Numerical Methods
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Lecture 30
Numerical differentiation part-VI (Undetermined coefficients unequal intervals)

Welcome to the lecture series on Numerical methods and currently we are discussing here numerical differentiation. So basically we are just dealing here this numerical differentiation based on different interpolation formulas like some of the finite different operators, some of the like unequal spaced intervals. So today we will just discuss about this numerical differentiation based on undetermined coefficients and some of these approximations using unequal intervals based on general formulas like Taylor series expansion.

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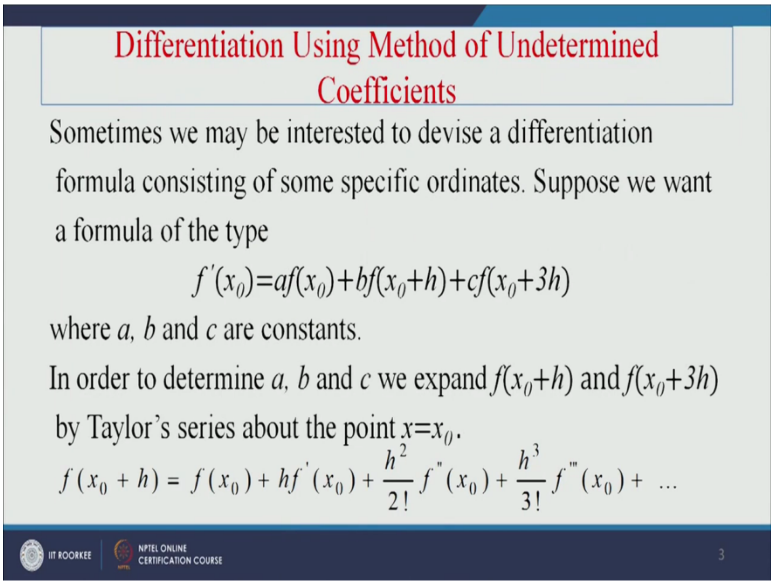

So first we will just go for this differentiation using method of undetermined coefficients, sometimes we may be interested to devise a differentiation formula considering some of the specific ordinates. This means that if some of the special coordinates are consisting of certain differentiation formulas or certain differentiation approximations it has asked you to evaluate so we can just do that one.

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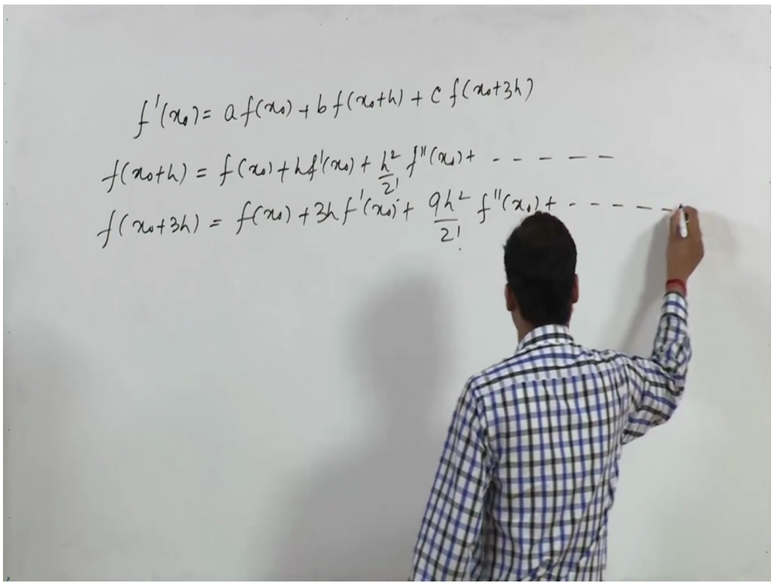

Suppose a formula is written in the form like f dash of x 0, this = a f of x 0 + b f of x 0 + h + c f of x 0 + 2 h here sorry 3 h suppose. So how we can just determine these coefficients a, b, c from this derivative here? Specifically if we want to determine these coefficients a, b, c, we have to expand this f of x 0 + h and f of x 0 + 3 h by Taylor series expansion here about this point x = x 0. So the basic idea is that sometimes if these nodal points are defined and if the functional values are given at that nodal points so how we can just evaluate these derivatives at that point suppose.

And here the hint is given that suppose your first order derivative is prescribed at a particular point that is in the form of like f of x 0 + b f of x 0 + h + c f of f of x 0 + 3 h here and it has asked you to like find these coefficients for this derivative formula here. So if you will just expand this Taylor series expansion of f of x 0 + h at point x 0 then we can just write this Taylor series expansion as f of x 0 + x h f dash of x 0 + h square by factorial 2 f double dash of x 0, so likewise we can just write. Similarly if you will just expand f of x 0 + 3 h here, this can be written as f of x 0 + 3 h f dash of x 0 + 9 h square by factorial 2 f double dash of x 0 + all other terms.

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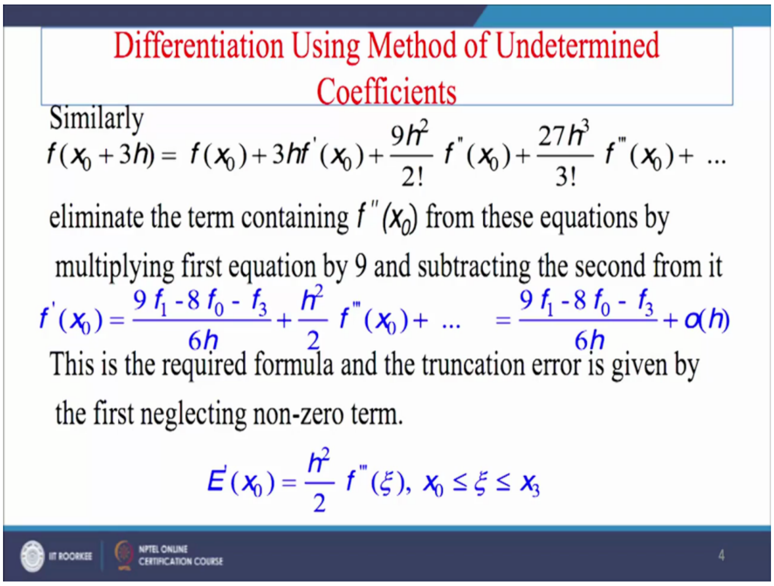

So if we will just eliminate here the terms containing f double dash of x 0 from these 2 equations by suppose multiplying this first equation by 9 and subtracting the second from 8. If we will just do that one so if you just see here so we have to multiply here 9 and then if you just subtract, this term will be just go out and from there itself we can just evaluate f dash of x 0 in terms of f of x 0 and f of x 0 + h and f of x 0 + 3 h and all other remaining terms that is of higher powers of h containing f triple dash of x 0 and all other higher order derivatives there.

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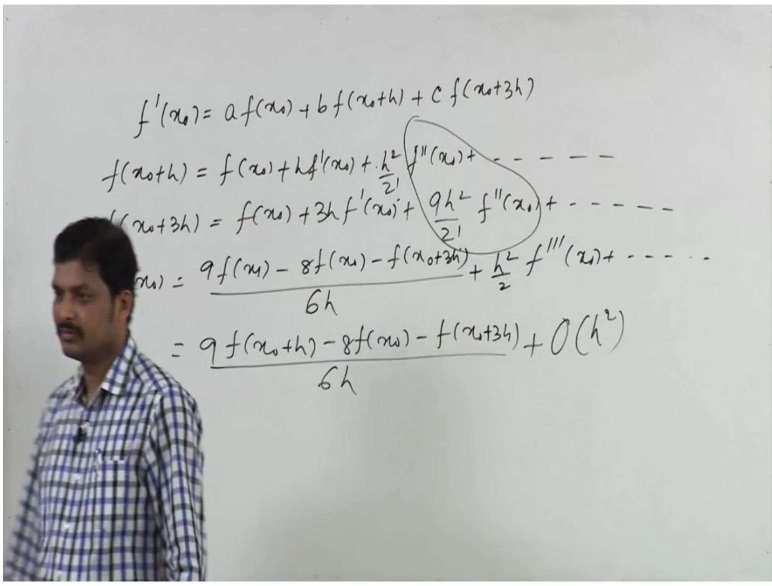

So if you just multiply this equation 1 by suppose 9 and from the second equation if you just subtract here then we can just obtain f dash of x 0 as 9 f of x 1 – 8 f of x 0 – f of x 0 + 3 h here divided by it can be written as 6 h + h square by 2 f triple dash of x 0 + all other terms are there. So if you will just consider this high-powered terms as order of h here then we can just write this expression as f of x 0 + h – 8 f of x 0 – f of x 0 + 3 h divided by 6 h + order of h square here or order of h you can just also write sometimes since higher powers of h can be neglected if the h size is very small here.

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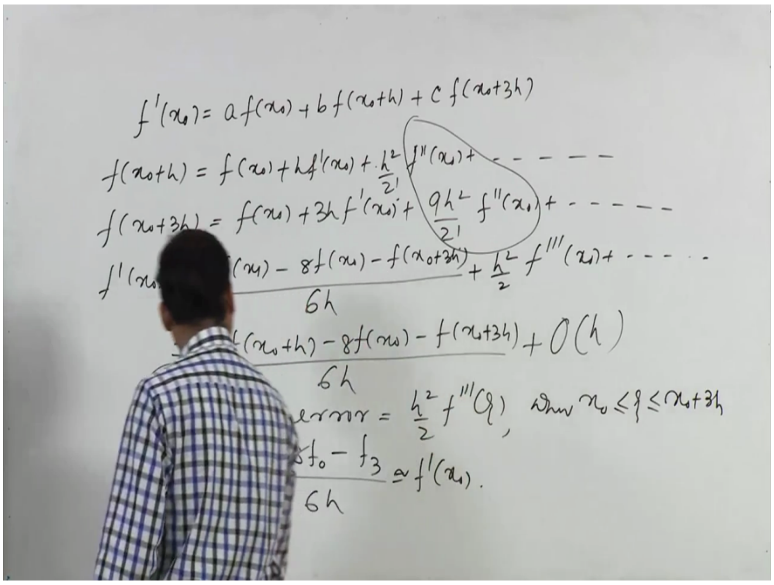

And if we will just consider this (())(5:37) error term here so the (())(5:39) error term can also be written in the form like this = h square by 2 f triple dash of Zeta where Zeta lies between x 0 to x 0 + 3 h. Obviously in a compatible way if we want to write so we can just write this formula that is in the form of like 9 f 1 – 8 f 0 – f 3 divided by 6 h, this = or approximates to f dash of x 0 here. So this above formula can also be obtained by equating these coefficients after expanding the function by Taylor series expansion also sometimes we can do that one.

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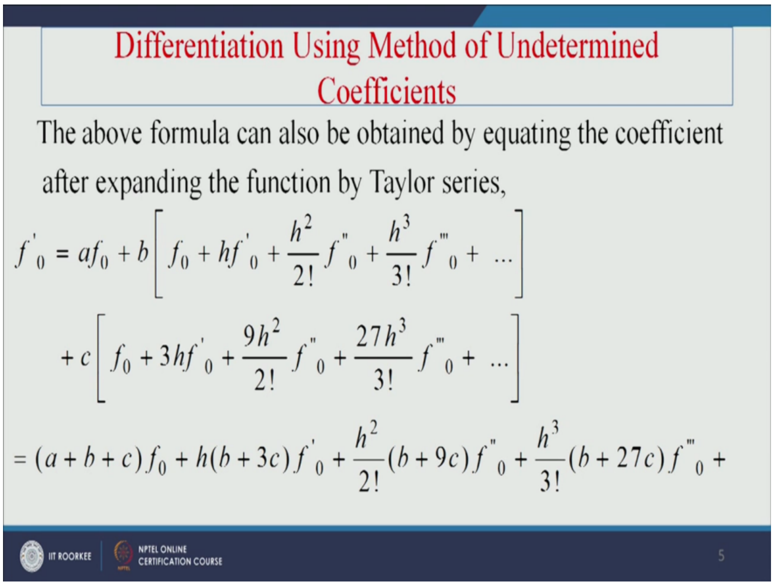

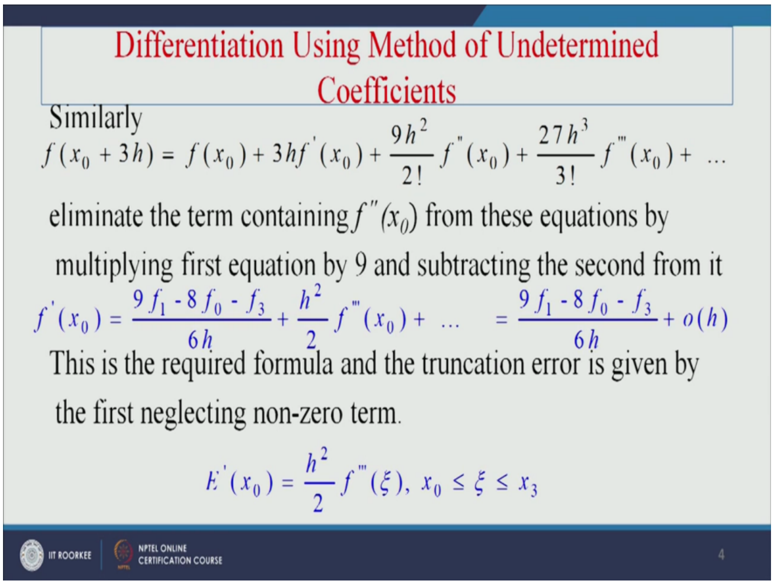

But in this form especially if you just write here like 9 f 1 – 8 f 0 – f 3 by 6 h so we have to consider this h size h size should be small there yeah so then these higher powers of h can be neglected. So if you just go for this other method of determining these coefficients, if you will just expand this one by Taylor series, if you just see the first-term a f of x 0 we will just keep that one as a f 0 here and the second term we can just write this one in the Taylor series expansion form and third term we can just write it in Taylor series expansion form, then both sides we will just equate the coefficients and if we will just solve these equations then we can just find the values of a, b, and c there.

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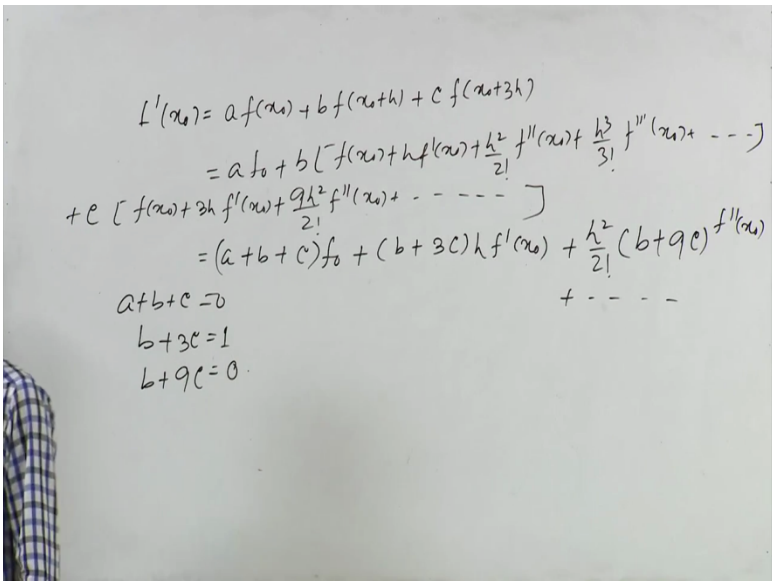

So for that if we will just rewrite this equation in the form as this one this = a f 0 here so then b of f of x 0 + h f dash of x 0 + h square by factorial 2 f double dash of x 0 + h cube by 3 factorial f triple dash of x 0 + all other terms. Similarly + c into f of x 0 + 3 h f dash of x 0 + 9 h square by factorial 2 f double dash of x 0 + all other terms. If you will just equate the first coefficient here like f 0 coefficient then right-hand side we do not have any or in the left-hand side we do not have any coefficient of x 0 there. So that is why we can just write this coefficient as 0 if you will just consider only f 0 coefficients from the right-hand side there.

So first if you just separate all these coefficients from the right-hand side then first one it can be written as a + b + c here into f 0, second one if you just see here that is b + 3 c h f dash of x 0 + h square by 2 factorial if you will just see here so that is b coefficient it is just taking b + 9 C here into f double dash of x 0 here + remaining terms are there. So if you will just equate these coefficients, so first coefficient we can just write a + b + c this = 0 here, next coefficient we can just write B + 3 C this = 1 here and the next coefficient if you just write B + 9 C this = 0 here.

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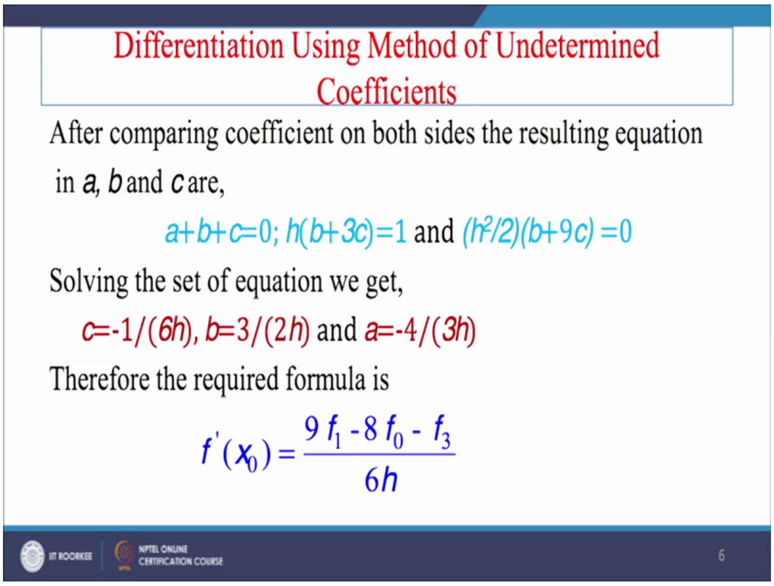

So if you will just solve these 3 equations since we have here 3 constants so that is why if we will just solve these 3 equations then we can just obtain that one the solution as c = – 1 by 6 h and b as 3 by 2 h and a as 9 4 by 3 h here. If you will just put all these coefficients then directly we can just obtain this formula for f dash of x 0 as since f dash of x 0 that is written as – 4 by 3 h f 0 + b as here 3 by 2 h f 1 + C as here – 1 by 6 h here f 3. And if you will just solve this equation then we can just obtain this formula as 9 f 1 – 8 f 0 – f 3 by 6 h here that is the basically that formula we have obtained in the earlier formulation also.

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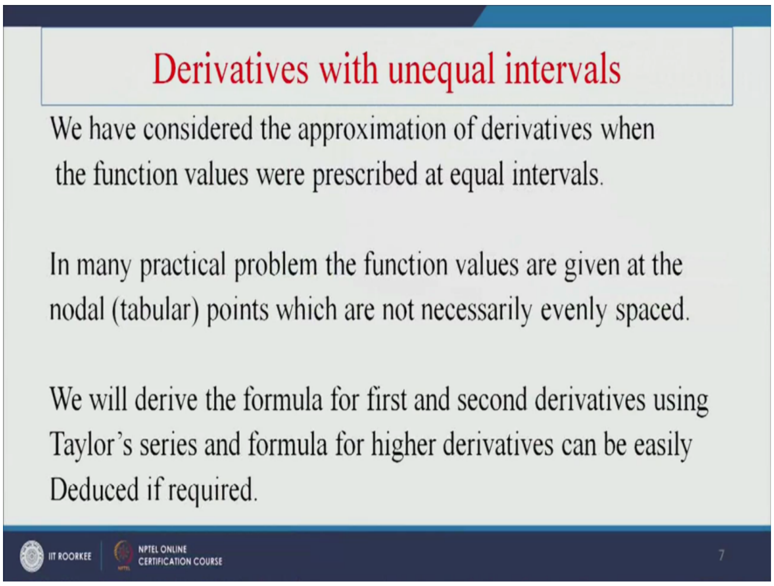

Next we will just go for this derivative that deals with like unequal intervals here. So already we have considered this approximation of derivatives and the functional values were prescribed at equal intervals at the beginning we have already discussed in previous lecture. And in many practical problems usually we can just get that one the function values are given at the nodal points which are not necessarily evenly is spaced. And if we will not use any interpolation formula like Lagrange interpolation or divided difference Newton's divided difference interpolation formula how we can just find this differentiation using directly this Taylor series expansion if we have unequal spaced points that we will just discuss now.

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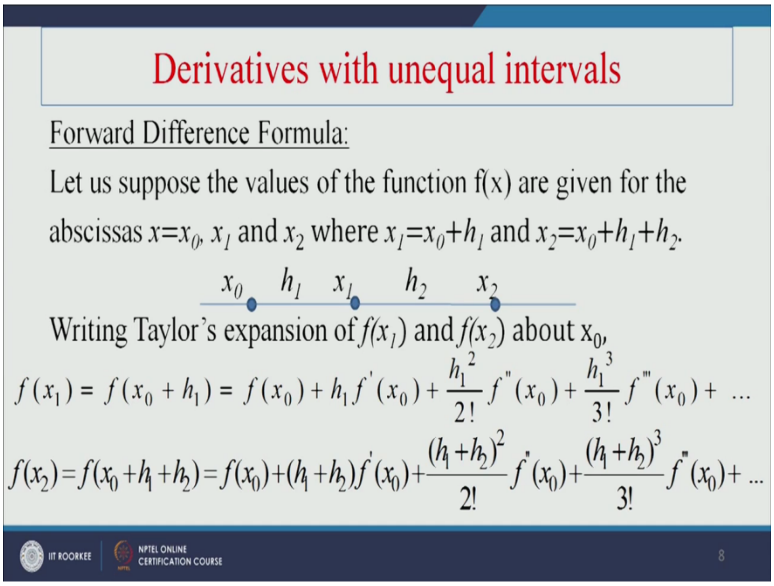

So if we just go for that formulas, first if we will just discuss about like forward difference formula here. Let us suppose that the functional values f of x are given and (())(12:20) like x = x 0, x = x 1 and x equal x 2 suppose. And if the space points are situated at h 1 distance from x 0 the first point and second point if it is spaced at suppose x 1 to x 2 this space size is x 2 here, so we can just write these 3 point suppose, the (())(12:44) are prescribed at x = x 0, x 1 and x 2 here and the points are placed like starting point is x 0 here and at a point h 1 distance suppose x 1 is placed and at distance of h 2, x 2 is placed there.

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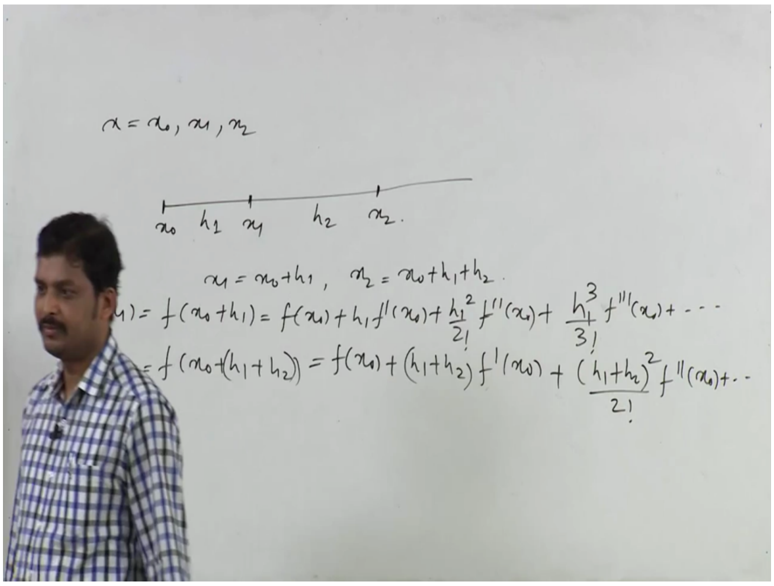

So if we will just write here in terms of x 0; x 1 can be written as x 0 + h 1 here and x 2 can be written as x 0 + h 1 + h 2. If you just do the Taylor series expansion at f of x 1 and f of x 2 about x 0 here, so then we can just write f of x 1 this can be written as f of x 0 + h 1 here and that can be written as f of x 0 + h 1f dash of x 0 + h 1 square by factorial 2 f double dash of x 0 + h 1 cube by factorial 3 f triple dash of x 0 + rest of the points. Similarly if you will just write here f of x 2 here, this can be written as f of x 0 + h 1 + h 2 here and obviously if you just keep this one as in the form of h here like x 1 + h 2 this can be written as f of x 0 + h1 + h 2 f dash of x 0 here + h 1 + h 2 whole square divided by 2 factorial f double dash of x 0 so likewise we can just write.

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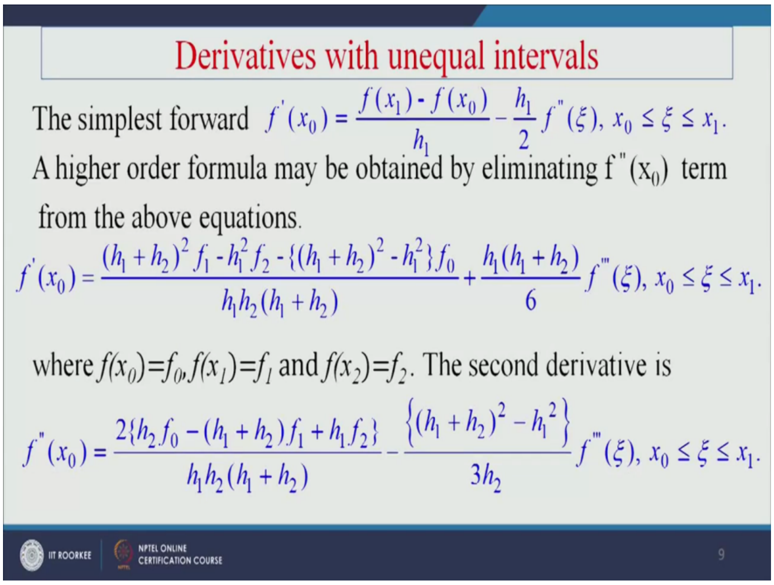

And if we will just try to evaluate suppose f dash of x 0 from these 2 equations so from the first equation we can just write f dash of x 0 this = f of x 1 so this equation if individually we try to evaluate f dash of x 0 then we can just write this one as f of x 1 – f of x 0 divided by h 1 + if you just see here 1 h1 it can be taken out so – h 1 by 2 factorial f double dash x 0 rest of the terms are there.

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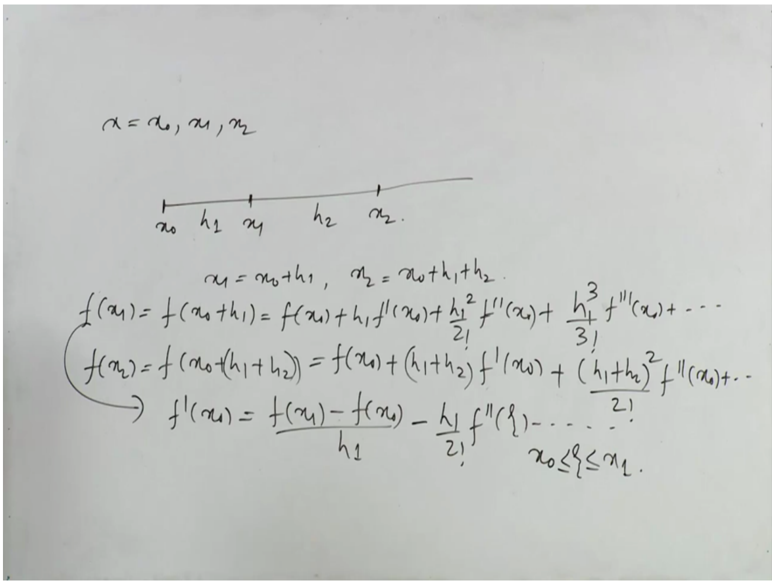

And if we want to write that in terms of error term here then we can just replace this x 0 point as Zeta, where Zeta should lies between x 0 to x 1 here. And suppose if it is asked to obtain a higher order formula so then we can just eliminate f double dash of x 0 from this first 2 equations like whatever we have just defined here f of x 1 and f of x 2 from these 2 equations.

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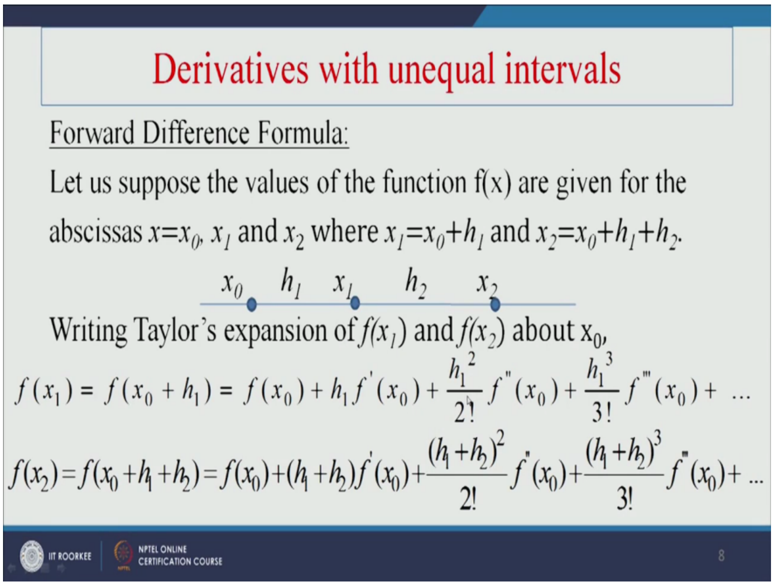

So from then we have to multiply like first equation if you will just multiply here, h 1 + h 2 whole square into the first equation and second equation if you will just multiply h 1 square, so then if you just subtract these 2 equations then we can just obtain the values of f dash of x 0 in terms of f of x 1, f of x 2, f of x 0 and the higher powers of h and the higher powers of derivatives there. So if you will just multiply these 2 terms by eliminating f double dash of x 0 from these 2 equation then we can just obtain this derivative as f dash of x 0 this is = h 1 + h 2 whole square f 1 – h 1 square f 2 – h1 + h 2 whole square – h 1 square into f 0 + h 1 into h 1 + h 2 by 6 f triple dash Zeta that is the error term there and the first term whatever we have just written that is divided by h 1 h 2 into h1 + h 2 here.

And if you just see here, this error term contains only f triple dash Zeta since we have eliminated f double dash of zeta from this term here, where we are usually given these symbols are denoted this x 0 is f of x 0 and f 1 as f of x 1 f 2 of f of x 2 there. And the second derivative if we want to find from these 2 equations here from these 3 equations like f of x 1 f of x 2 so then we have to eliminate f dash of x 1 from the equation or f dash of x 0 from that equation.

So if we will just eliminate f dash of x 0 by multiplying suppose h1 + h 2 in the first equation and second equation only h 1 it can be multiplied and both these equations after this multiplication if both these equations will be subtracted then we can just obtain the second order derivative in terms of the f 0, f 1 and f 2 there, where all the coefficients will involve in terms of h 1 and h 2 only and higher powers of terms will be involved these derivatives of third order of higher powers where f dash of x 0 is just eliminated from these 2 equations, so if we just go for these unequal spaced points and in a backward difference formula form suppose. Suppose the values of f of x are given like x 0, x of – 1 and x of – 2 like our Newton backward difference formula.

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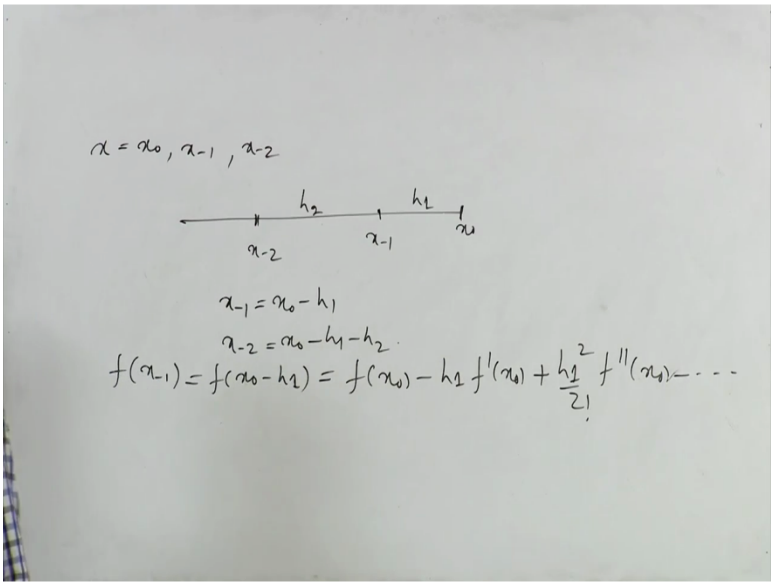

So if you just go for this backward difference formula here, we can just write this nodal points that is in the form of x = x 0, x of – 1, x of – 2. So usually we have to go back from x 0 in a line if you just define x 0 is the last point like our Newton's backward difference formula. So the immediate previous point it will be x of – 1 it can be placed at h 1 distance and if you just consider here x of – 2 here and which is placed at a distance of h 2 from x of – 1 here then we can just write x of – 1 as x 0 – h1 here and x of – 2 it can be written as x 0 – h 1 – h 2.

And then we can just use this Taylor series expansion at f of x of – 1 there so then we can just write f of x of – 1 as f of x 0 – h 1 here. And if we will just write in Taylor series expansion that can be written in the form of f of x 0 – h 1 f dash of x 0 + h 1 square by factorial 2 f double dash of x 0, so likewise we can just try it, next term is – here. So similarly if you will just write this f of x of – 2 here then we can just write that one as f of x 0 – h 1 – h 2 and it can be written as f of x 0 – h 1 + h 2 f dash of x 0 + h 1 + h 2 whole square by factorial 2 f double dash of x 0 + all other terms. So if we want to find this derivative of f at point x 0.

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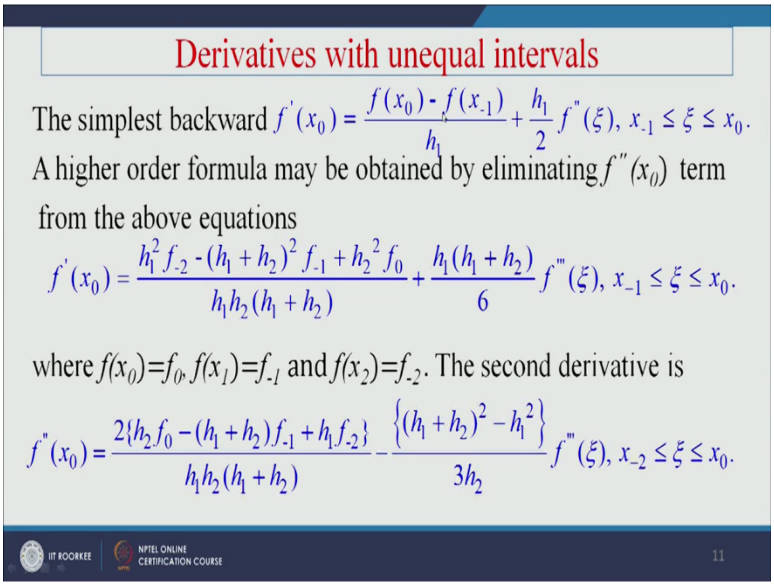

So from the first equation we can just write that one as f of x 0 – f of x of – 1 divided by h 1 that is first order derivative at x = x 0 and the remaining term that will be represented in the form of h 1 by 2 M double dash of zeta here where zeta should be lies between like x 0 to x of – 1 here. Similarly higher order formula may be obtained by eliminating f double dash of x 0 from the above equations by eliminating f dash of x 0 from both these equations sorry f double dash of x 0 from both these equations here.

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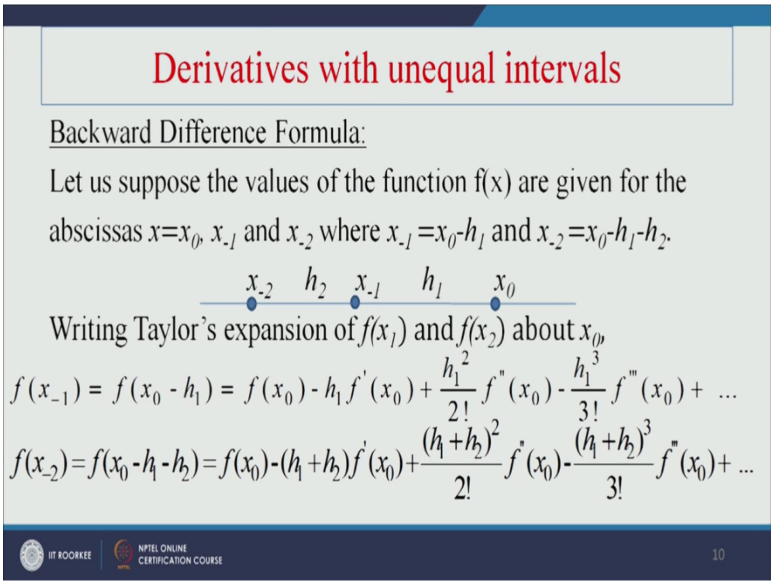

So that is why if we want to eliminate f double dash of x 0 from these 2 equations then we have to multiply h 1 + h 2 whole square in the first equation and h 1 square in the second equation. And if we will just multiply and subtract then we can just obtain this formula for f dash of x 0 here in terms of higher powers of h and the remainder term specially it can come in the higher powers of higher derivative powers of f here so that is why this first remainder term it is just coming in the order of 3 here and this remainder term is usually written as h 1 + h 1 into h 1 + h 2 by 6 f triple dash of zeta where zeta should be lies between like x 0 to x of – 2 here. Sorry since the points are existing between x 0 to x of – 1 so zeta should be lies between x 0 to x of – 1 here.

Similarly if we want to evaluate the second order derivative then we have to eliminate the first order derivative term from both these equations sp that is why we can just multiply in the first equation here h 1 + h 2 and in the second equation only h 1 and if we just subtract then we can just obtain the second order derivative in the form of h 1 and h 2 f 0 f of – 1 f of – 2 here.

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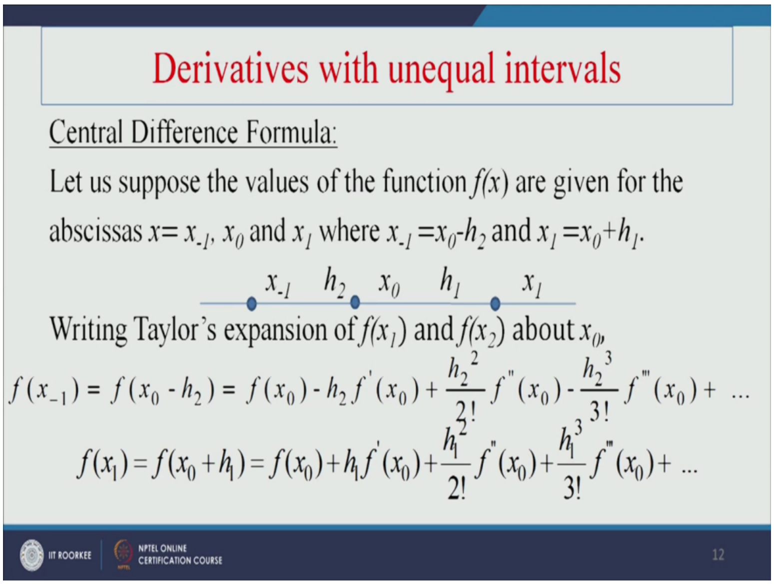

Next we will just go for like central difference formulas, in the central difference formula if we will just write these points that is in the form of we have like these points x 0 is their centreline so that is why we can just write x as x of – 1 and the previous point so a x 0 is the centreline here.

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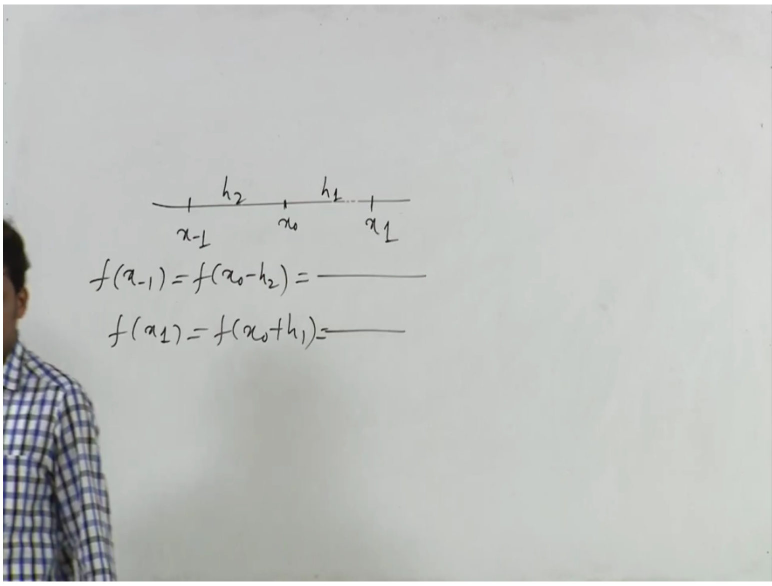

So that is why we can just write h 1as x 1 here and if we will just write like x of – 1 here that is h 2 space here, at that point we can just use this Taylor series expansion as f of x of – 1 as f of x 0 – h 2. Similarly at f of x 1 we can just use f of x 0 + h 1 here and if we will just expand in Taylor series expansion at both these points like x = x 0 there then we can just obtain this Taylor series expansion at f of x of – 1 and f of x 1 there.

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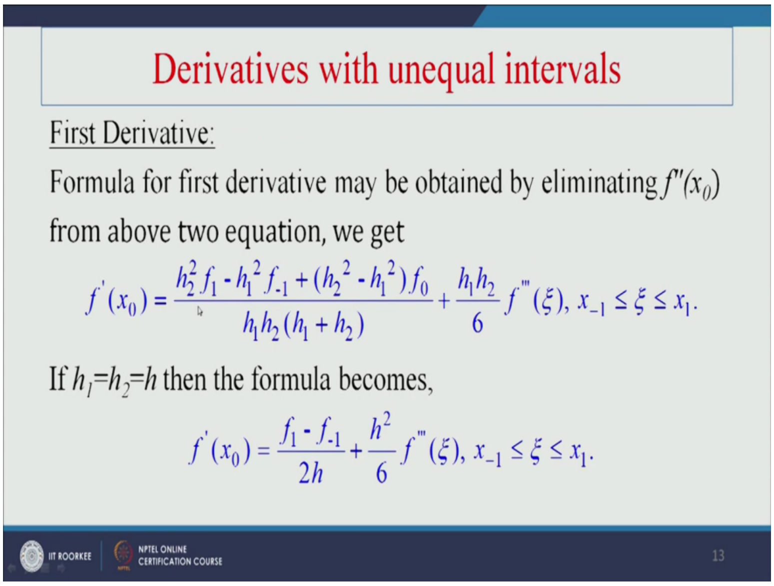

To obtain this first derivative here, so if we will just eliminate f double dash of x 0 from both these equations then we can just obtain the derivative at or first order derivative of f at x 0 there so this first order derivative at x 0 for f it can be written as h 2 square f 1 – h 1 square f of – 1 + h 2 square – h 1 square whole into f 0 divided by h 1 h 2 into h 1 + h 2 + h 1 h 2 by 6 f triple dash of zeta, where zeta lies between x of – 1 to x 1 here since this is a central difference formula here. And sometimes if we will just use suppose h 1 = h 2 both spaces are equal then your formula can be rewritten as f dash of x 0 = f 1 – f of – 1 by 2 h, usually this is the central difference formula for all other formulas also.

This means that if you just expand f of x + h and f of x – h and if you will just add both these terms, the same formula you can just obtain also. So that is why this f dash of x 0 it can be written in the form of f 1 – f of – 1 by 2 h and this remainder term that can be written in the form of like h square by 6 f triple dash of zeta, where zeta should be lies between like x of – 1 to x 1 here. And if you will just go for second order derivatives here, second order derivative means we have to eliminate f dash terms from both these equations there.

So if we want to eliminate both these equations of the terms of like f 0 dash then we have to multiply in the first equation h 1 square by 2 factorial or h 1 square if you just multiply and the second equation if you just multiply h 2 square then we can just eliminate sorry if you will just multiply in the first equation like h 1 and second equation like h 2 then we can just eliminate f dash of x 0 from both these equations and we can just obtain this formula for second order derivative in central difference approximation and this formula is can be written as 2 into h 2 f 1 – h 1 + h 2 f 0 + h f of – 1 divided by h 1 h 2 whole into h 1 + h 2 in the remainder term that will be represented in the form of like – 1 by 6 h 1 cube + h 2 cube by h 1 + h 2 f fort of zeta.

Since third order term it will just associate with some of these constant terms here that is why that cannot be neglected so that is why this one more term it will be associated like – 1 by 6 h 1 – h 2 f triple dash of x 0 there. Since central different approximation we are just approximating at these 2 points like 3 points we are just considering for this one. And in this form also if you just consider this h 1 = h 2 = x suppose then we can just write f double dash of x 0 it can be written as f 1 – 2 f 0 + f of – 1 by h square. And this error it will just occur in the order of h square here and that can be written as – h square by 6 f fort of zeta, where zeta should be lies between – x of – 1 to x 1 here, thank you for listening the lecture.