Numerical Methods
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Lecture No 3
LU Decomposition

Hello everyone so this is the 3rd lecture of this course and today I am going to introduce you another direct method for solving linear system that is called LU decomposition, so again the idea of this method LU decomposition is the same which we were having in case of Gaussian elimination that to convert the coefficient matrix in triangle matrix however in Gaussian elimination we were converting or reducing our matrix into an upper triangle matrix then we were using back substitution. Here the idea is to write the coefficient matrix as a product of lower and upper triangular matrices and then solve the linear system of equation using forward followed by the back substitutions.

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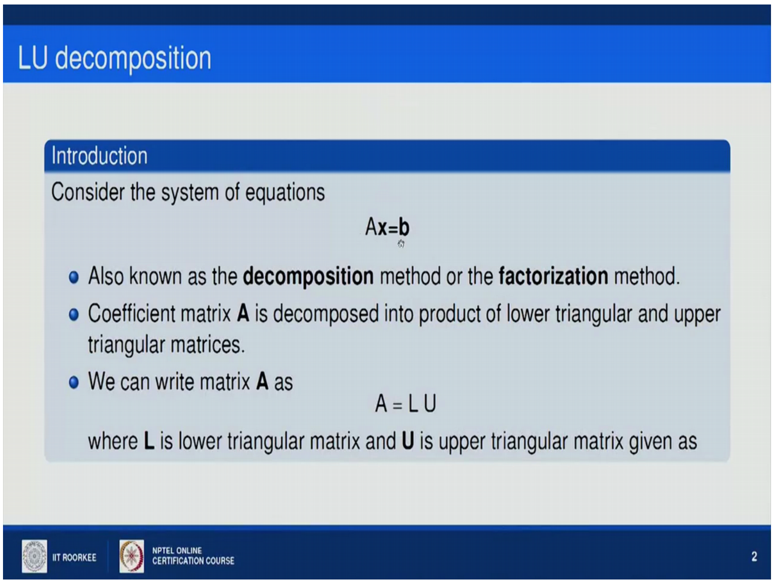

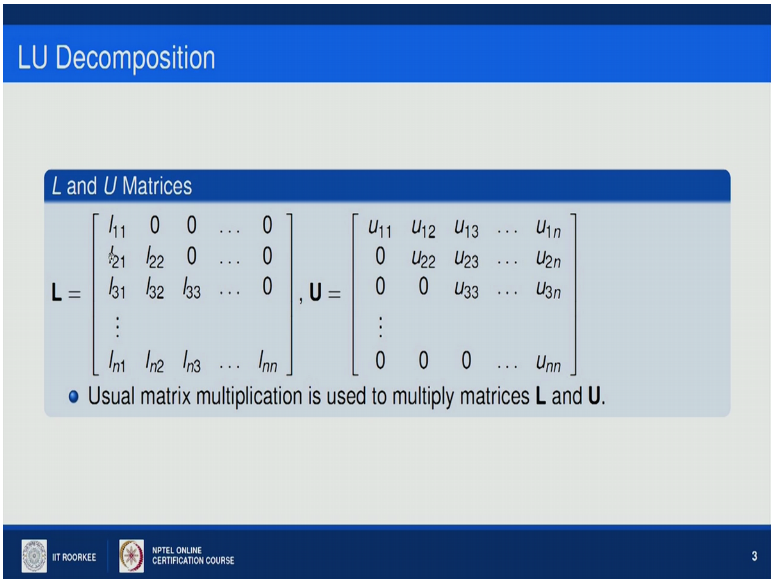

So consider the system of equation A x equal is to b this also known as the decomposition or factorization method, so the coefficient matrix is decomposed or factorized into product of lower triangular and upper triangular matrices L and U, so A equals to L and U, L into U L is lower triangle matrix and U is the upper triangle matrix, so these 2 matrices are given as, as you can see it is n by n matrices and it is a lower triangular matrix because all the entries about the main diagonal are 0. Similarly U is an upper triangular matrix and here all the entries below the main diagonal are 0. We are using the usual matrix multiplication to multiply the matrices L and U.

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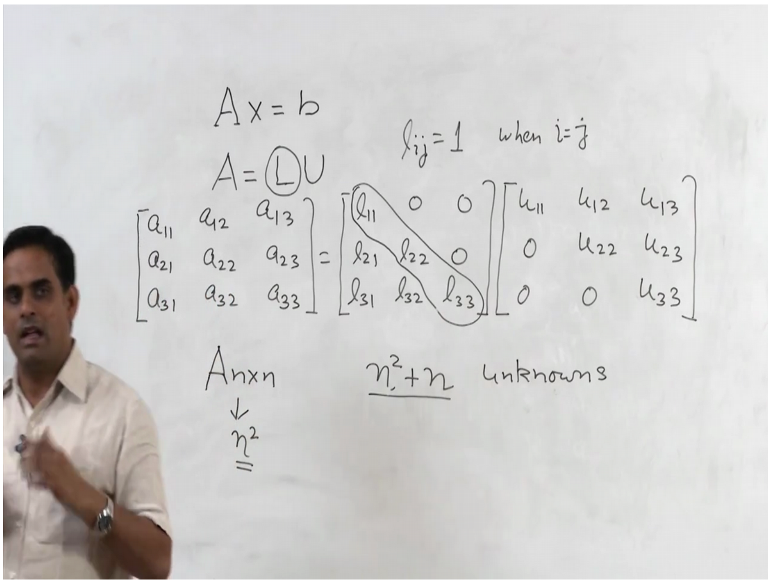

So basically what we are doing we are taking the system A x equals to b but what we are doing at this moment we will focus on the decomposition of this coefficient matrix is the product of 2 matrices L and U, so if we are having again a 3 by 3 system a 11, a 12, a 13, a 21, a 22, a 23, a 31, a 32, a 33 this can be written as the product of a 3 by 3 lower triangle matrix, so this is the lower triangle matrix L into an upper triangle matrix U, u 11, u 12, u 13, 0, u 22, u 23, 0, 0, u 33. Now if you look in the left-hand side we are having total 9 entries, these 9 entries are known to us however in the right-hand side we are having total 12 entries, 6 from the matrix L and 6 from the matrix U.

Hence we multiply these 2 matrices and try to find out the values of all lij and uij, we will not be able to do because here we are having only 9 entries, so 9 equations and 12 unknowns, so what is the solution? So what is the solution to this problem? In general like can say this problem is like that if A is n by n matrix then I will be having total n square plus n unknowns because you can see in 1^{st} row I will be having one, in 2^{nd} row 2, in 3^{rd} row 3, in nth row n, so it will be some 1 plus 2 plus 3 up to n that will be basically n into n plus 1 by 2.

So n into n plus 1 by 2 from the lower triangle matrix and in 2n plus 1 by 2 from the upper trying matrix total will become n square plus n unknowns including all lij and uij. While for a n by n matrix say I will be having only n square entries which is known to me. So somehow I need to reduce n unknowns, so what I will do? If I write the diagonal entry, entries of either from l or diagonal entries of u as 111 then the trick will work, so what I will do? Either I will choose these entries as 111, so lij equals to 1 when i equals to j, so what will happen? The n entries will become less here, so n number of unknowns be reduced, so n square unknown n square entries, I will get a unique solution or the LU factorisation or instead of this I can take all the diagonal entries of my upper triangular matrix as 1.

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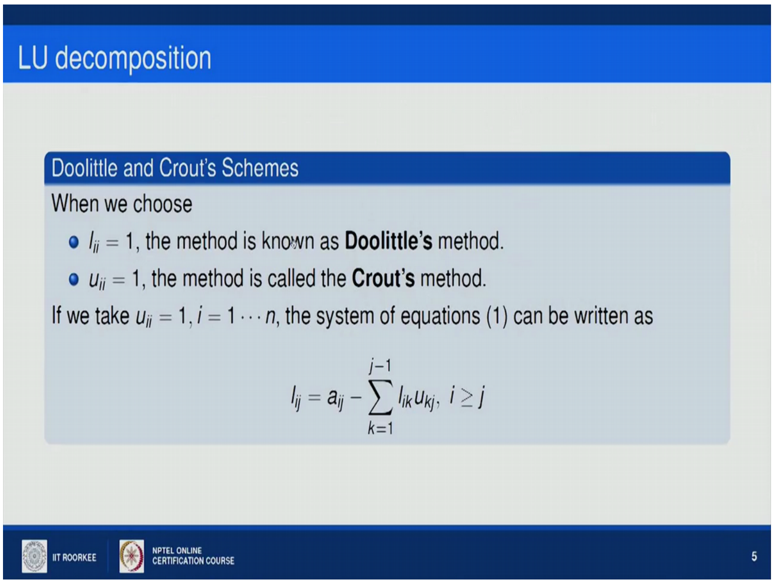

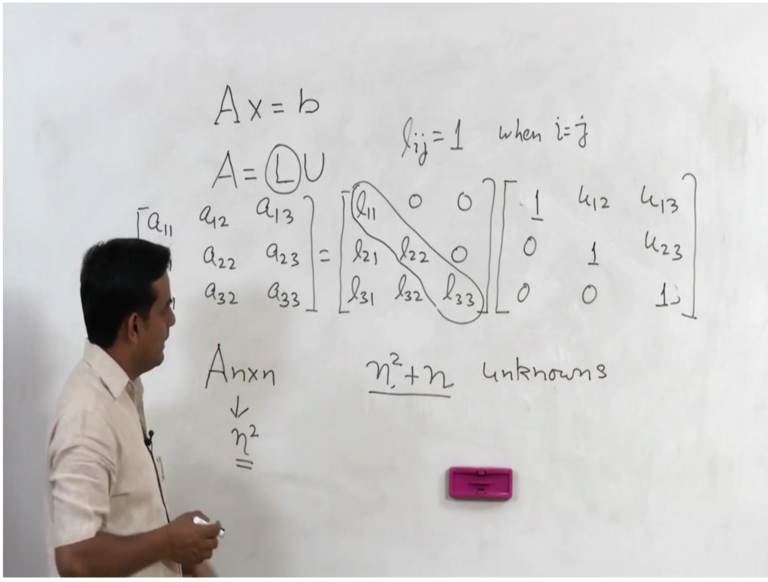

So if I take the diagonal entries of lower triangle matrix as 1 the method is known as Doolittle method. If I take the entries of upper trying you matrix as 1, entries means diagonal entries, the method is called as Crout's method. Now let us take the entries of upper trying you matrix the entries those are a diagonal, when diagonal is 1 then total 6 unknowns from here, 3 unknowns from here and 9 entries then what I will do? I will find out the values of all l ij and u ij.

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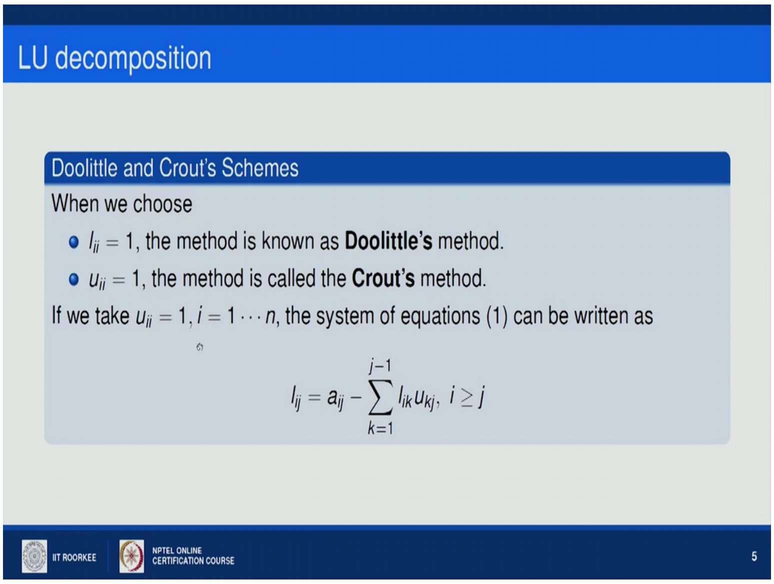

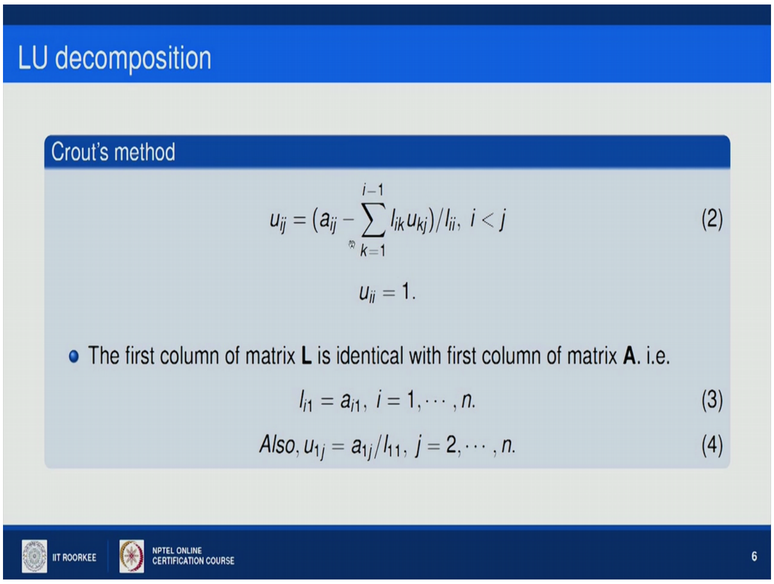

So if we take u ii equals to 1 from equals to 1 to n means all diagnosed entries of upper triangle matrix as 1, the system of equation one can be written as l ij equals to a ij minus summation k equals to 1 to j minus 1 l ik U kj, whenever I is greater than equals to j. This can be written in this way whenever i is less than say by the equation 2, if and all rest of the u ii equals to 1. When i is less then j U ij is given by this equation when i is equal to or greater than j l ij is given by this equation, so this equation will give all the entries of lower triangle matrix.

This equation will give you all the entries of upper triangular matrix those are above the main diagonal and the entries of the upper triangular matrix with basically the diagonal entries are 1. Then what we will do? From the 1st column of the matrix L, we can find out the entries like l 11, l 21, l 31 and l n1 because the 1st column of the lower triangular matrix will be identical to the matrix A. After doing this after finding l i1 for i equals to 1 to n, what we will do? We will go to 1st row, in 1st row we can calculate all u 1j that will be basically a 1j upon l 11 where j equals to 2 to n. Here we are taking j from 2 because u 11 is already we have fixed as 1.

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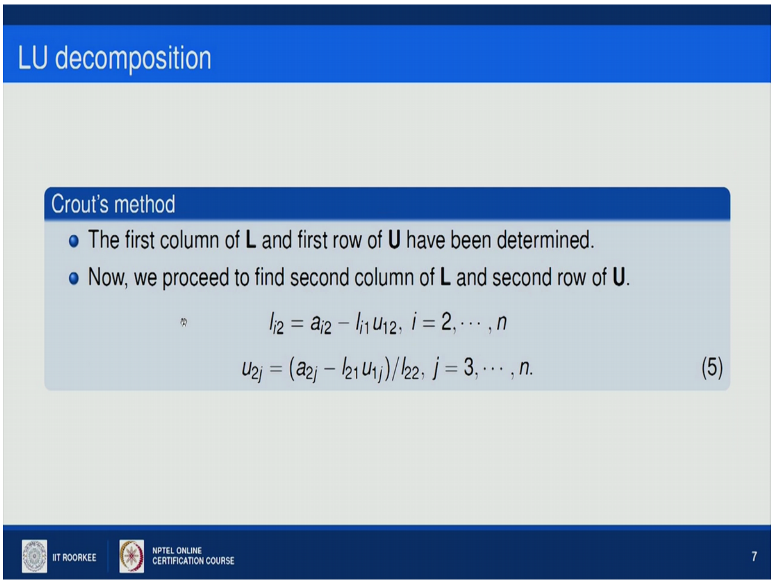

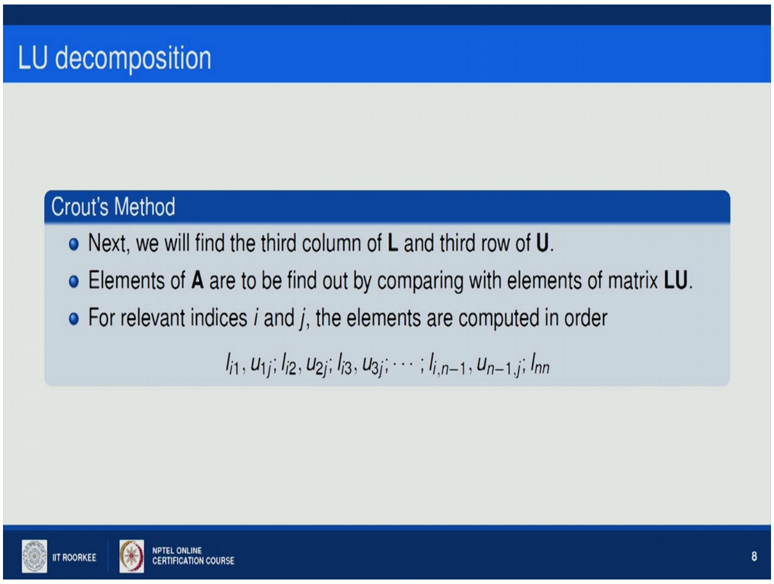

So 1st column will give you the entries like l 11, l 21, l 31 and so on the 1st row will give you the entries like u 11 is 1, u 12, u 13 up to u 1n and then what we will do? Then we will go to 2nd column, in 2nd column will give me the entry like l i2 equals to a i2 minus l i1 U 12, i equals to 2 to n and then 2nd row will give the entries of 2nd row of u by the equation 5 and then we will go like 3rd column 3rd row, 4th column 4th row and so on and we will be able to get all these entries l ij and U ij.

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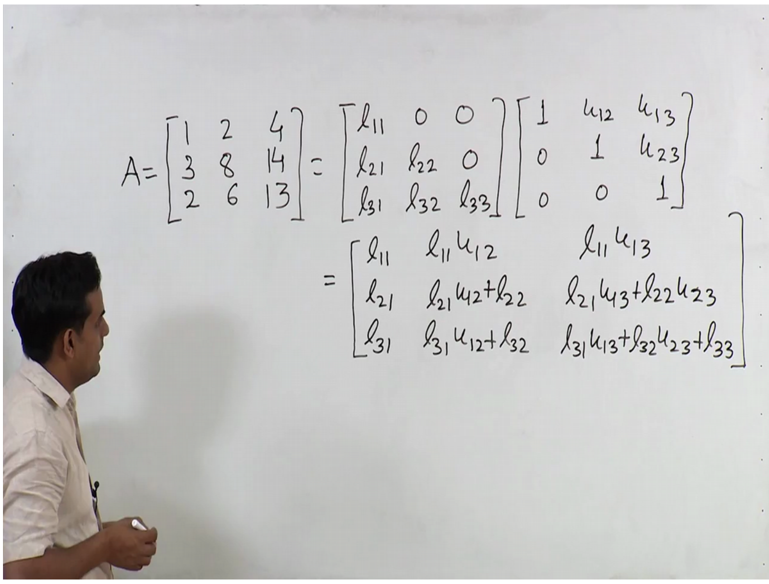

So here we will take an example of this method okay so take a 3 by 3 matrix say which is given as 1, 2, 4 and then the 2nd row of this matrix is 3, 8, 14 in the last row of this matrix is 2, 6, 13. Now in Crout's method when we are decomposing this as a lower triangular matrix and upper triangular matrix A equals to l 11, 0, 0, l 21, l 22, 0 and then l 31, l 32, l 33, so this is a lower triangular matrix l into an upper trying matrix u, so here am taking a main diagonal elements of u as 1, so 1, u 12, u 13, 0, 1, u 23, 0, 0, 1. Now if I multiply these true matrices then the 1st element will be l 11 in the 2nd element will be l 11 u 12 and the 3rd element will be l 11 u 13.

Similarly from the 2nd row l if I multiply with 1st column this element will be l 21, this element will be l 21 u 12 plus l 22 and this entry will be l 21 u 13 plus l 22 u 23. In the last row of the product matrix this element will be l 31, this element will become l 31 u 12 plus l 32 and finally the last entry of this matrix will be l 31 u 13 plus l 32 u 23 plus l 33. So this matrix equals to A now comparing these 2 matrices and the comparison will be done based on the strategy I have earlier explained and initially we will compare the elements of 1st column.

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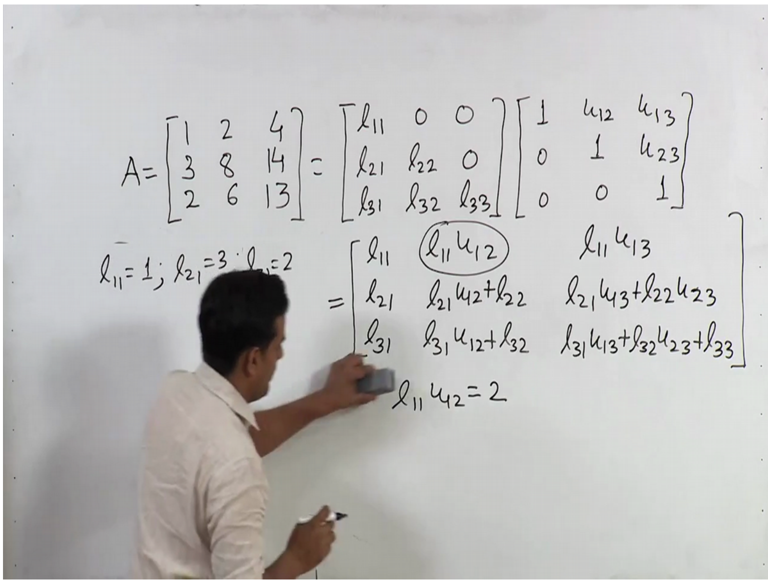

So when I compare the elements 1st column of this matrix with this one I will get l 11 equals to 1, l 21 equals to 3 and finally l 31 equals to 2. So after comparing the elements of 1st column now will compare the elements of 1st row, so I will take this element and this element is l 11 into u 12 and in this matrix this is 2 since l 11 is 1, so u 12 comes out to be 2, so here u 12 is 2. Now I will take this element, so when I will compare this element with this one l 11 is already one so I will get u 13 as 4. Now I will compare the elements of 2nd column, so for this I will take this particular element, so l 21 u 12 plus l 22 equals to 8 and l 21, l 21 is 3 into u 12 is 2 plus l 22 is 8.

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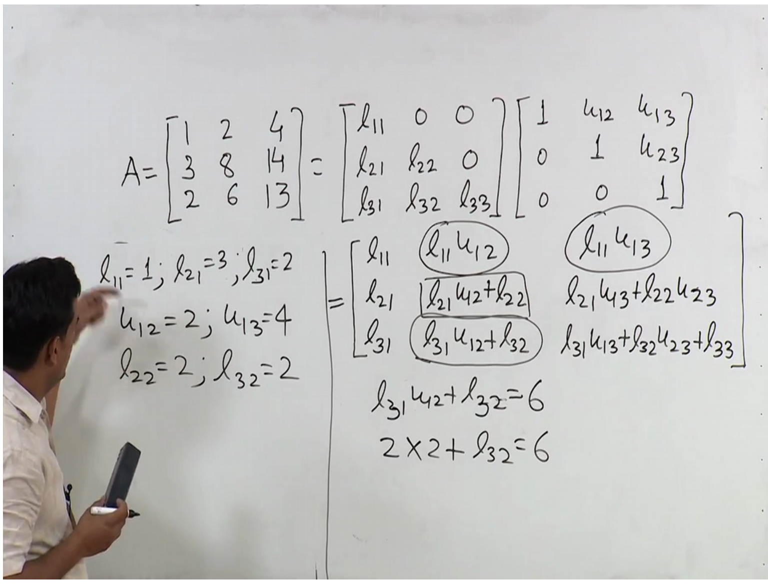

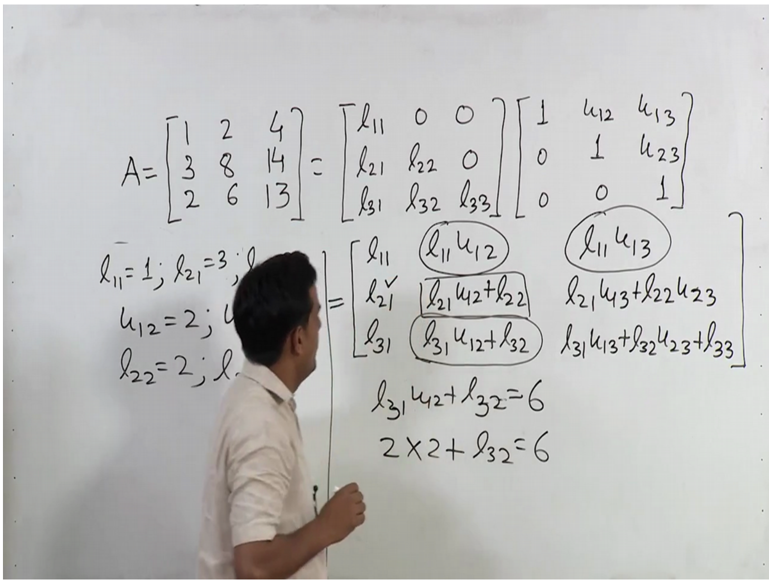

So from here I got l 22 equals to 8 minus 6 that is 2. Now I will take this element, so this gives me l 31 u 12 plus l 32 and from here it is 6, l 31 is 2 u 12 is 2 plus l 32 equals to 6 and from here I got l 32 equals to 2. So out of the 9 elements I got 7 elements just by comparing 2 columns and one row.

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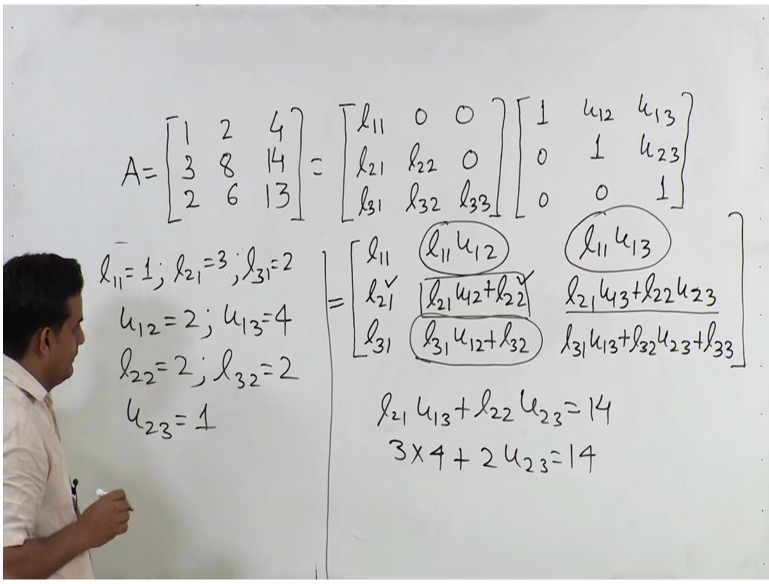

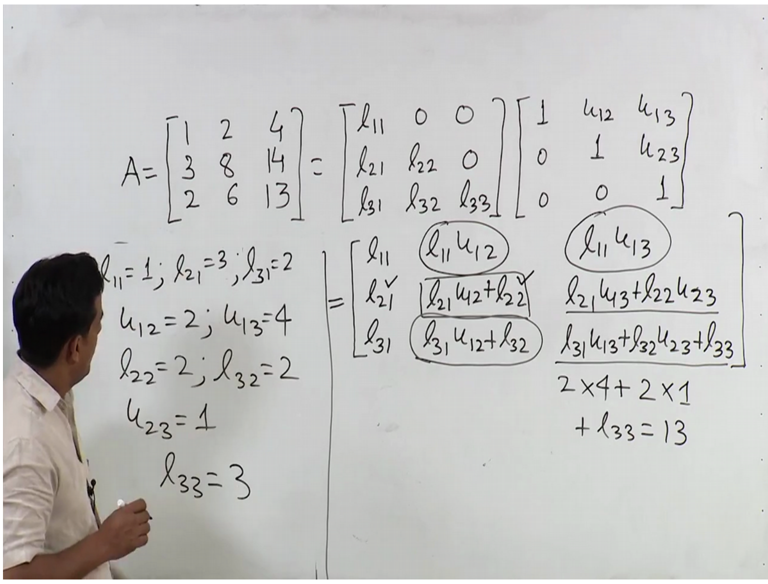

Now I will compare 2nd row so from 2nd row this element is already I have taken in comparison this one have taken so now I will go for this element. So when I will compare this element it is l 21 into u 13 plus l 22 u 23 and from here it is 14 so l 21 is 3 u 13 is 4 plus l 22 is 2 u 23 equals to 14. So from here I will get u 23 equals to 2, so u 23 equals to 1 and finally I will compare this element, so this element is l 31, so l 31 is already known to us it is 2 in to u 13 which is 4 plus l 32, l 32 is 2 into u 23 which is 1 plus l 33 equals to 13, so 8 plus 2 10 so from your I got l 33 as 13 minus 10 that is 3.

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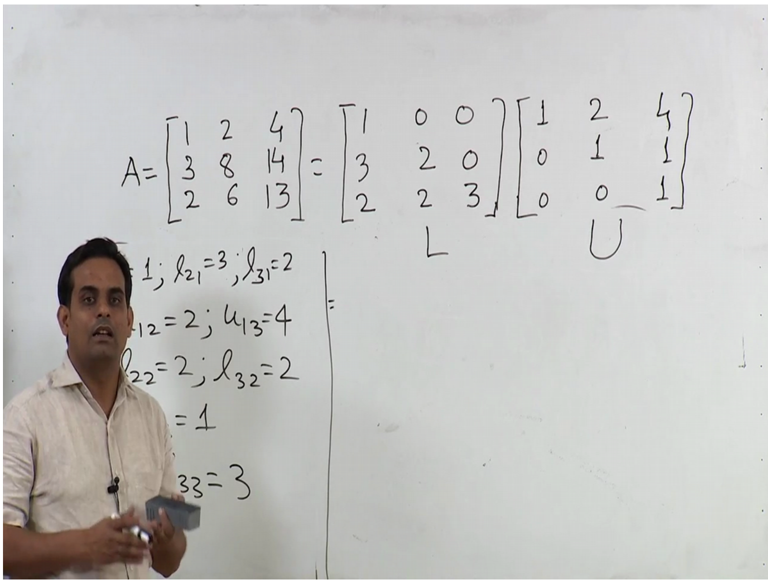

So in this way I calculated all the 9 entries, so now if I write these 9 entries the decomposition or LU decomposition of this matrix will be 1, 3, 2 then l 22 is 2, l 32 is 2, l 33 is 3, so this is the lower triangular matrix and the upper triangular matrix is u 12 is 2, u 13 is 4 and finally u 23 is 1. So hence this is an example of LU decomposition of a given matrix A when the diagonal entries are the diagonal entries of the upper triangular matrix are 1.

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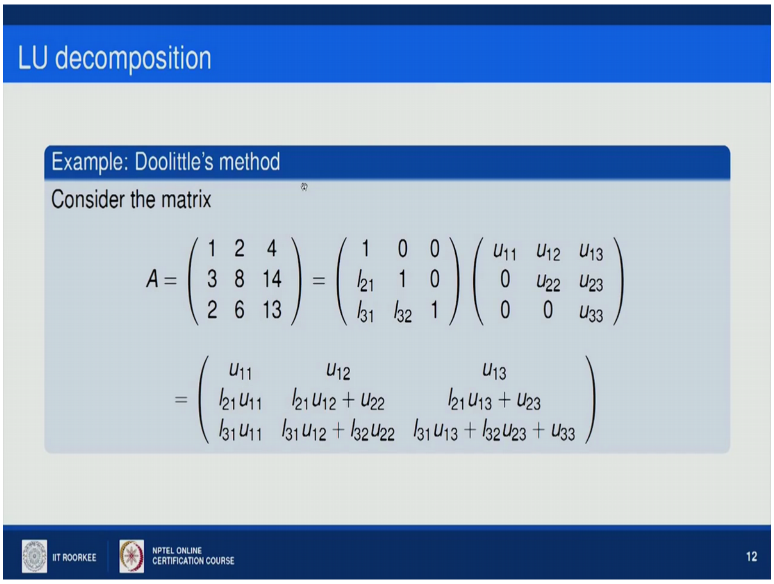

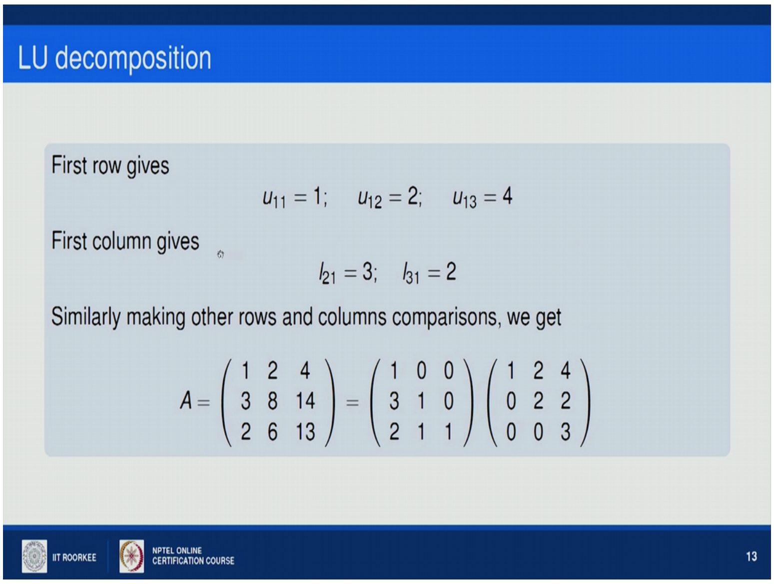

So this is the example of Doolittle method so as you can see u 11 will be 1 u 12 will be 2 u 13 will be 4, so this is coming from the 1st row similarly 1st column gives me l 21 equals to 3 and l 31 equals to 2 similarly making the other comparisons I will be able to decompose the matrix say which is the same matrix as I have taken in Crout's method equals to product of these 2 matrices.

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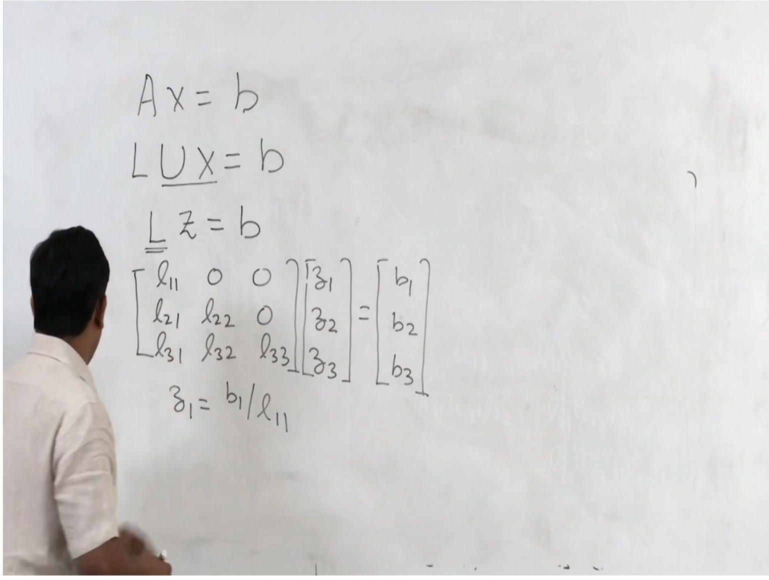

Now this is about the factorisation or decomposition, now question how to solve a linear system using the concept of factorisation, So let me explain it so basically we are having Ax equals to b, so I have decomposed A as L into U so LU into x equals to b. Now what I will do let us assume that this Ux equals to z, so what I will be having Ux will be a column vector so let us say it is z 1, z 2, z n, so Ux equals to z and if I substitute Ux equal to z, the original system will become Lz equals to b.

Here you can note down that L is a lower triangular matrix, so what I will do it will be something like that l 11, 0, 0, l 21, l 22, 0, l 31, l 32, l 33 into z 1, z 2, z 3 this equals to b 1, b 2, b 3, so what I can do it is a lower triangular matrix so from the 1st equation it gives z 1 equals to b 1 upon l 11 directly. Now what I will do in the 2nd equation I will substitute the value of z 1 and I will get the value of z 2 similarly from the 3rd equation I will substitute the value of z 1 and z 2 and I will get the value of z 3, so this is something I am doing like forward substitution, so making use of forward substitution I am getting the values of z 1, z 2, z 3, so once z 1, z 2, z 3 or up to z n if we are having n dimensional matrix n by n matrix then I can find out the unknown factor z.

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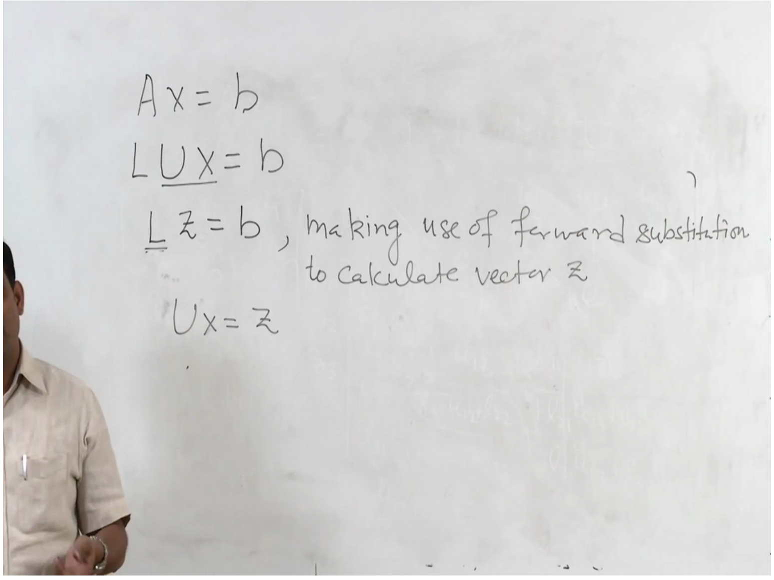

So here what I am doing I am making use of forward substitution and I have to calculate vector z. Once I know these vector z I know that Ux equals to z, U is an upper triangular matrix z is known to me by this step I can find out the value of x by making use of back substitution as we have done in Gaussian elimination, so from the last equation I can get the value of x n I will substitute the value of x n in penultimate equation from there I will get the value of x n minus 1 and so on. Finally substituting the value of x n, x n minus 1 up to x 2 from the 1st equation I will get the value of x 1 and this method is called Crout's method for solving the linear system of equation, so you are what we are doing 1st we are decomposing our coefficient matrix as the product of lower triangular and upper triangular matrix and then we are using forward substitution and finally back substitution.

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Linear System

Example

Consider the system of equations

$$\begin{bmatrix} 1 & 2 & 4 \\ 3 & 8 & 14 \\ 2 & 6 & 13 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 4 \\ 12 \\ 11 \end{bmatrix}$$

Triangularization Method

Example

This system can be written as in triangular matrices form

$$\begin{pmatrix} 1 & 0 & 0 \\ 3 & 1 & 0 \\ 2 & 1 & 1 \end{pmatrix} \begin{pmatrix} 1 & 2 & 4 \\ 0 & 2 & 2 \\ 0 & 0 & 3 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} 4 \\ 12 \\ 11 \end{pmatrix}$$

Let

$$\begin{pmatrix} 1 & 2 & 4 \\ 0 & 2 & 2 \\ 0 & 0 & 3 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} z_1 \\ z_2 \\ z_3 \end{pmatrix}$$

Triangularization Method

Example cont...

$$\begin{pmatrix} 1 & 0 & 0 \\ 3 & 1 & 0 \\ 2 & 1 & 1 \end{pmatrix} \begin{pmatrix} z_1 \\ z_2 \\ z_3 \end{pmatrix} = \begin{pmatrix} 4 \\ 12 \\ 11 \end{pmatrix}$$

Using forward substitution, we get $z_1 = 4$; $z_2 = 0$; $z_3 = 3$. Now, solving $Lx = z$ using back substitution

$$\begin{pmatrix} 1 & 2 & 4 \\ 0 & 2 & 2 \\ 0 & 0 & 3 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} 4 \\ 0 \\ 3 \end{pmatrix}$$

We get $x_1 = 2$; $x_2 = -1$; $x_3 = 1$.

This is the example which I have taken earlier, so suppose we are having this system of equation 1, 2, 4, 3, 8, 14, 2, 6, 13, so 3 equations with 3 unknowns the matrix the coefficient matrix is same which have taken earlier, so I can write this as a product of L into U x equals to b. Now assume this equals to z 1, z 2, z 3. So what will happen if I substitute this by z 1, z 2, z 3 the original system can be written like this. From here using the forward substitution I can get z 1 equals to 4 z 2 equals to 0 and z 3 equals to 3.

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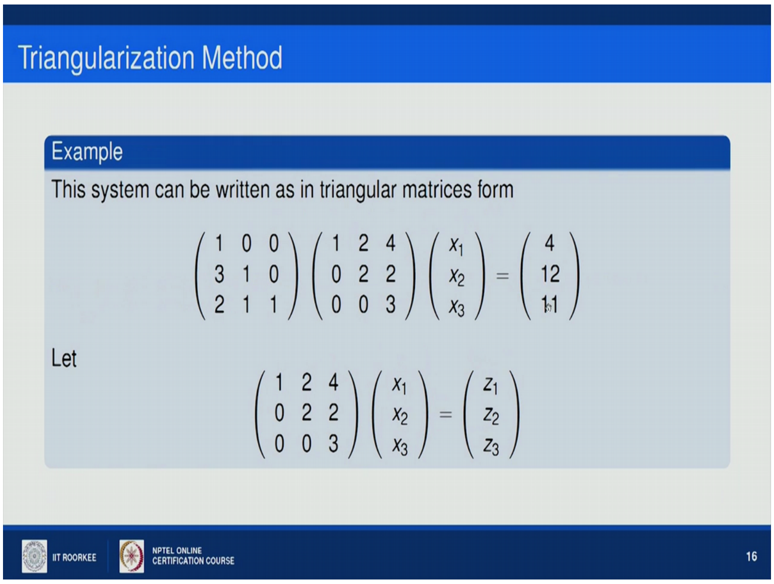

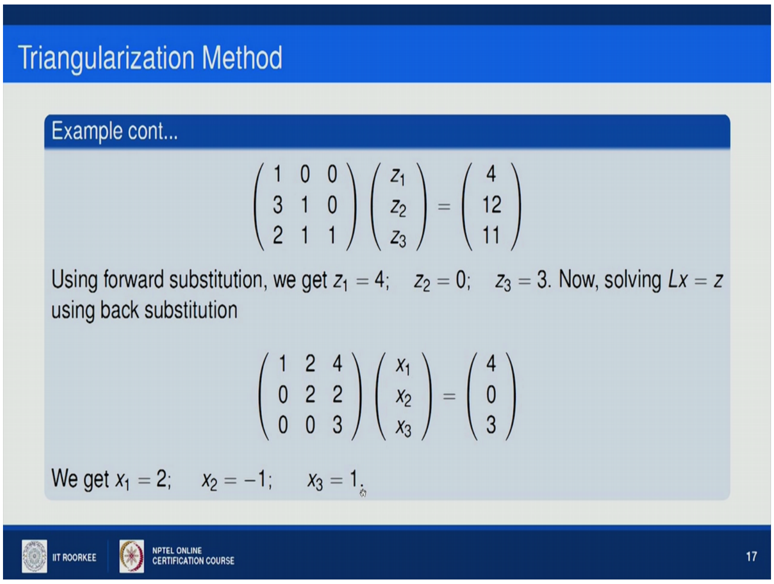

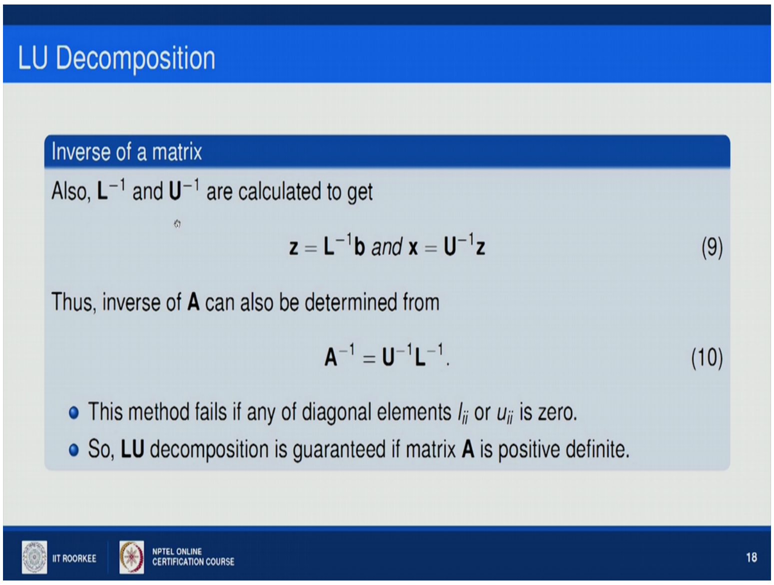

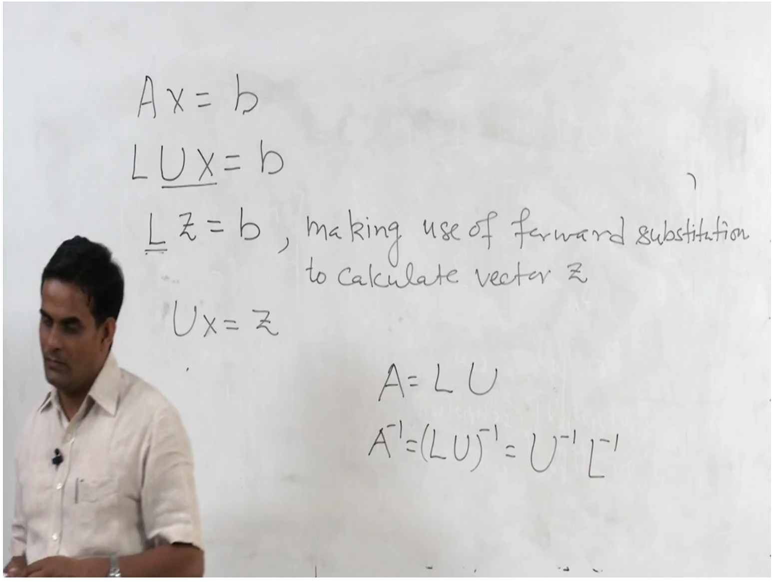

What I will do now I will put the value of z 1, z 2, z 3 here and I will get the values of x 1, x 2, x 3 which is coming out like x 1 equals to 2, x 2 equals to minus 1 and x 3 equals to 1 which is the solution of the system. We can also find out the inverse using the LU decomposition, so for finding the inverse you know that we can write A equals to L into U, so A inverse will become LU inverse that is basically U inverse into L inverse. Now question is whether this method will always work? No.

This method fails if any of the diagonal elements either from the lower triangular matrix or from the upper triangle matrix, in the 2 method is 0 because what will happen for calculating the values of other variables the diagonal elements used to come in denominator and if it is 0 we cannot find a finite value or the other variable. So what is the condition, sufficient condition for this method? LU decomposition is guaranteed to give you a solution if the matrix A is a positive definite matrix.

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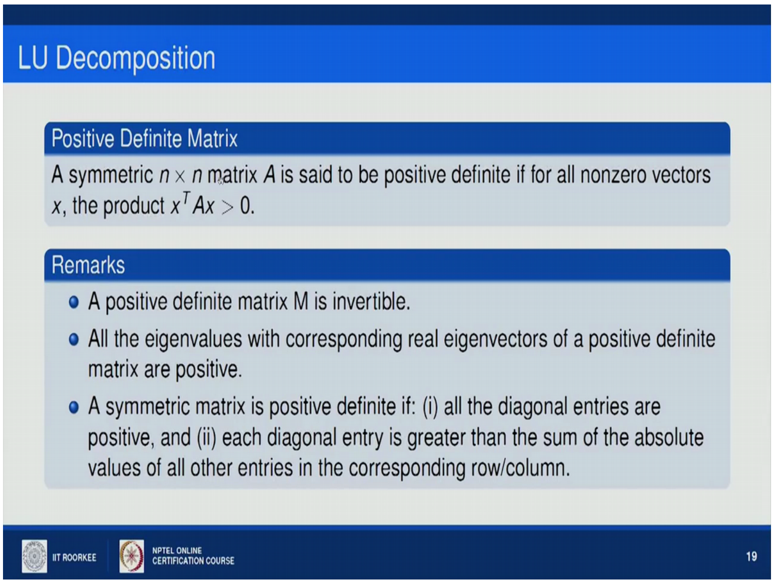

Now what we mean by a positive definite matrix? A symmetric n by n matrix A is said to be positive definite if you take a nonzero vector x, the product of x transport A into x comes out positive. There are some other property of the positive definite matrix like a positive definite matrix will be always invertible means determinate will be positive or nonzero. All the eigenvalues with corresponding real eigenvector of a positive definite matrix will be positive. A symmetric matrix is positive definite if all the diagonal entries are positive and each diagonal entry is greater than the sum of the absolute value of all other entries in the corresponding row and columns.

There is one more method for the symmetric matrix if the coefficient matrix is a symmetric matrix that is called Cholesky method is basically since the coefficient matrix is the symmetric we can write it as the product of L into L transport, so if L is the lower triangular matrix L transport will become an upper triangular matrix, so in other way we can write as the product of an upper triangular matrix U into U transports. Hence in decomposition we need to find out only one matrix instead of L and U either L or U and then the rest of the process will be similar to the Crout's or Doolittle method.

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Linear system

Cholesky Method

(13) can be written as

$$\mathbf{L}^{\mathbf{T}}\mathbf{x} = \mathbf{z} \tag{14}$$

$$\mathbf{L}\mathbf{z} = \mathbf{b} \tag{15}$$

- The values $z_i,\ i = 1, \cdots, n$ are obtained from (15) by forward substitution.
- $x_i,\ i = 1, \cdots, n$ are determined from (14) by back substitution.
- Alternatively, we can also find $\mathbf{L}^{-1}$ and obtain

$$\mathbf{z}=\mathbf{L}^{-1}\mathbf{b} \tag{16}$$

and, $\mathbf{x}=(\mathbf{L}^{\mathbf{T}})^{-1}\mathbf{z} = (\mathbf{L}^{-1})^{T}\mathbf{z}$

Linear system

Cholesky Method

The inverse of **A** can be obtained from (11)

$$\mathbf{A}^{-1}=(\mathbf{L}^{-1})^{T}\mathbf{L}^{-1} \tag{17}$$

The elements of lower triangular matrix **L** may be obtained as

$$l_{ii} = \left(a_{ii} - \sum_{j=1}^{i-1} l_{ij}^2\right)^{1/2},\ i = 1, 2, \cdots, n$$

Linear system

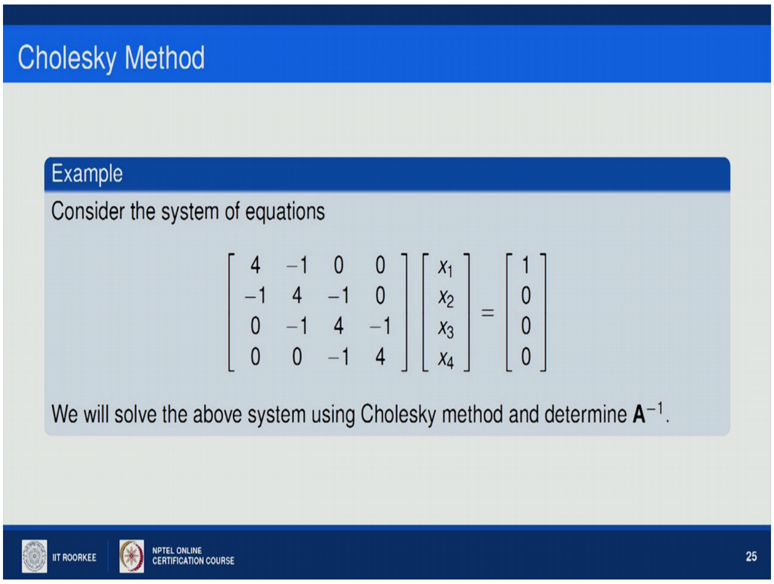
Cholesky Method

$$u_{ij} = \left(a_{ij} - \sum_{k=j+1}^{n} u_{ik}u_{jk}\right)/u_{jj},$$

$$i = n-2, n-3, \cdots, 1;\ j = i+1, i+2, \cdots, n-1$$

$$u_{ii} = \left(a_{ii} - \sum_{k=i+1}^{n} u_{ik}^2\right)^{1/2},\ i = n-1, n-2, \cdots, 1$$

$$u_{ij} = 0, i > j \tag{19}$$



Like if you are writing A equals to L into L transport the system of equation say x equals to b can be written as L into L transport as x equals to b, just as you L transport as x equals to z, so this will give Lz equals to b, so from here you will calculate z and then you can substitute back the value of z here to find out x, if L is non-singular you can calculate z as L inverse b and then finally x as L transport inverse z that is L inverse transport z similarly you can use Cholesky method to find out the inverse of A matrix, so inverse is given by L inverse transport into L inverse, so this is the working process equations to find out the entries in the Cholesky method. It will be the same as we have did in Crout's method and only change will be due to the symmetric property of A. Consider this example it is a 4 by 4 system the coefficient matrix is a tri-diagonal matrix it is a symmetric matrix.

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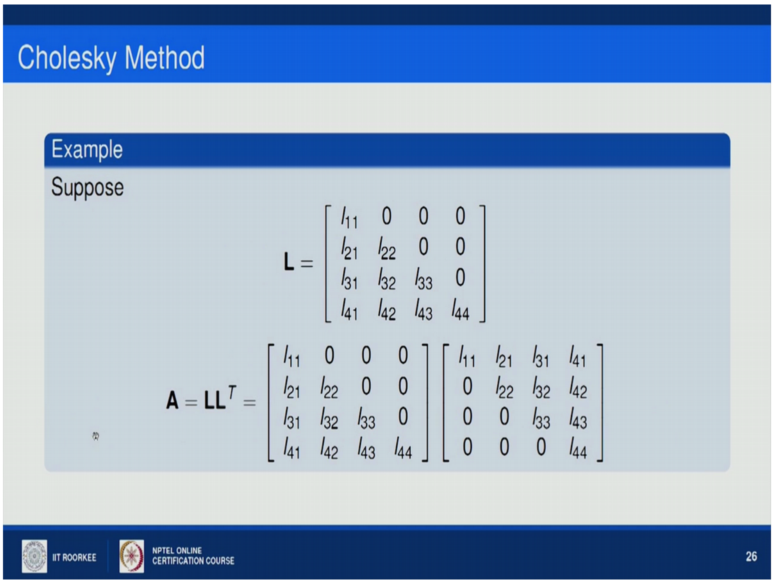

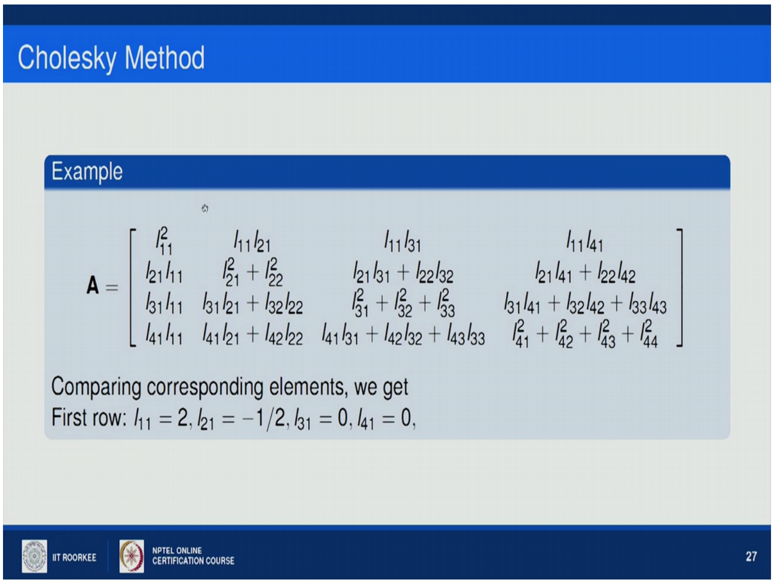

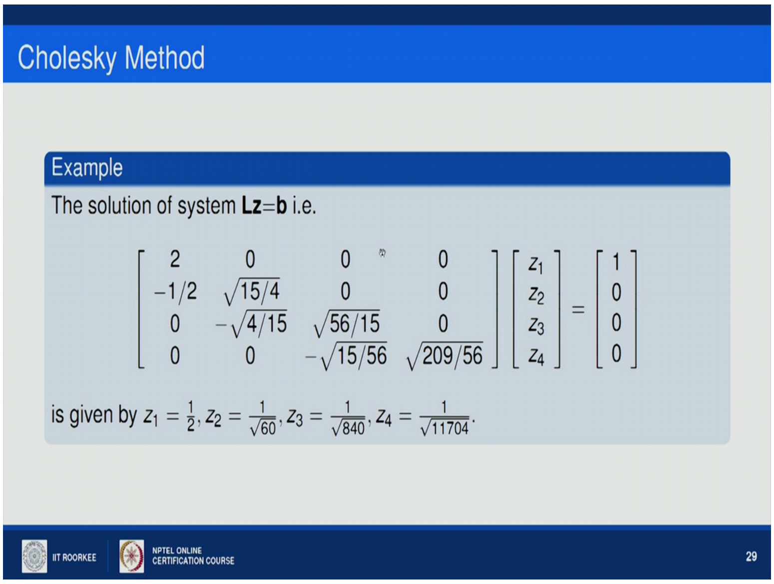

So if we solve this system using Cholesky method I will take L as the lower triangle matrix LL transport will become like this, so here this will be the transport of L. Products can be written as in this way after comparing the entries of A with the entries of this matrix L comes out in this way, okay. From here I will get the values of z 1, z 2, z 3, z 4. Once I will get z 1, z 2, z 3, z 4 I can solve L T x equals to z using the back substitution and I will get the values of x 1 as 56 upon 209, x 2 as 15 upon 209, x 3 as sorry x 3 is 4 upon 209 and x 4 as 1 upon 209.

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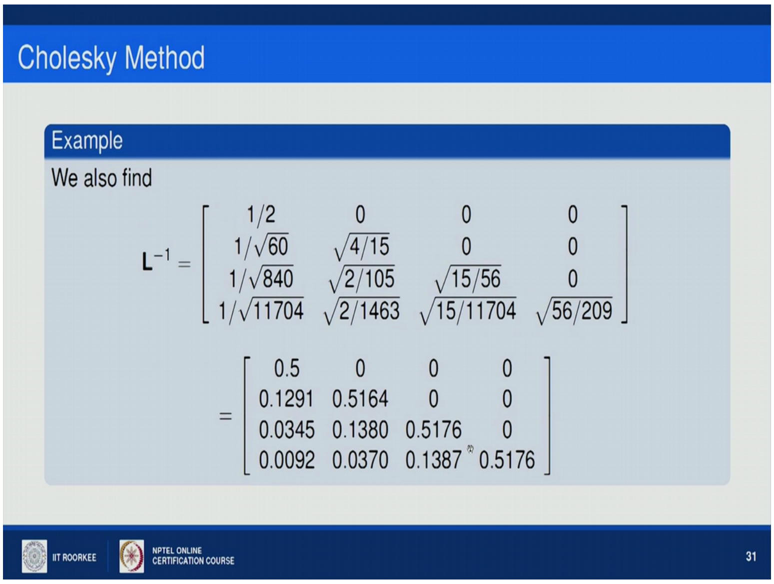

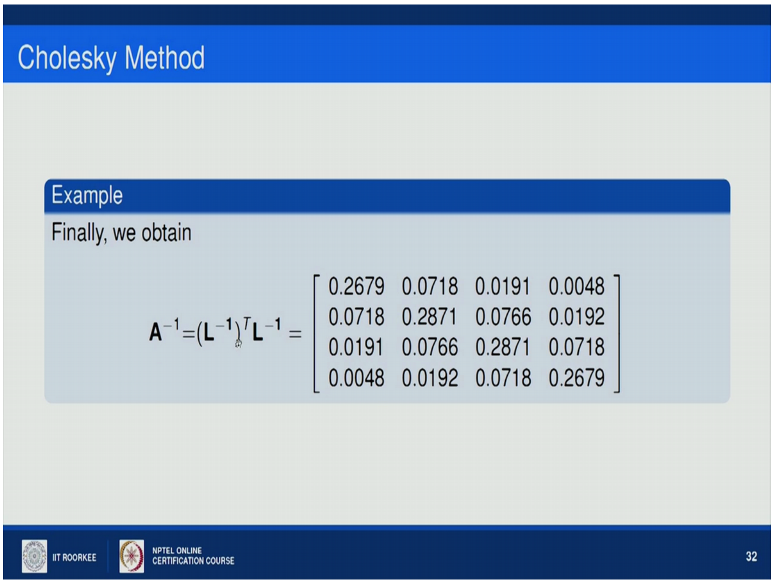

If I want to find L inverse the same process L inverse comes out in this way which is equal to this particular matrix and A inverse will become L inverse transport into L inverse that this is the matrix A inverse. So in this lecture I discussed about method based on triangular matrices for solving the linear system of equation. Here first I discussed about the Crout's method and Doolittle method and finally I have given a small explanation to Cholesky decomposition which is basically in the case when A is a symmetric matrix. In the next lecture I will go to other category for solving the linear system of equation and that category is called Iterative methods. So far I have discussed direct methods and Iterative methods are basically having few advantages over the direct methods, so thank you very much for listening this lecture.