Numerical Methods
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Lecture No 31
Numerical Integration Part 1

Welcome to the lecture series of numerical methods. In the last lecture we have discussed about numerical differentiation and today we will start about numerical integration. So in numerical integration we will just deal different kinds of interpolation formulas and how we can just integrate this polynomial or this function with using this interpolation formulas.

So specifically the advantage of this numerical integration is that if we will have this specific nodal points with its tabular values then at that point if the function is not known to us explicitly we can use this numerical integrations to evaluate this functional values at all of this point or whatever wherever this points are asked to evaluate.

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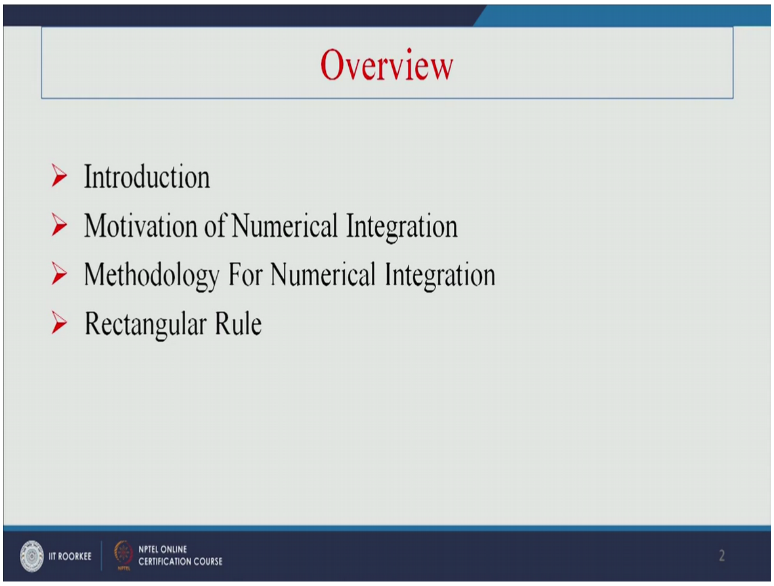

So first we will just go for this introduction that where we are just observing this difficulties of the solving problems in integration and why we are just going for this numerical integration. Then we will just go for this methodology for numerical integrations and obviously for this lecture we will just go upto the rectangular rule that how we can just use this rectangular rule to evaluate this integration.

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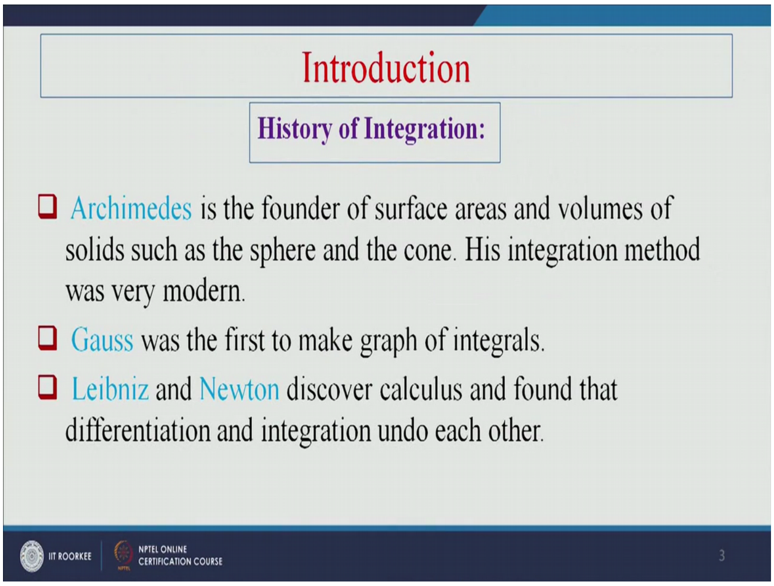

So specifically if you will just go for the history of integration you can just find that Archimedes has developed this integration to find the surface area and volumes of solids and this integration method was very modern nowadays. And since earlier it was just developed to calculate this area or volume or solids volume of solids in the spear or in cones. Gauss was first introduced to make the integrals in a graphical sense this means that graphs for integrals he has developed and Leibniz and Newton discovered the calculus and found this differentiation in integration which are converts to each other.

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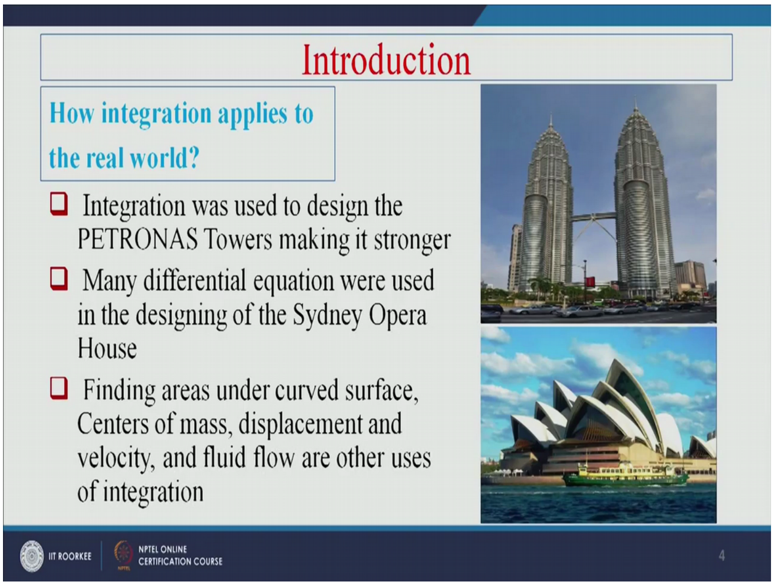

So integration was used to design or it can be applicable to many fields of science and engineering if you will just see. And here I am just showing some of this modern infrastructures where this integration was use like PETRONAS Tower and Sydney Opera House or finding this curves under the Centre of mass, displacement, velocity and fluid flows, specifically without integration and derivative we cannot solve such type of problems.

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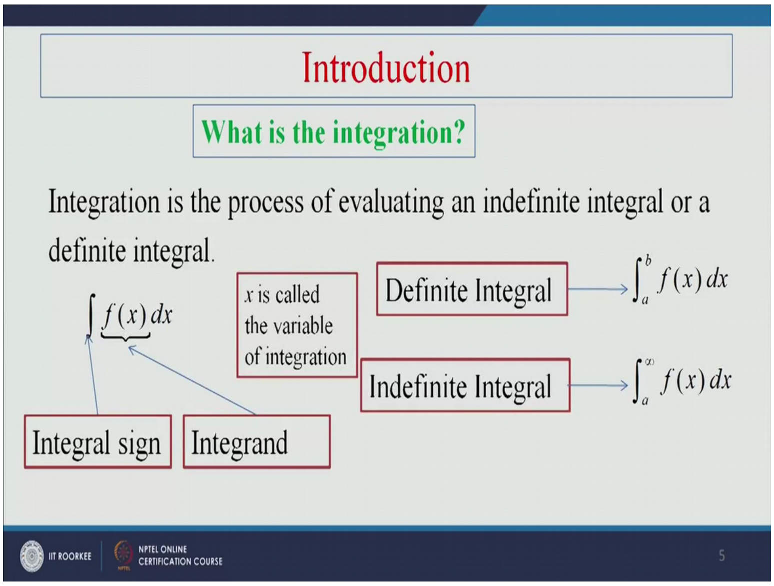

So if you will just go for this integration, if somebody asked what is integration, so then especially we are just putting curve sign and which is just giving a parametric region where we want to evaluate this integration or calculate this areas or surfaces or volume bounded within that surface. And if we are just defining this integrals, so integrals are of 2 types, first it is called definite integrals, where this end point should be finite and we can just easily position this points. And next is that indefinite integrals if some ranges are infinite then especially it is called indefinite integral, where it is difficult to evaluate the functional values within that range.

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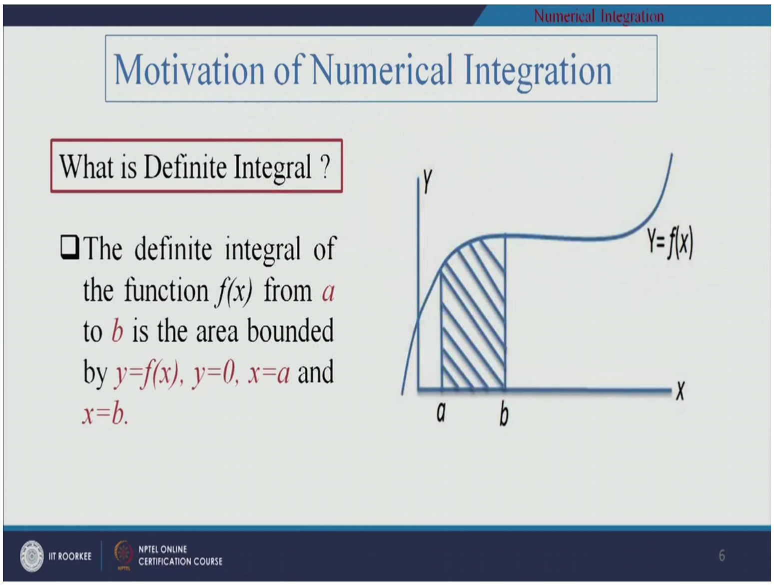

So in such cases so if you will just go for like finite definite integral or finite range integrals. So especially we are just defining a range there, y has a range like y equals to like 0 to some range suppose and x has a range like x equals to 0 to suppose a or x equals to a to b and within that point how this y curve is approximating or how this variation of y is occurring that we tried to estimate with the axis bounding region with this curve there. This means that if y equal to f of x is a function is defined and how this area is occurring or how this aerial interaction or this enclosure is occurring within this x-axis for 2 fixed points that can be evaluated by using integrations here.

So why we are just going for this numerical integration, if the function f of x is not given as I have told you but the values are given at a discrete points then we can just use numerical integration to find this values of the function within a given range. Sometimes if some of this complicated functions if it is given like E to the power minus x square suppose within the range of like 0 to 1 dx. So it is very difficult to carry out and in that case if you will just go for numerical integrations it is very easy to evaluate the integration’s.

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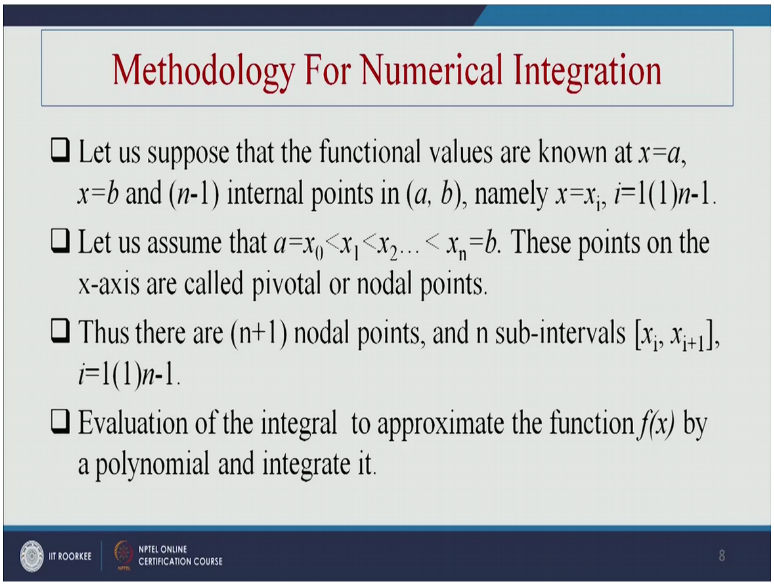

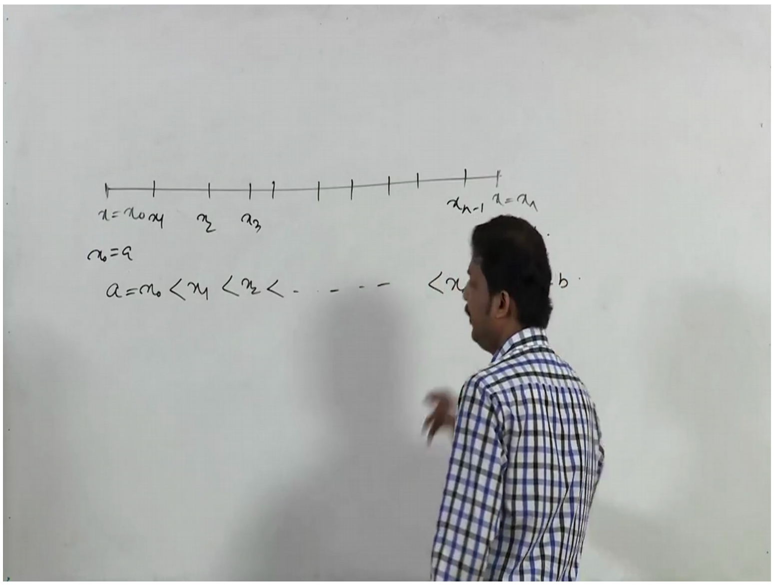

Let us suppose that the functional values are known at x equals to a and x equals to b and suppose a and b is divided into n equals this means that n minus 1 internal points in a, b. Suppose this internal points we are just naming as x 1, x 2, to x n minus 1, so if you will just visualize this one in a graphical sense here suppose we will have this points like x equals to x 0 to x equals to x n suppose then we can just divide this range x equal to x 0 equals to x 1 into suppose n's of intervals.

So we will have now this minus 1 internal points like x 1 to x 2, x 3 upto x n minus 1, obviously can just write or sometimes for convenience we are just expressing x 0 equals to a and x n equals to b. Then we can just define this points are as a equals to x 0 less than x 1 less than x 2 less than upto less than x n minus 1 less than x n this equals to b. So if we are just approximating any function with a polynomial approximation within this range suppose x 0 to x n here then we can just obtain a polynomial within that range.

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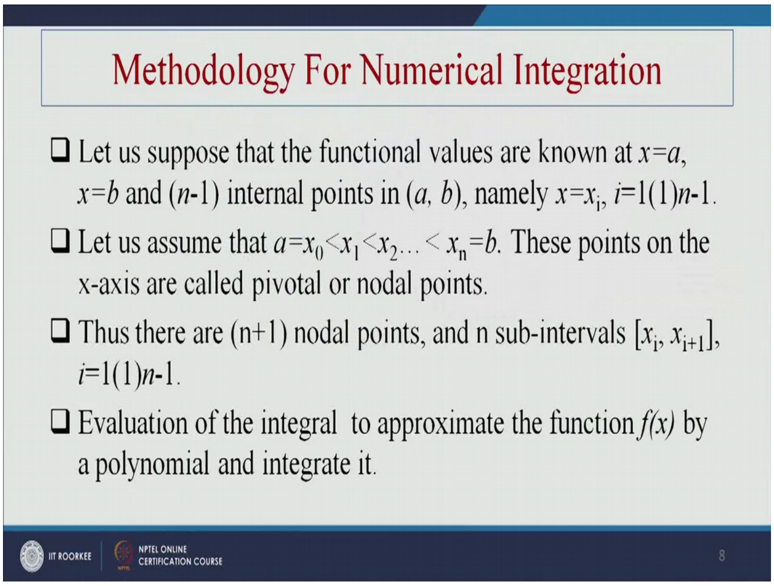

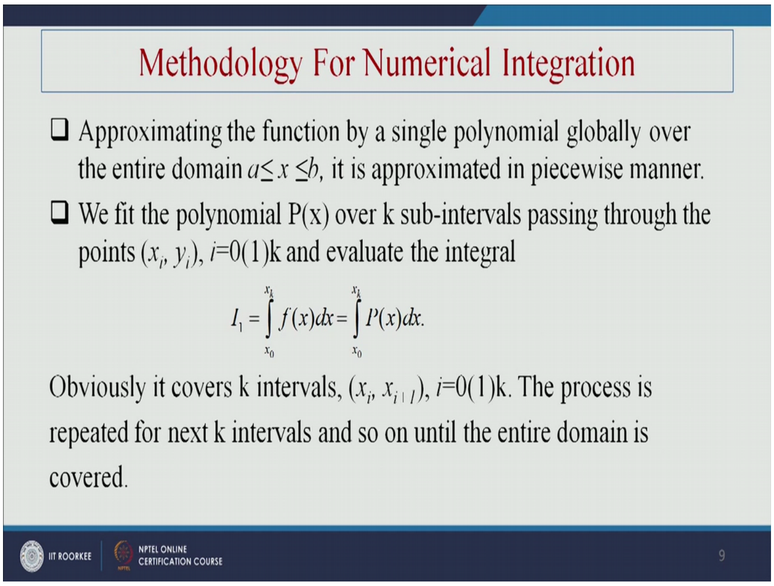

So in that range we will have n plus 1 nodal points with n's of intervals like x i, x i plus 1, if i is varying from 1 to n minus 1 here. So if you will just approximate this function with a polynomial or we want to fit the polynomial p of x over suppose k's of intervals here passing through the points like x i, y i here, i equals to 0, 1, 2 upto k and evaluate this integrals.

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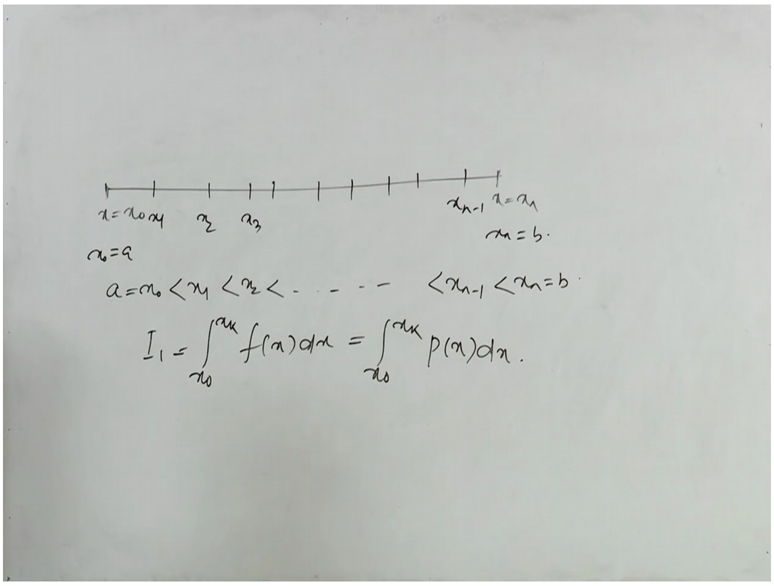

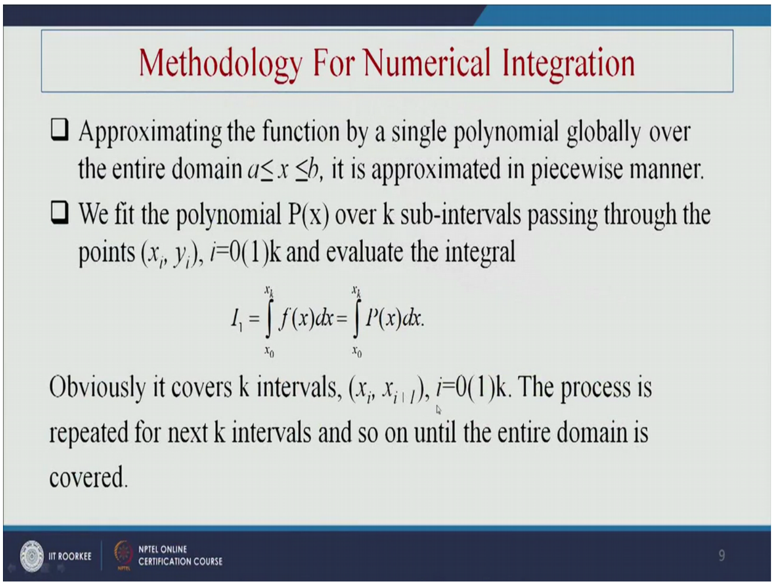

So then we can just write i 1 as x 0 to x k, f of x dx or we can just write this one as x 0 to x k, p of x dx here. Obviously if it covers k intervals like x i to x i plus 1 for i equals to 0, 1, to k here and this process is repeated for all the intervals this means that first it covers k intervals then in the repeated process in the next interval also it can covers like k intervals again, until over the entire domain should be covered off.

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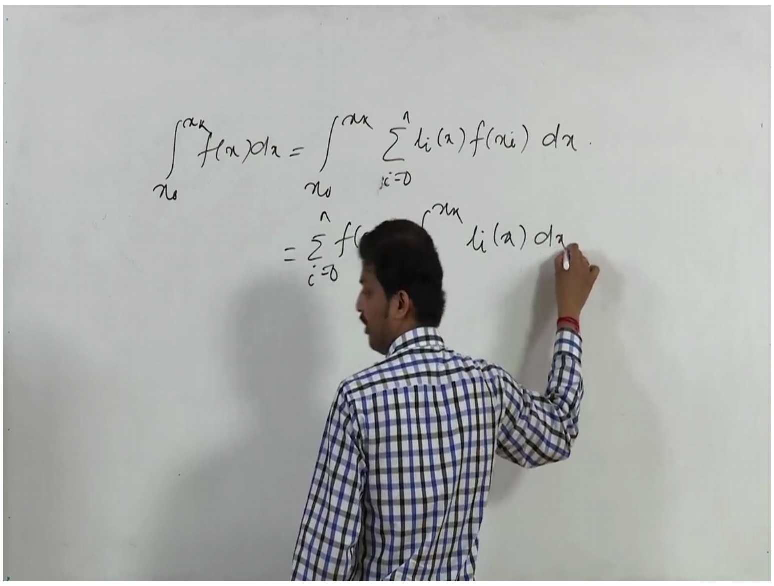

Then we will just go for the discussion of this numerical integration in a different interpolation formulas, first will just go for this Lagrange's formula here. So if we will just write this Lagrange's formula, this Lagrange's formula for this polynomial approximation is usually written in the form of summation of like i equals to 0 to n, l i x, f of x i, where l i x is called Lagrange's polynomial coefficients.

And usually this l i x is expressed in the form of like omega of x divided by x minus x i into omega dash of x i there. Specifically if we will just write in a elaborated form then we can just write l i x as x minus x 0, x minus x 1 to x minus x i minus 1, x minus x i plus 1 upto x minus x n divided by x i minus x 0, x i minus x 1 upto x i minus x i minus 1, xi minus x i plus 1 upto x i minus x n here.

If you will just write this integration here this integration can be written as integration of x 0 to x 1, f of x dx suppose. This can be written as or if we want to write this one as x 0 to x k suppose in each of this intervals if you will just evaluate x 0 to x k, f of x into dx this can be written as x 0 to x k summation of i equals to 0 to n, l i x, f of x i into dx here or it can be written in the form of summation of i equals to 0 to n, f of x i integration of x 0 to x k, l i x dx here.

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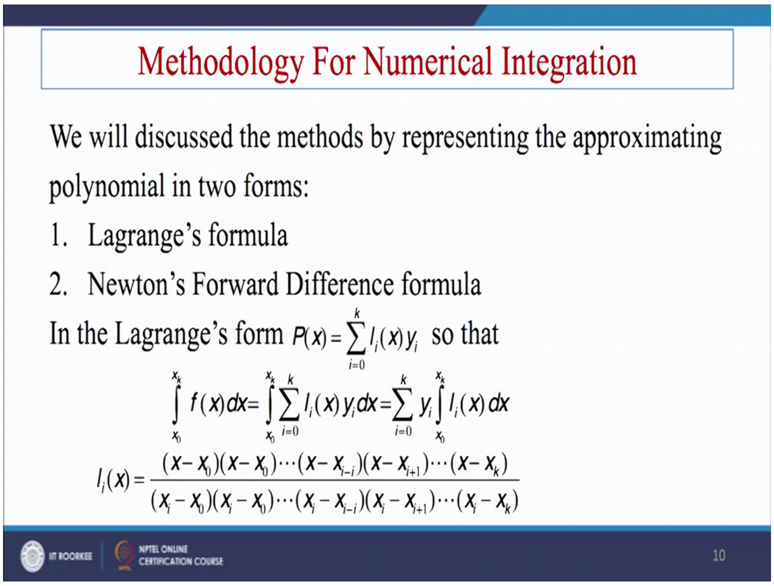

Already it is known that l i x can be expressed as the coefficient of x here which can be written in the form of either in the form of omega x by x minus x i into omega dash of x i there. So if we will just implement the numerical integration for all this product terms there then we can just evaluate this numerical integration for this polynomial there.

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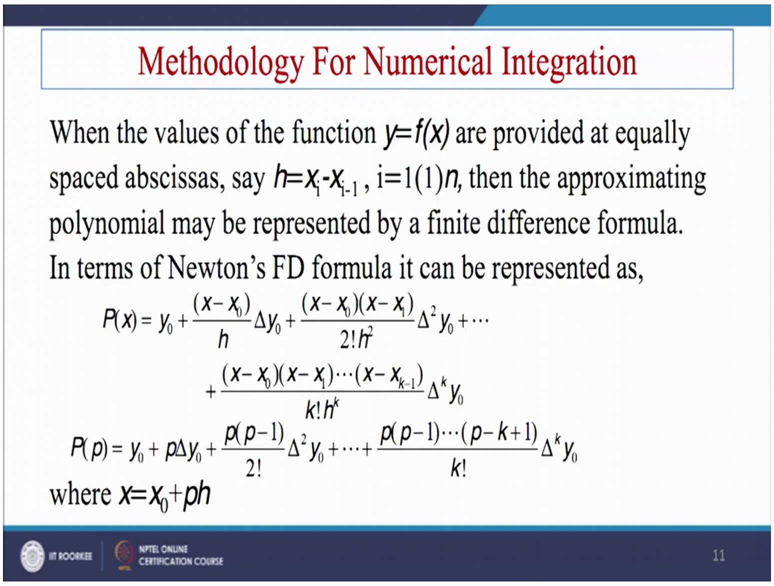

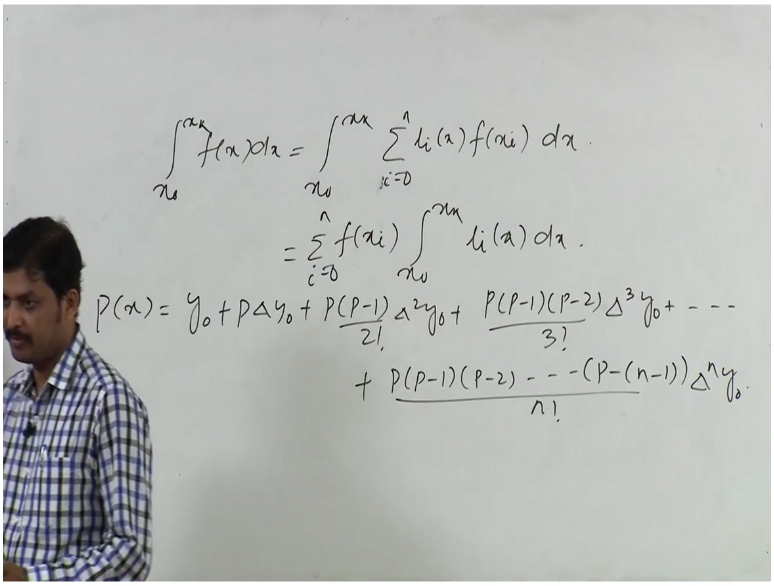

So next we will just go for this as numerical integration using Newton's forward difference formula, when the values of function y equals to f of x are provided suppose at equally spaced abscissas then we can just use this Newton's forward difference formula. And in finite different approximation usually this Newton's forward difference formula in polynomial form are written as y 0 plus p delta of y 0, p into p minus 1 by factorial 2, del square of y 0 plus p into p minus 1, p minus 2 divided by 3 factorial, delta cube of y 0 plus upto finite number of terms that is as p into p minus 1, p minus 2 upto p minus n minus 1 divided by n factorial, nabla to the sorry delta to the power n of y 0 here plus the remainder term.

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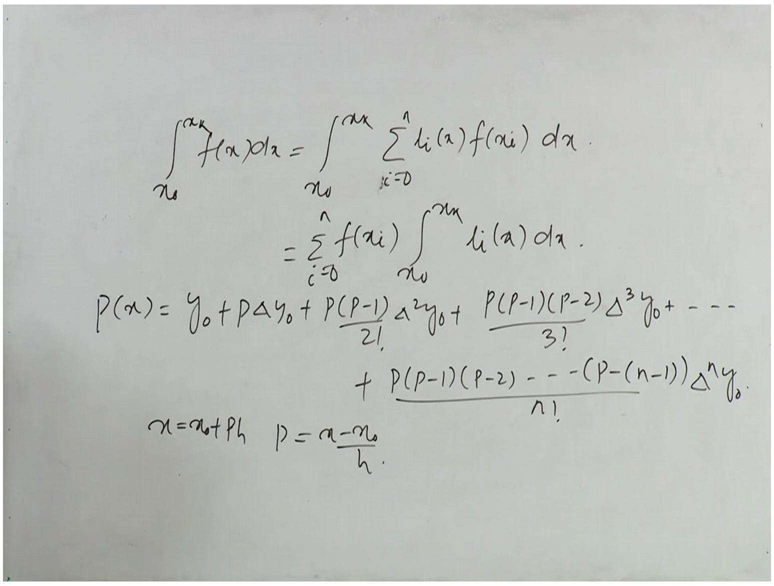

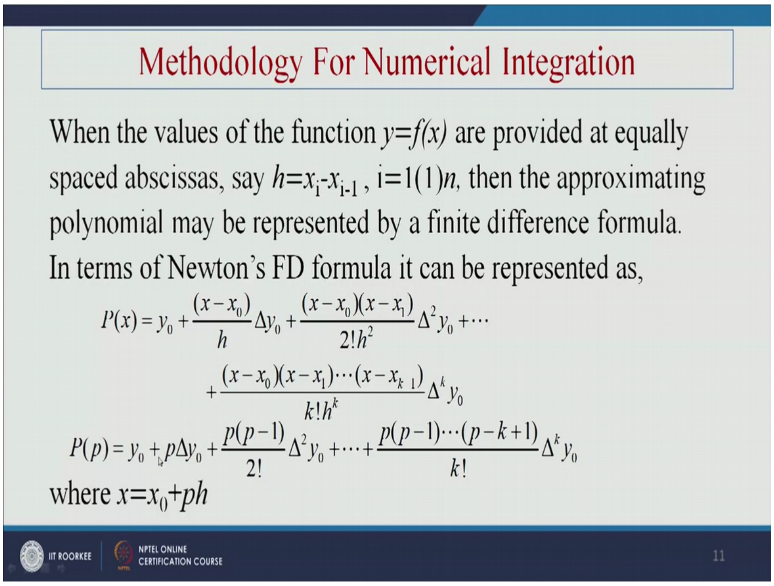

So if you will just replace in terms of x here usually we are just replacing here x equal to x 0 plus p h here. So that is why p can be written as x minus x 0 by h here. And if you will just replace this p value in terms of x here then p of x can be written as y 0 plus x minus x 0 by h into delta of y 0 then x minus x 0 into x minus x 1 by 2 factorial, h square, del square of y 0 plus all other terms in the same fashion.

So we will just write this polynomial in terms of p usually we can just denotes that one in terms of p here that is p of p can be written as y 0 plus p delta of y 0, p into p minus 1 by 2 factorial, del square of y 0 upto p into p minus 1 upto p minus k plus 1 by k factorial, del to the power k of y 0 there.

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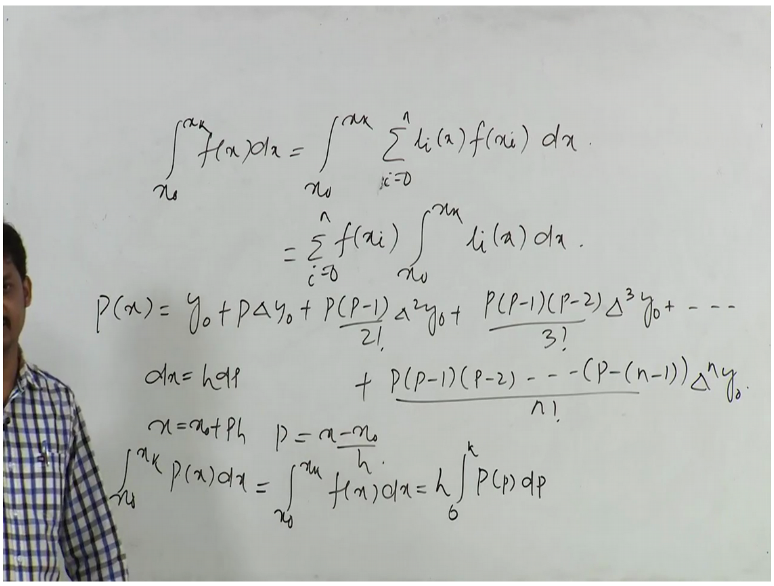

So if you will just use this integration for this formula here then we can just write integration of x 0 to x k, p of x dx here or it can be written as x 0 to x k, f of x dx. This equals to we can just write h into integration of like 0 to since x 0 to x k here, 0 to k we can just write and p of p into d p here.

Since if you will just see here dx is nothing but h d p if you will just write this one, so that is why this h is coming out and at this point x equals to x 0, if i will just put p equals to 0 and if i will just put x equals to like x k that is nothing but x 0 plus k h, so that is why p equals to k there. So that is why we are just changing this intervals that is in this form here.

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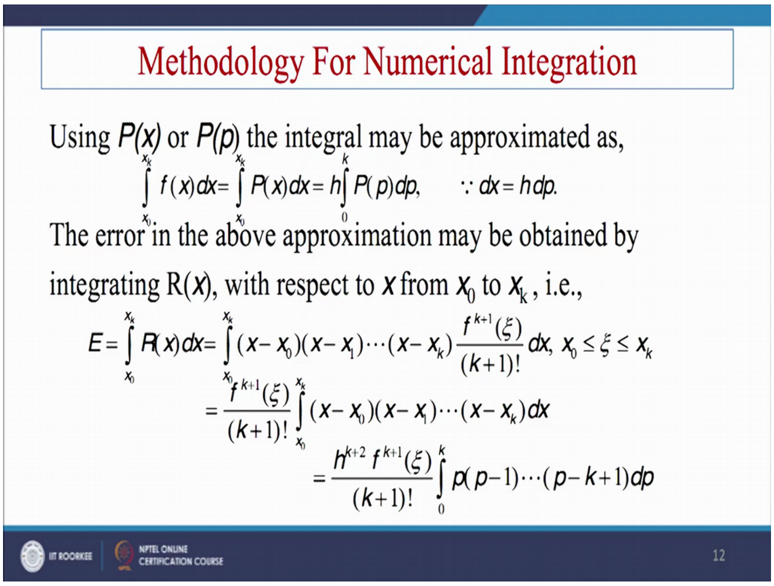

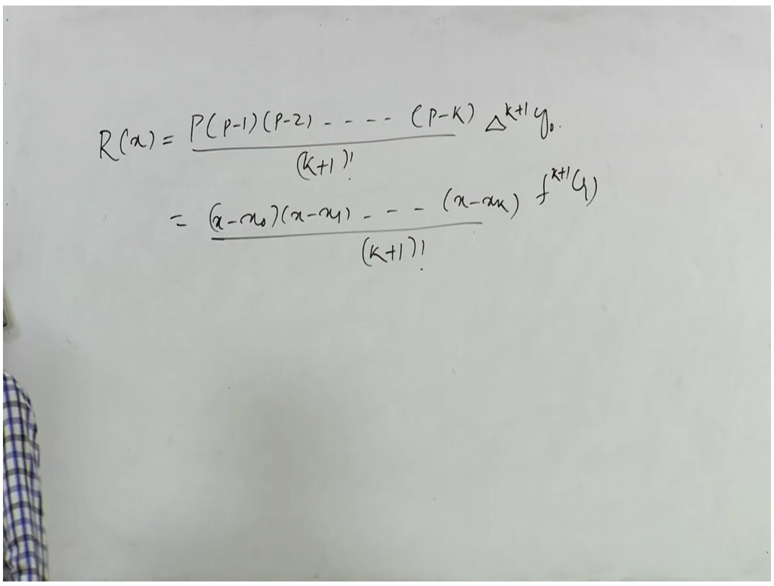

So in this regard also we can just write this error term also, so usually this error terms are usually written in the form of like x minus x 0. So for this Newton's divided difference formula if you will just write this error term that can be written as, so this error term this Newton's interpolation formula R of x can be written as p into p minus 1 into p minus 2 upto p minus k if you will just write to k-th term here.

So we can just write in this form or divided by you can just write k factorial here and del to the power sorry this is k plus 1 factorial here and del to the power k plus 1 y 0 here. And in terms of x if you will just write this can be written in the form of like x minus x 0, xi minus x 1 upto x minus x k divided by k plus 1 factorial and it can be written also h term it is also there, so that is why we can just write this one as f to the power k plus 1 zeta divided by k plus 1 factorial here.

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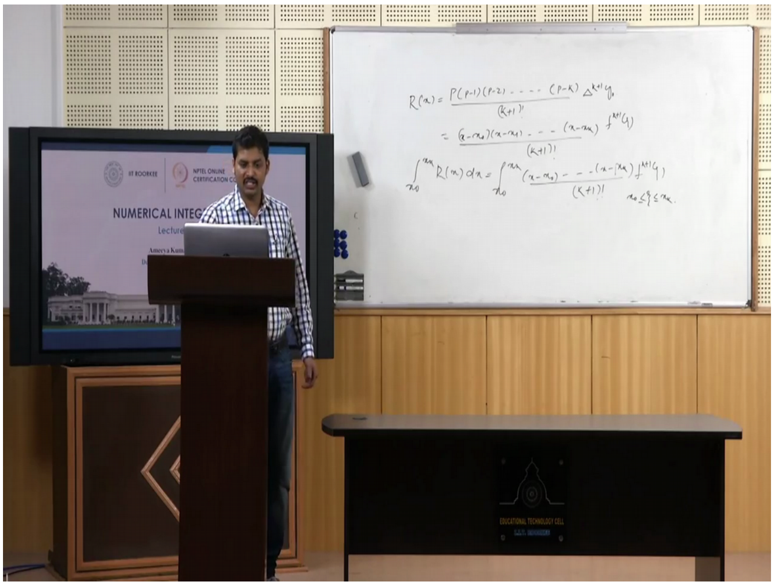

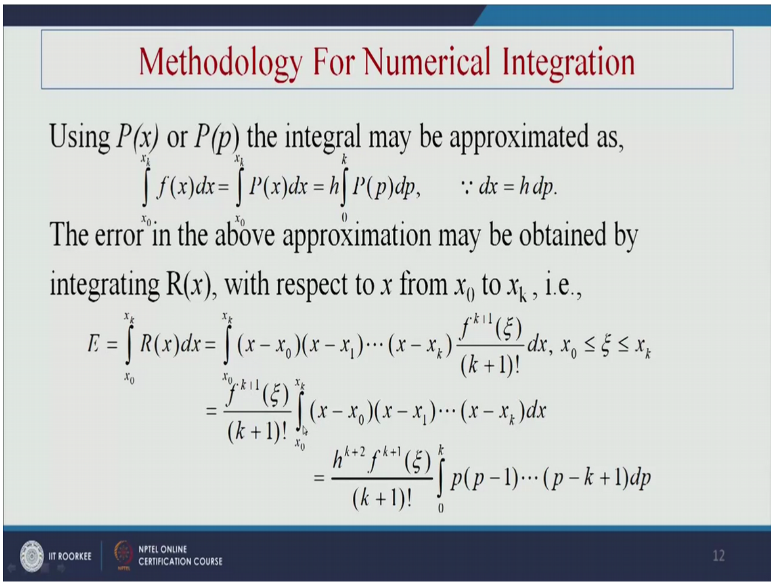

If you just integration this one within this range of x 0 to x k here, so this can be written as x 0 to x k, R of x dx this can be written as x 0 to x k here like x minus x 0 to x minus x k here divided by k plus 1 factorial, f to the power k plus 1 zeta here, where zeta should be lies between x 0 to x k here. So if we will just take out this term like f of the power k plus 1 zeta by k plus 1 factorial as that are constant so it can be taken out, so inside this integrations we can just write x 0 to x k, x minus x 0, x minus x 1 to x minus x k into dx here.

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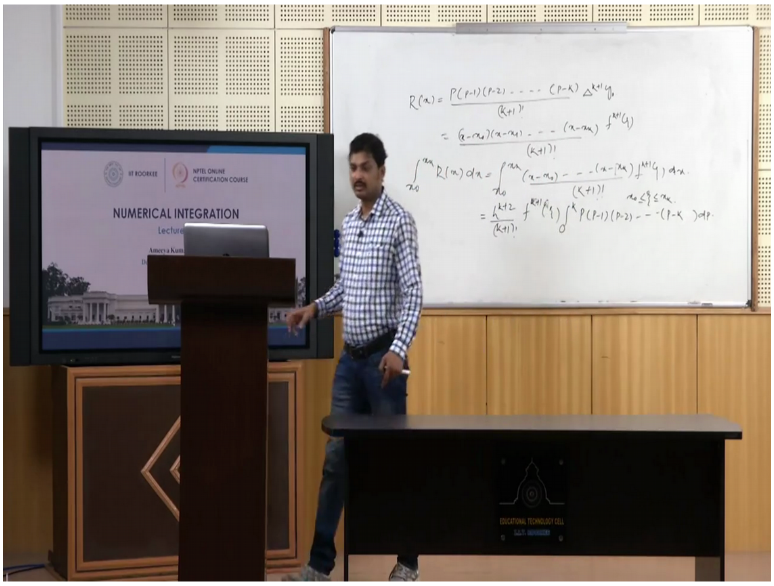

So if you will just eliminate the dx as in the form of h d p here, so then x minus x 0 and x minus x 1 in terms of p here, so we can just write that the total term can be written as h to the power k plus 2, this can be written as a h to the power k plus 2 divided by k plus 1 factorial, f to the power k plus 1 zeta, integration from 0 to k, p into p minus 1, p minus 2 upto p minus k plus 1 here into d p. Sorry this should be p minus k upto that one since x minus k is there.

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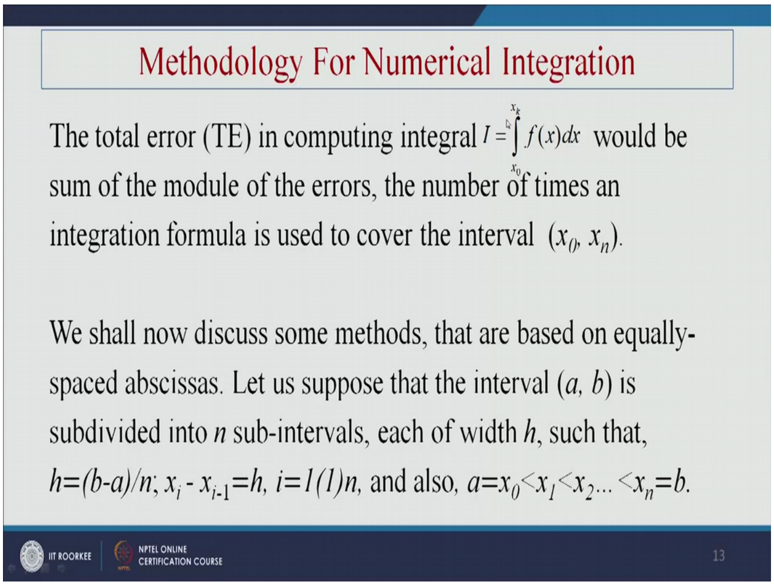

So this is the integration formula, so if you just go for this like total error in computing the integral i equals to x 0 to x k, f of x into dx, then sum of the model of errors that is the number of times it is just occurring in each of intervals that can be taken off and sum of all this terms to cover off this error terms in each of this intervals x 0 to x n. So now we shall discuss some of the methods that are based on equally spaced abscissas, suppose that interval a, b is subdivided suppose n's of intervals here each of width suppose h here then we can just define h equals to b minus a by n here.

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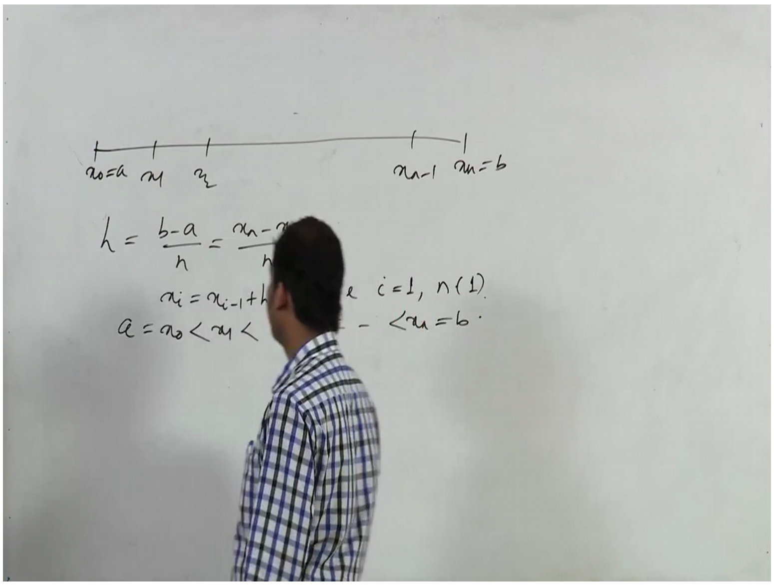

So suppose in a graphical sense if you just visualize we can just define x 0 to x n upto x n equals to b here, so then we can just define if all this points are equally spaced here then we can just define h equals to b minus a by n or we can just write x n minus x 0 by n here. And each of this points since it is equi-spaced here we can just write x i equals to x i minus 1 plus h here, where i is varying from like 1 to n here incremented by 1.

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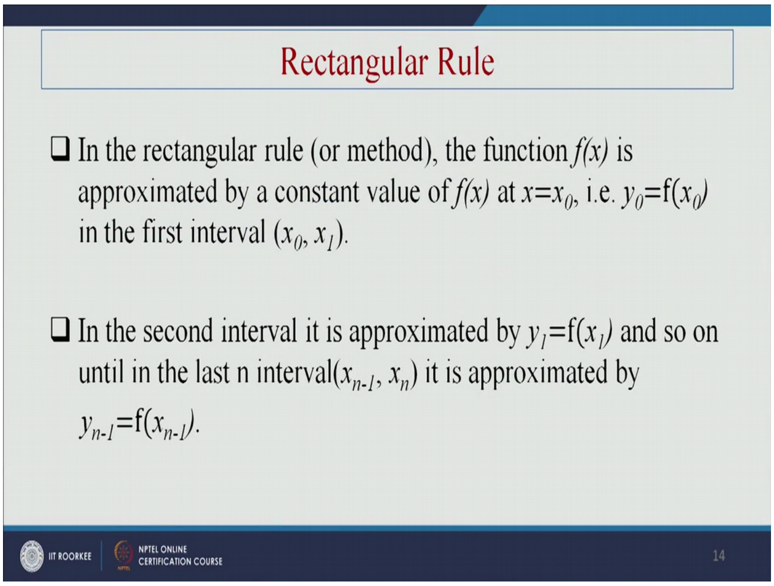

So and we can just define this points that is in the form of like a equals to x 0 less than x 1 upto less than x n equals to b here. And if you just go for this rectangular rule suppose, in the rectangular rule form usually this method is applicable whenever we have a constant value of function which attempts its maximum value within that range. This means that if suppose we will have a integration range from x 0 to x 1 then we can just assume that y 0 or this functional values at x 0 can be considered as a constant value for f of x within the whole interval there.

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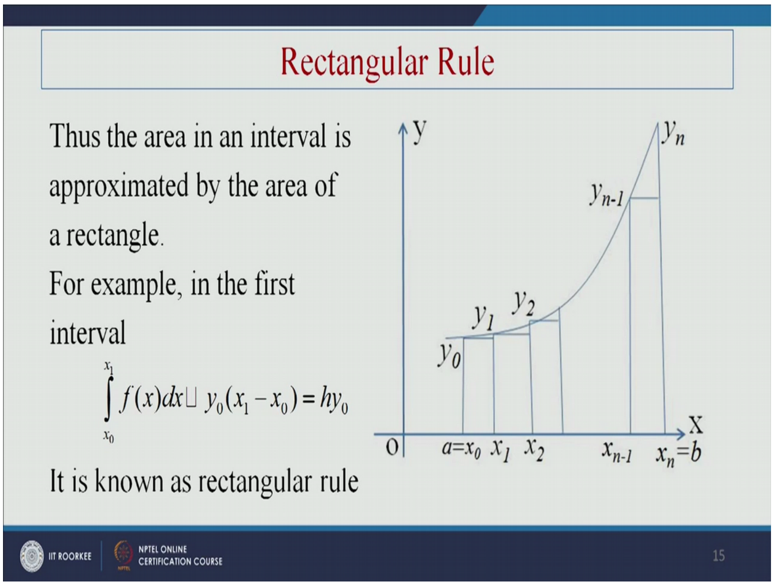

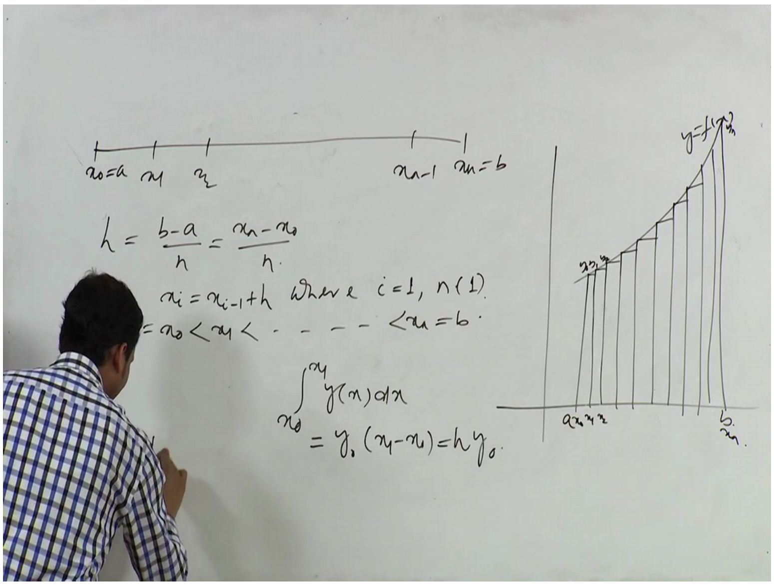

And in the 2nd interval the same fashion we can just approximate that one y 1 as f of x throughout the interval and in that case we can just write this integration as in the form of like a real cross-section and within that if you will just divide each of this intervals in a rectangular sections like if suppose we will have this integration range y equals to f of x is given to us and it is asked to evaluate this integration within this range a to b suppose we can just subdivide this total regions into n's of intervals here. And if you will just take this small section as a straight line or this perpendiculars we can just draw off at each of this points here since sufficient close intervals we are just considering, so that is why the error of approximation should be minimized in this sense here.

If you will just consider the starting point as x 0, next point is x 1 and next point is x 2, likewise the last point is x n here and corresponding y values like y 0, y 1, y 2 likewise the last point will be y n here, then we can just consider this integration within this range like x 0 to x 1, y 1 of x dx or f of x dx which can take the maximum value y 0 within this range x 0 to x 1 here. Then we can just write this integration as y 0 into x 1 minus x 0 here and this can be written as h, y 0 here.

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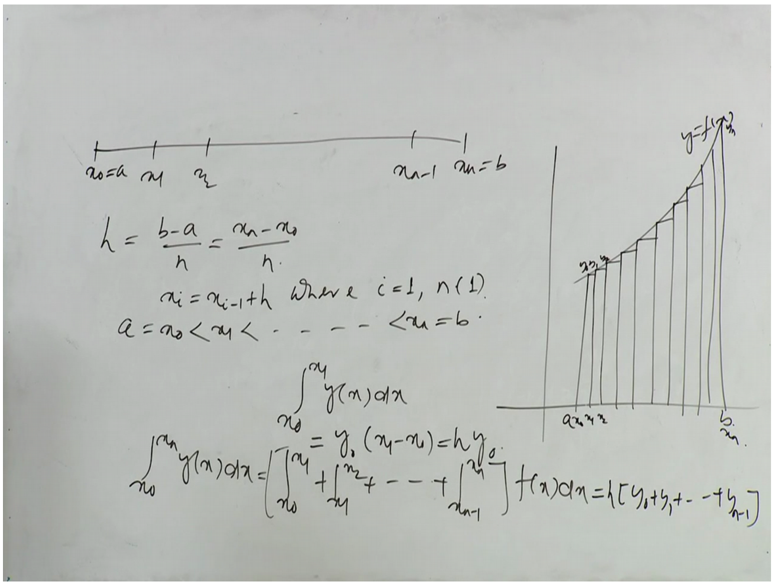

So similarly we can just define this integration range like x 1 to x 2, x 2 to x 3 in the same fashion and adding of if you just add up all this areas like x 0 to x n suppose y of x dx or f of x dx this can be written in the form of like x 0 to x 1, x 1 to x 2 upto x n minus 1 to x n, f of x dx here. Then each of this intervals just first interval we can just obtain h of y 0, 2nd we can just obtain h of y 1 and the 3rd one we can just obtain that one h of y 2. So likewise if you will just sum of all this terms here then the final integration we can just get it as h into y 0, y 1, upto y n minus 1 here.

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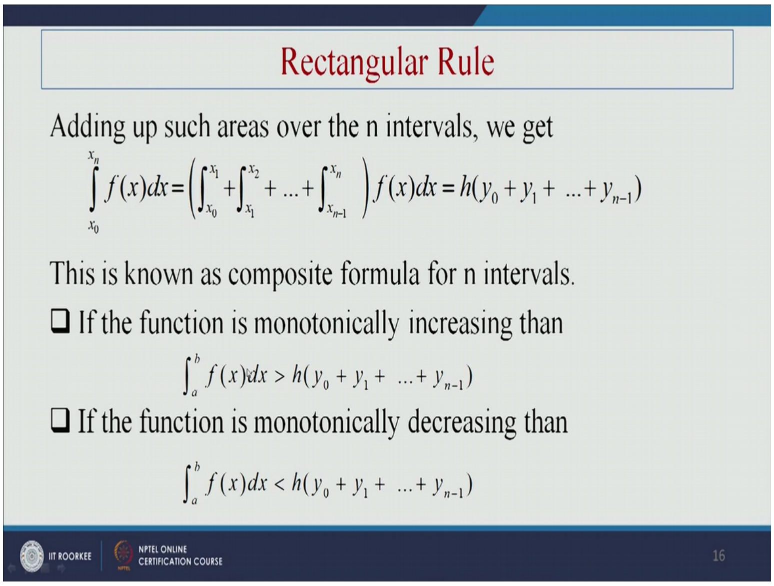

And if the function is suppose monotonically increasing then we can just write integration a to b, f of x dx is greater than h of y 0 plus y 1 to y n minus 1. If suppose it is monotonically decreasing then we can just define as integration a to b, f of x dx is less then h into y 0 plus y 1 to y n minus here. So then we will just go for like rectangular rule used for Lagrange's formula.

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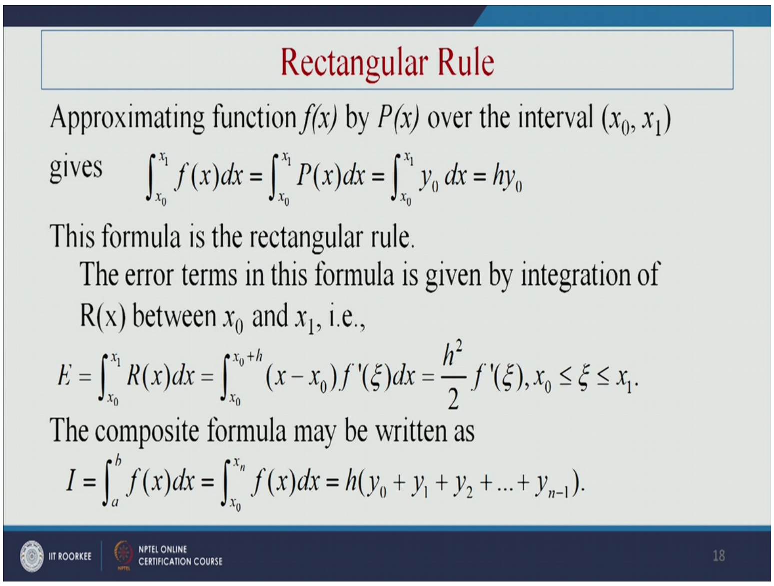

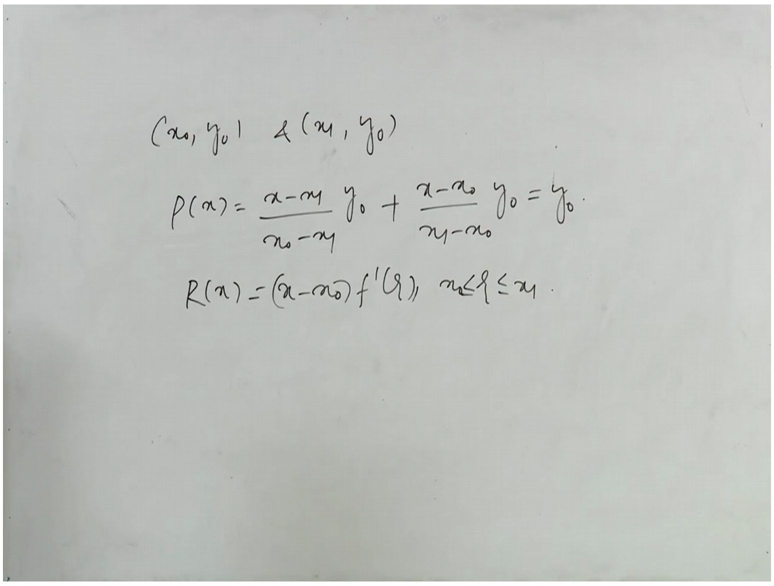

So in the Lagrange's formula suppose if the interval is given like x 0 to x 1 and it is asked to fit with a polynomial passing through the points like x 0, y 0 and x 1, y 0 suppose since the functional values are same at both this points. So then we can just define this polynomial as we will have this points like x 0, y 0 and x 1, y 0 here, this functions values will remain constant at both this points.

Then we can just define this Lagrange's polynomial as x minus x 1 divided by x 0 minus x 1, y 0 plus x minus x 0 divided by x 1 minus x 0, y 0 here, so then this total polynomial can be written as y 0 here. And if you will just approximate this error for this polynomial then this error term R of x can be written in the form of like x minus x 0, f dash of zeta, where zeta should be lies between x 0 to x 1 here.

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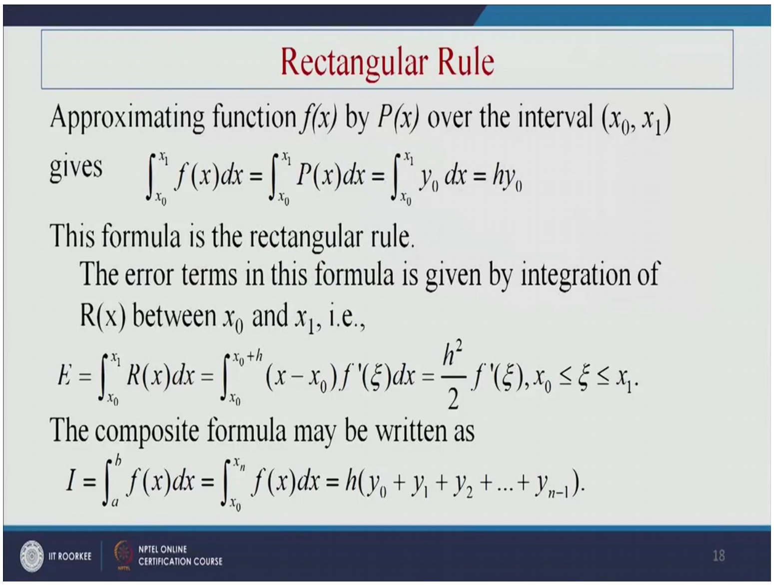

And if you will just integrate this linear interpolating polynomial for Lagrange's method then we can just write this one as x 0 to x 1, p of x dx, this can be written as y 0 into x 1 minus x 0 that is nothing but h of y 0 here. And this formula is the rectangular rule especially it is called and if you will just approximate the error terms also, this error terms can also be written as E equals to x 0 to x 1, R of x into dx here.

And that can be written as x 0 to x 0 plus h, x minus x 0, f dash of zeta into dx here, which can be written as h square by 2, f dash of zeta, where zeta should be lies between x 0 to x 1. And in the same form we can just obtain this composite formula for this linear interpolating polynomial in Lagrange's method as a to b, f of x dx that as x 0 to x n, f of x dx, this equals to h into y 0 plus y 1 plus y 2 upto y n minus 1 here.

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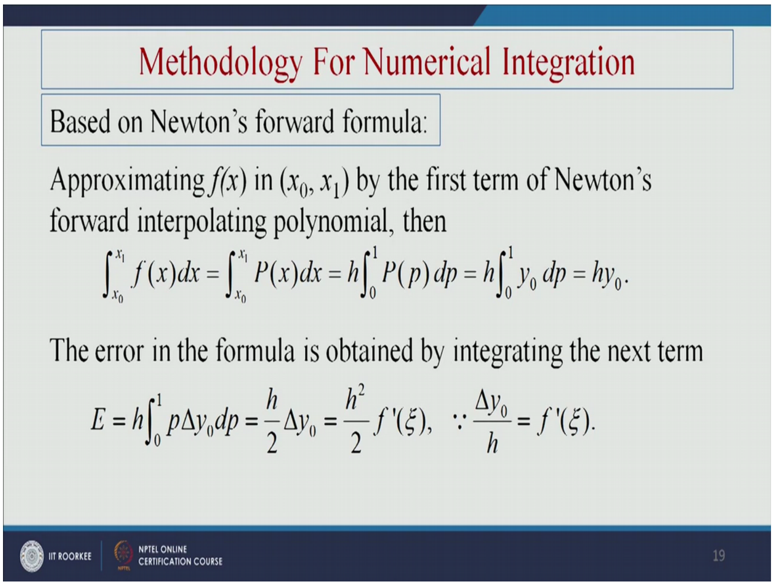

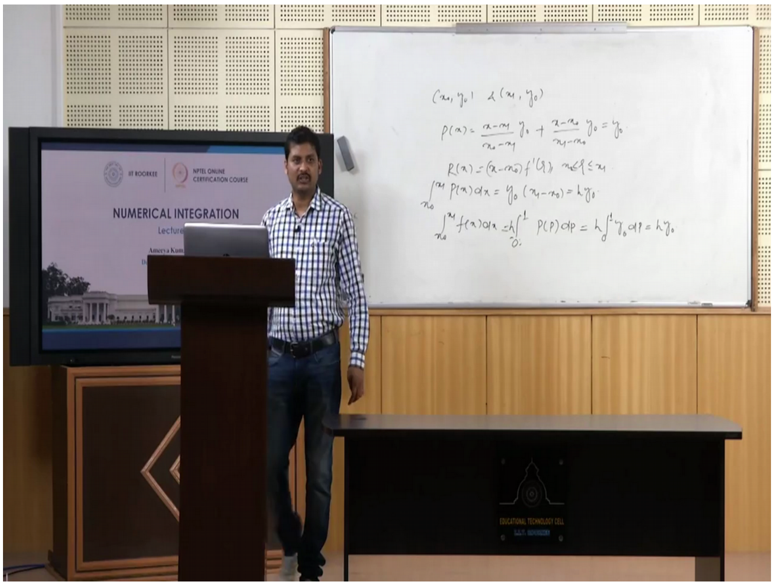

So based on Newton's forward formula if you will just go for this calculation is here like if we will have this function f of x in x 0 to x 1 by terms of Newton's forward interpolating polynomial here, usually we can just write x 0 to x 1, f of x dx here or that can be written in the form of x 0 to x 1, p of p into dx here. And obviously this integration range for the p of p can be transformed to 0 to 1 and dx can be written as h d p here.

So that is why we can just write this one as h, 0 to 1 into your functional value y 0 into d p here and this specially can be written as h y 0. And in a similar fashion also we can just determine the error for this formula that can be written as E equals to h, 0 to 1, p delta of y 0 into d p, so which can be written as h square by 2, f dash of zeta.

Since we have known that del y 0 by h is equals to f dash zeta, so in both this method we are obtaining this same formula for rectangular rule. Thank you for listening this lecture on numerical integration, next lecture we can just continue for this numerical integration based on Newton's forward difference formula and all other interpolation formulas.