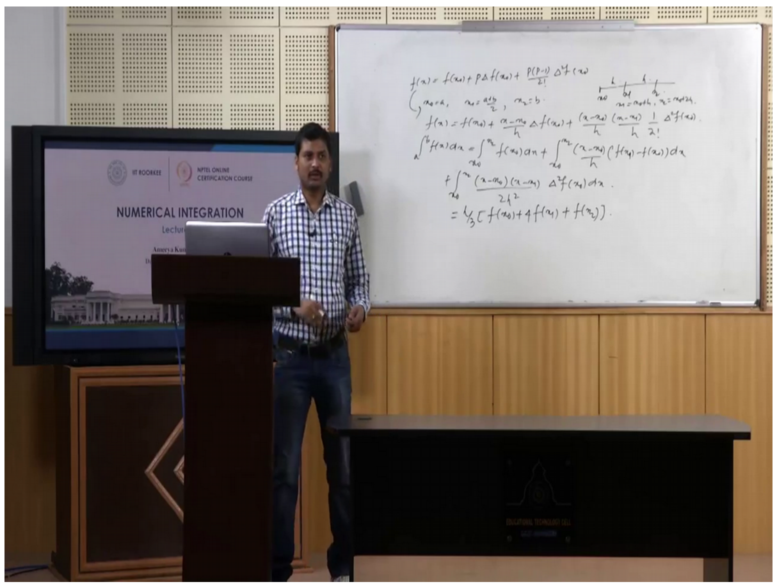

So if you will just integrate them in the final form if you just put all these values here that we have just derived here that is x 2 equals to x 0 plus 2 h x 1 equals to x 0 plus h here we can just obtain that one as h by 3 f of x 0 plus 4 f of x 1 plus f of x 2.

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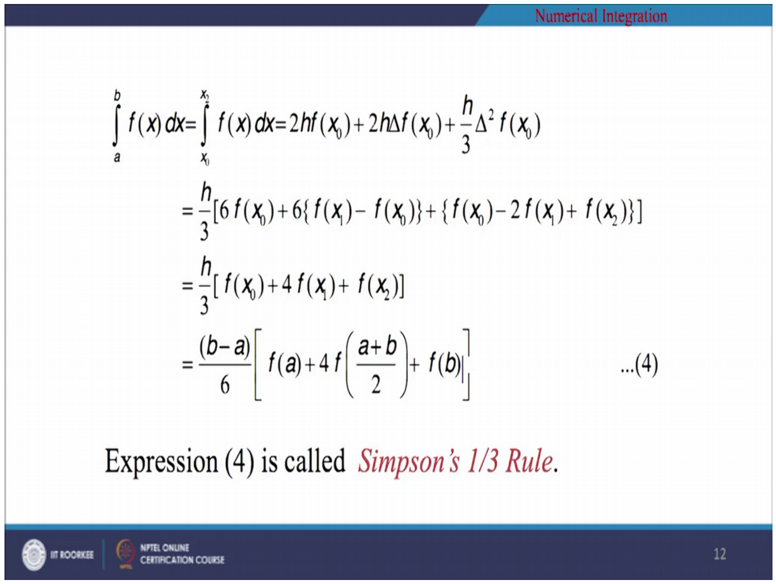

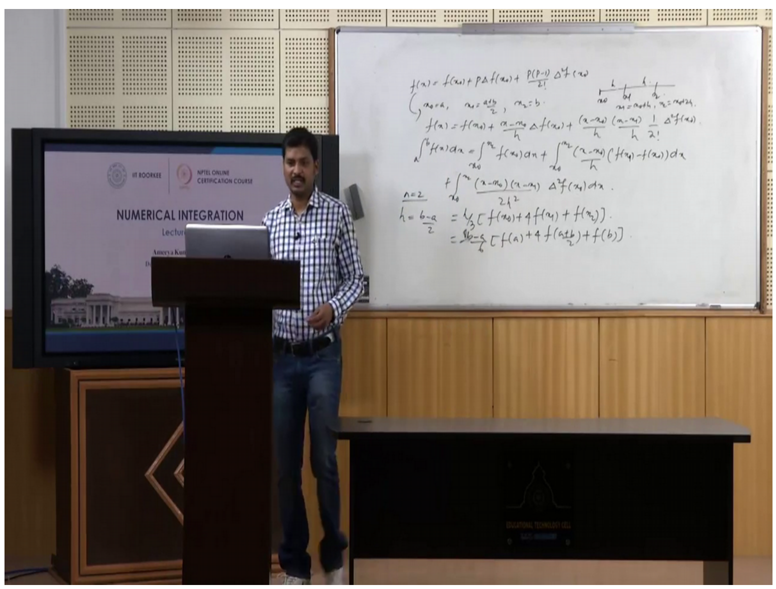

So just a simplification if you just do then you can just obtain this formula and directly also we can just write since h is expressed here that is b minus a by 2 since if you will just see here h is nothing but b minus a by 2 here since n is 2 here.

So that is why we can just write h by 6, sorry this is b minus a by 6 f of a plus 4 f of a plus b by 2 plus f of b here in terms of a and b if we want to write this formula so this can be represented in this form here.

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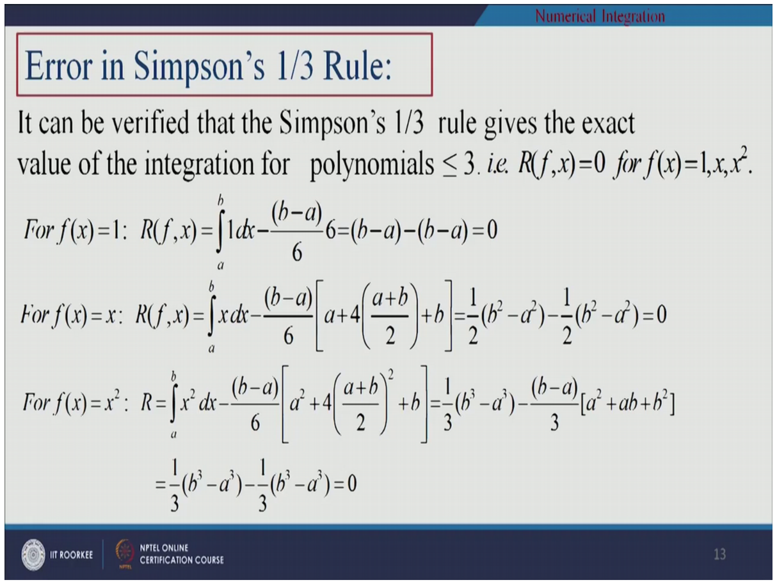

This is called 1 by 3 Simpson's rule and if you will just go for this error calculation of this Simpson's rule we have to consider that this function is exact for polynomials of degree 0, 1 and 2 there this means that it can just provide the exact value this means that this integration minus this formulated value whatever this formula we have derived here this difference will be 0 for 1 x and x square this gives the exact polynomial of degree less or equal to 3 here.

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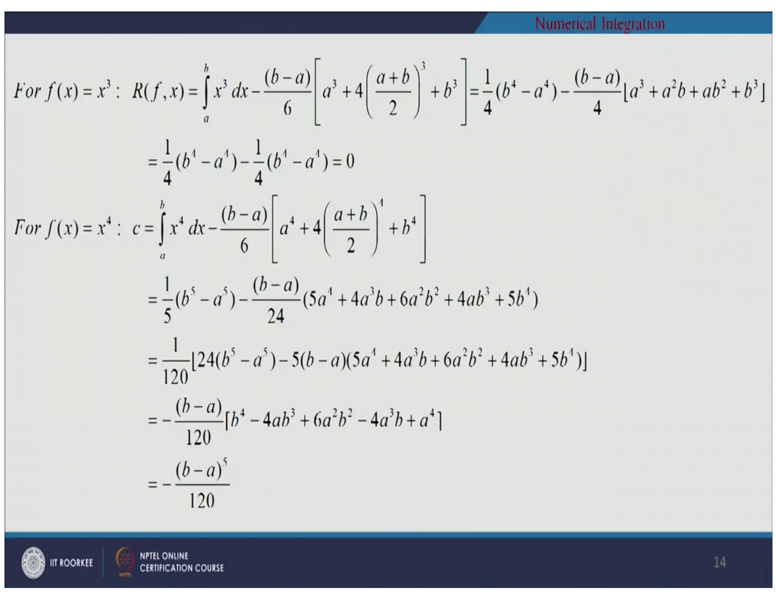

So for degree 4 we can just get exactly this error term for this function here if you just put here f of x equals to 1 for this error approximation that I can just see that R of f of x equals to 0 like our earlier calculation if you just see.

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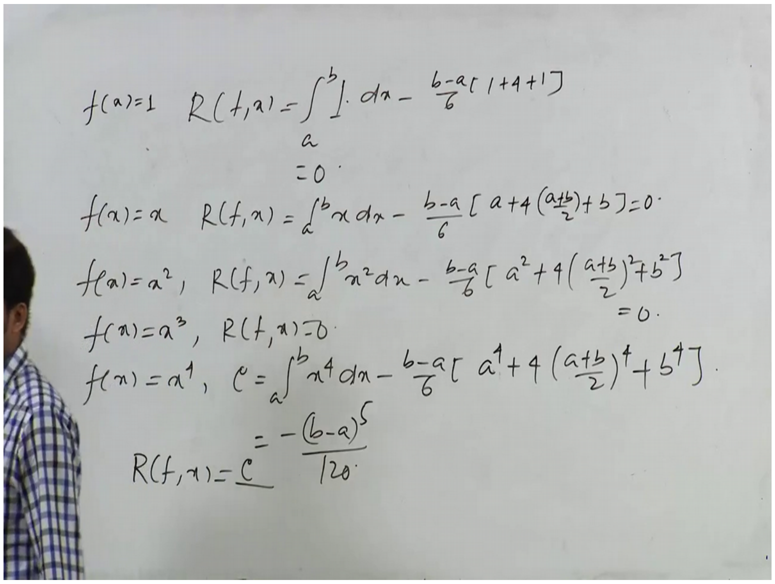

For f of x equals to 1 if you will just put a to b 1 into dx here, that is for f of x equals to 1 if you will just write this error term here R of f of x this can be written as like a to b 1 into dx minus your formula that is as b minus a by 6 so f of a plus 4 f of a plus b by 2 plus f of b.

Then if you will just write this one that is 1 plus 4 plus 1 here this will just give you 0 value. Similarly if you will just consider here f of x equals to x suppose R of f of x can be written as a to b x dx minus b minus a by 6, so first term if you just write here as a plus 4 a plus b by 2 plus b here. So this will just obviously gives you also a 0 value, similarly if you will just consider f of x equals to x square this can also be written as r of f of x here a to b x square dx minus b minus a by 6 a square plus 4 a plus b by 2 whole square plus b square here this will also give you 0 value here.

If you just consider like a f of x equals to x cube since directly if you just put like a polynomials of degree or 3 x is less or equal to 3 here exactly 3 should be the error term here but specially for degree 3 we are just getting 0 value here so that is why we have just to consider the immediate next term for this error here that is f of x equals to x to the power 4 if you just put here f of x equals to x to the power 4 here that can be written as c equals to if you just see here f of x cube, R of f of x equals to 0 here and for f of x equals to x to the power 4 we can just compute c term here so that is a to b x to the power 4 dx minus that is b minus a by 6 here and your terms a to the power 4 plus 4 into a plus b by 2 whole to the power 4 plus b to the power 4 here.

And if you just like do some simplifications we can just obtain this value as minus of b minus a whole to the power 5 by 120. So then the total error term we can just write R equals to or R of f x equals to c byN plus 1 factorial here especially if you will just see or p plus 1.

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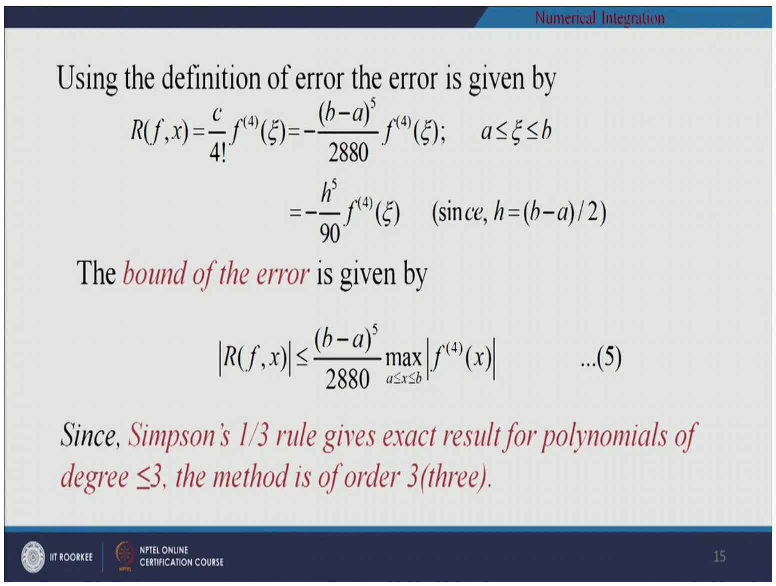

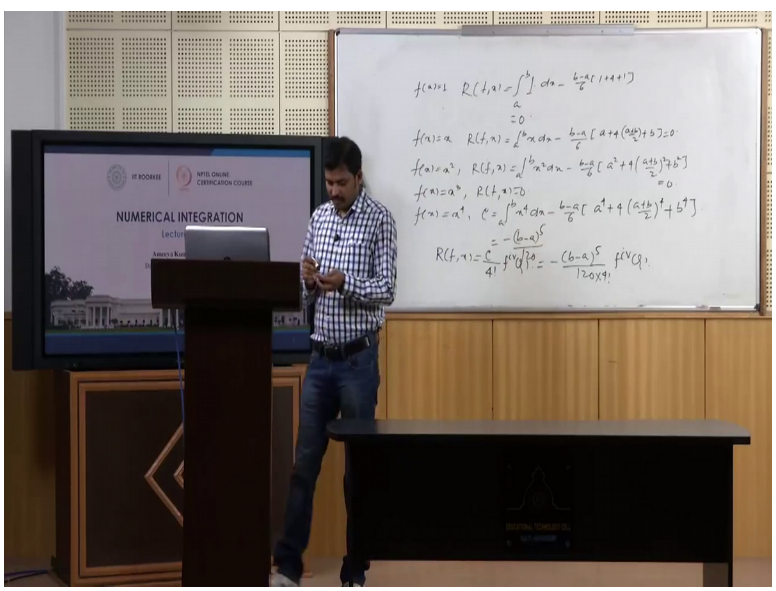

So c by 3 is your highest p term since we are just getting for degree 3 also we are just getting the exact polynomial function which is approximated with this f of x value. So that is why you can just consider p as 3 here c by p plus 1 so that why 4 factorial here so f to the power 4 zeta value here where zeta should be lies between a to b there so that why we can just write if you just put here that as minus of b minus a whole to the power 5 by 120 into 4 factorial f to the power 4 of zeta here.

And this total term that is just giving you here minus h to the power 5 by 90 f to the power 4 of zeta the bound for this error if you will just go for the positive values here like modulus of R of f of x this can be written as less or equal to b minus a whole to the power 5 by the total product that is just coming as 2880 into maximum of f to the power 4 x where x should be lies between a to b here since Simpson's 1 by 3 rule gives us the exact result for polynomials of degree less or equal to 3 the method is said to be of order 3 here that is why it is called 1 by 3 Simpson's rule here.

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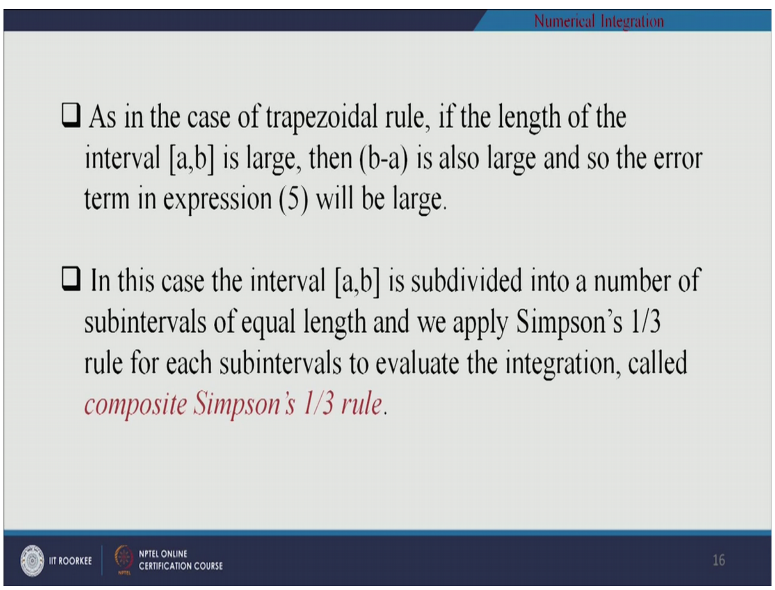

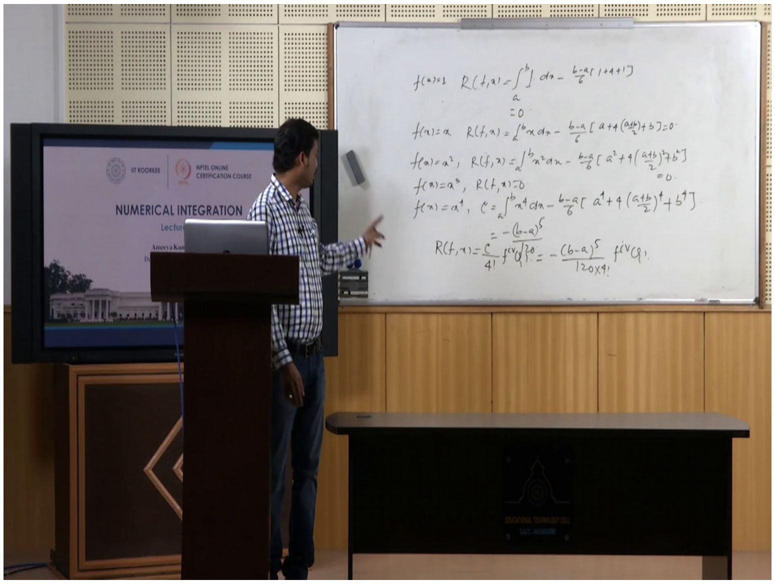

So as in the case of trapezoidal rule if the length of the interval a, b is large, then we can just find that b minus a is also large so the error term in expression 5 is also large. And in this case the interval a, b is subdivided into number of subintervals of equal to length and simply apply Simpson's 1 by 3 rule for each of this intervals to or the subintervals to evaluate this integration and it is basically called the composite Simpson's rule here. Basically in trapezoidal rule we are sorry the trapezoidal rule we have just considered a single interval there and if you just go for this like a your 1 by 3 Simpson's rule we have just considered 2 subintervals there.

So that is why (())(26:29) both these methods if you just compare like trapezoidal rule and 1 by 3 Simpson's rule here then we can just visualize this error will be less in comparison to trapezoidal rule is larger compared to the 1 by 3 Simpson's rule here. This means if you will just use 1 by 3 Simpson's rule we can just obtain these results maybe around 6 steps then it requires near about 12 steps to get this result for this trapezoidal rule there.

So the conclusion is that whenever we are just subdividing these domains into number of intervals then we are just getting these results in more accurate form compare to the less number of division of the intervals. So in trapezoidal rule if you just see we have just used N equals to 1 and if we are just extending that one to N equals to 2 there then this error is just getting reduced in R1 forms there. So that is why it is always impressive to choose this space sizes as larger to get this less error there, thank you for listening this lecture.