**Numerical Methods**
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**Lecture 29**
**Numerical differentiation part-V (Differentiation based on finite difference operators)**

Welcome to the lecture series on numerical methods, in this lecture series we are just discussing this numerical differentiation based on interpolation and in the previous lecture I have discussed this like differentiation based on this Newton's divided difference interpolation and this error differentiation. And in this lecture we will just continue about this differentiation using this Central difference approximation and some of this finite difference approximation like based on finite difference operators.

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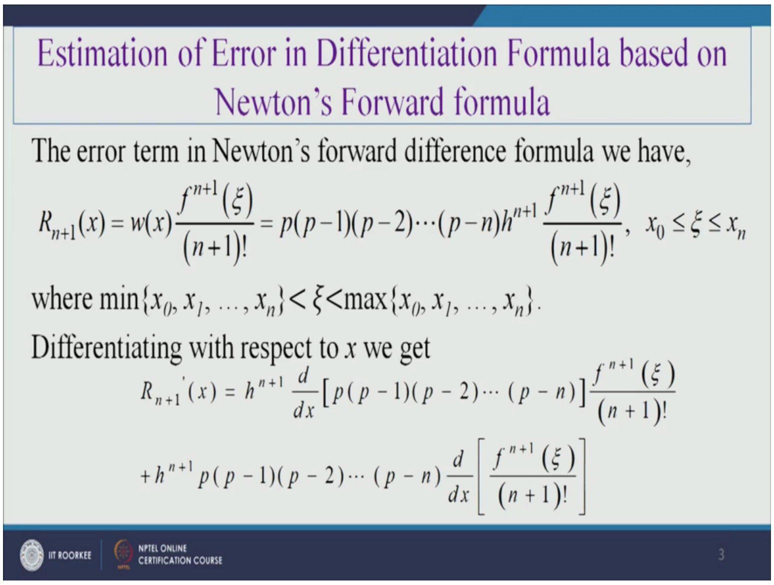


So if you will just go for this error term analysis based on this Newton's forward difference formula, so usually these error terms for this Newton's forward difference approximation is written in the form like R n + 1 x this = omega x f to the power n + 1 Zeta by n + 1 factorial, where this Omega x is usually written in the form like x – x 0, x – X1 to x – x n here.

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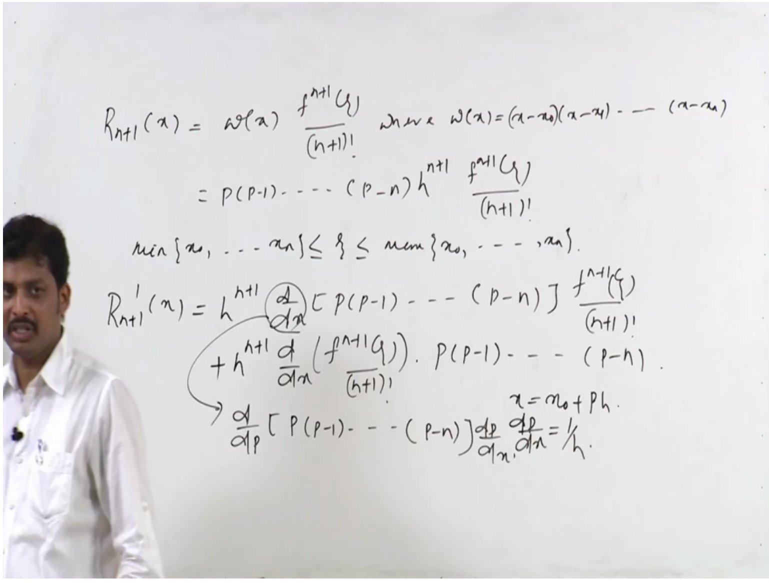


And in terms of p if you will just write then it can be written in the form like p into p – 1 up to p – n h to the power n + 1 and f to the power n + 1 zeta by n + 1 factorial here. Since usually we are just expressing p as x – X 0 by h and that is why obviously we can just replace that one as in the form of p – 1, p – 2 that I have discussed in the previous lectures. So this Zeta well it can be determined as minima of x 0 to x n this should be lies between zeta and maximum of x 0 to x n here. So if you want to differentiate this error term here then this error term R n + 1 x can be written as R n + 1 dash of x which can be expressed as h to the power since it is a constant here h to the power n + 1 d by dx of p into p – 1 up to p – n f to the power n + 1 Zeta by n + 1 factorial here.

+ if you will just consider here + h to the power n + 1 d by dx of your f to the power n + 1 zeta by n + 1 factorial into the product of p terms here that is p into p – 1 up to p – n here. Since Zeta is the function of x so that is why we have to differentiate this one with respect to x also here. So if you will just go for this like elimination of process here then we can just find that h to the power n it will just come since if you just see d by dx usually it can be written in the form like d by dp into dp by dx so dp by dx that will just give you 1 by h here. Usually we are just expressing x as x 0 + p h here so that is why if you will just write this one as d p by dx here, which can be written as 1 by h here.

Obviously, this differentiation can be replaced as d by dp of the total function p into p – 1 to p – n into dp by dx here. So that is why this h term is coming and the final representation it can be written in the form of like h to the power n if you will just write into d by dp of p into p – 1 up to p – n f to the power n + 1 Zeta by n + 1 factorial + the remaining terms.

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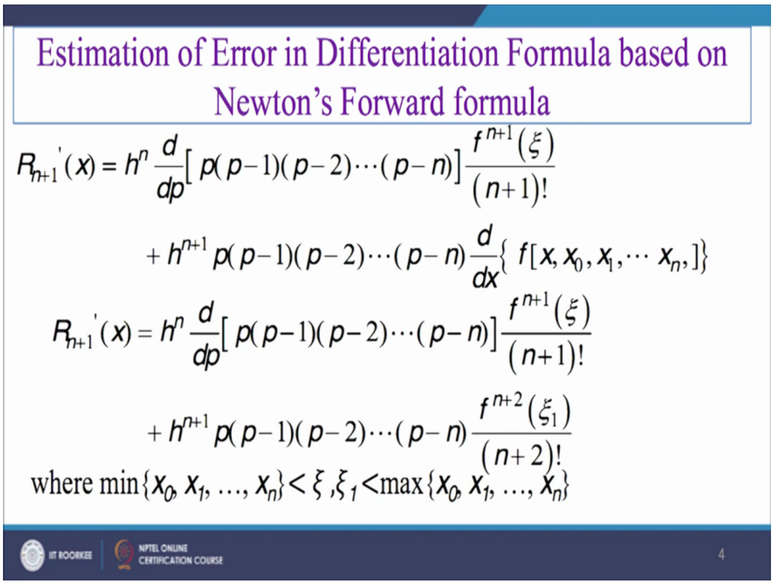


And if you will just apply this differentiation since in the earlier lecture we have derived this differentiation of f to the power n + 1 zeta there. So that is just completely expressed in the form of like divided difference from there, but here if we want to express this differentiation in the form of like Newton's forward difference operators here then this forward difference operators can be replaced here this f to the power n + 1 zeta as Del to the power n + 1 y 0 by n + 1 factorial here.

Obviously we can just express this one in Delta form here since in all of these lectures I have just explained that this tabular value is known to us but the function is not known explicitly to us, so that is why if these tabular values has been given to us directly we can just approximate this error term by considering all of these tabular values even if this differentiation can be approximated by using this forward difference table here. And if we will just write this difference terms here in terms of Newton's forward difference operator so this can be written in the form of like Del to the power n + 1 y 0 divided by n + 1 factorial into h to the power n + 1 here.

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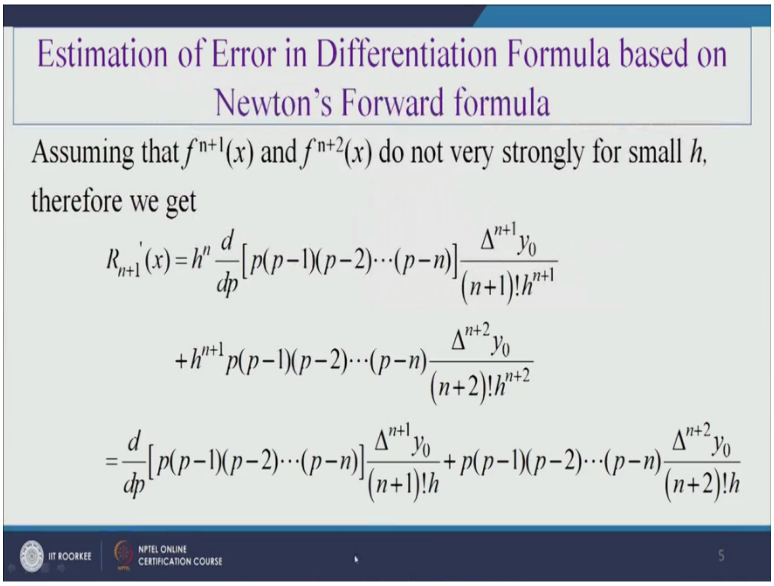


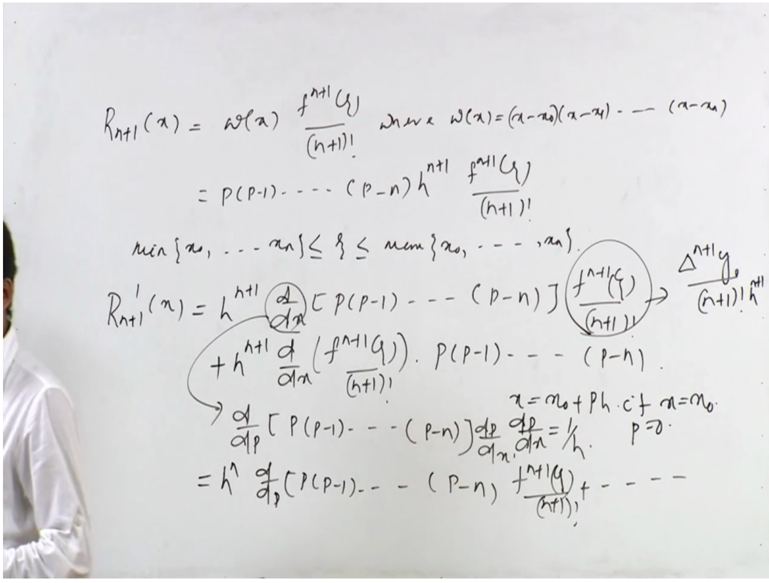

So once it can be expressed in this divided difference form sorry this forward difference from here, then we can just obtain this f to the power n + 2 zeta term since we will if you will just take the differentiation of f to the power n + 1 zeta by n + 1 factorial, so that will just give you f to the power n + 2 zeta by n + 1 factorial zeta dash x there so that is why it can be expressed in the form of like Del to the power n + 2 y 0 by n + 2 factorial h to the power and + 2 there. And obviously if you just write all these terms here then this can be expressed as d by dp of p into p – 1 into p – 2 up to p – n Del to the power n + 1 y 0 by n + 1 factorial into h + p into p – 1up to p – into Delta the power n + 2 y 0 by n + 2 factorial into h there.

Since h to the power n + 1 is there and it can just cancel it out with h to the power n + 2 and 1 by h is left over there. Since at x = x 0 obviously we are just obtaining p = 0 if x = x 0 there and at that point we can just write this error term differentiation that is R + 1 R of n + 1 dash of x 0 this can be written as – 1 whole to the power n factorial Del to the power n + 1 y 0 by n + 1 factorial h here.

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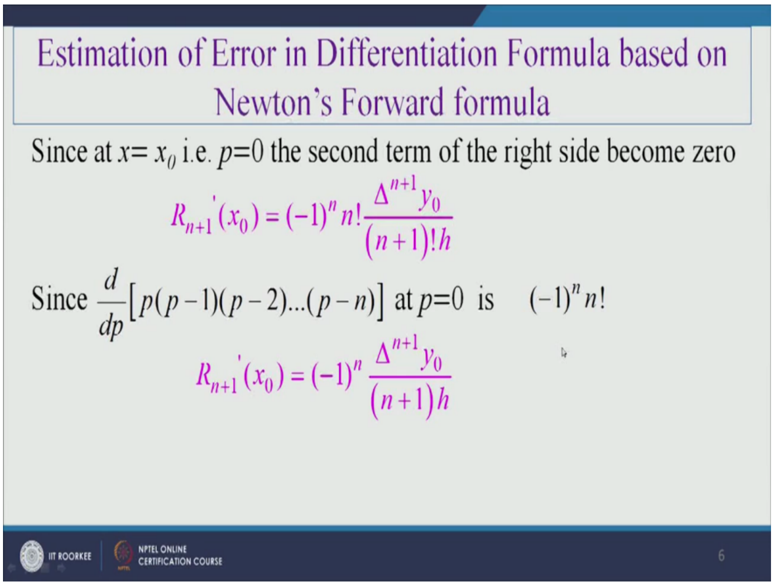


Since if you will just see this x function here, that is d by dp of p into p – 1 to p – n at p = 0 is nothing but – 1 whole to the power n n factorial here. And that is why this R n + 1 dash x 0 can be written as – 1 whole to the power n Del to the power n + 1 y 0 by n + 1 into h also sometimes. Since n + 1 factorial it just takes the product of n factorial into n + 1 so n + 1 remained over there. So based on this approximation we will just go for computation of errors in the Newton's forward interpolation formula.

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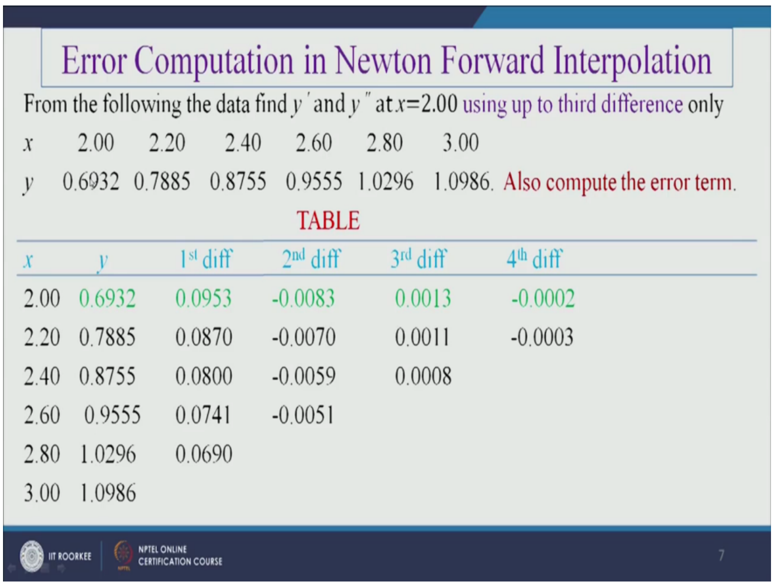

So for that if you just consider a table like x is prescribed at 2, 2.20, 2.40, 2.60, 2.80 and 3.00 here and corresponding y values if it is just to be given like 0.6932, 0.7885, 0.8755, 0.9555, 1.0296, 1.0296 here. Then the first difference if you just compute here that is the difference of 0.7885 – 0.6932 since already we have derived lot of forward difference approximation in tabular form so that is why it is easy to compute since we will just consider this difference of like 2 values corresponding tabular values like associated to each other there. So 0.7885 – 0.6932 this just gives you the value as 0.0953, difference of 0.5755 – 0.7885 this is just giving you the values as 0.0870, the difference of these 2 values 0.8755 – sorry this is 0.9555 – 0.8755 this is just giving you 0.0800.

Then 1.0296 – 0.9555 this is just giving you 0.0741, then the difference of 1.0986 – 1.0296 this is just giving you 0.0690 here. Similarly we can just obtain the second differences here, third differentiation and fourth differentiation.

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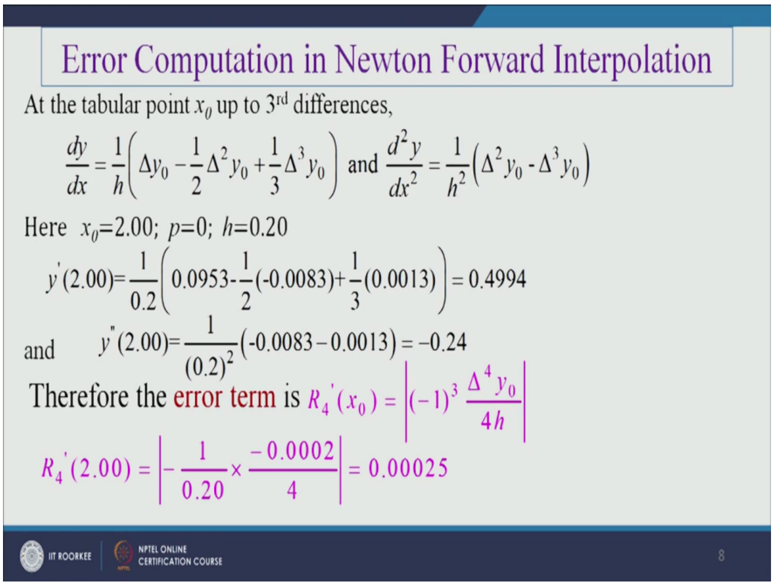


So based on this if the tabular point is asked you to compute up to third differences and then obtain this error term then we obviously we can just write this one as the fourth term there since the divided difference up to third order can be approximated in a polynomial form there, immediate next term can be considered as the error term for that position therefore the error term can be written in the form like R 4 of x here. So usually this R 4 of X 0 it can be written in the form of like if you will just see here, are 4 of x can be written as like Omega x f to the power 4 of zeta by 4 factorial so specially it can be written.

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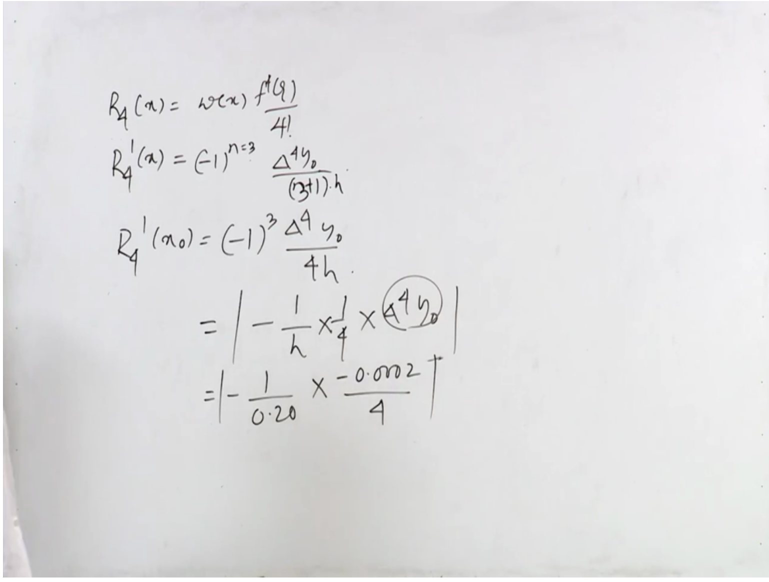


If you will just write in a complete form here, so are 4 of x can be written as Omega of x f to the power 4 of zeta by 4 factorial here. And in the differentiation form if you just write R 4 dash of x, this can be written as – 1 whole to the power n that is n = 3 there and then we can just write Del to the power 4 of y 0 by n + 1 this means n is 3 here so 3 + 1 into h here. So obviously if you will just put directly in the tabular values here, so are 4 dash of x 0 this can be written as – 1 whole to the power 3 Del to the power 4 of y 0 by 4 h here.

And at that point exactly if the error is asked suppose at x 0 = 2.00 here since the beginning of the tabular values is 2.00 so R 4 dash of 2.00 it can be written as like – 1 whole to the power 3 so that will just give you '–'sign here and this your h size is 0.20 here so that is why – 1 by 0.20 into 1 by 4 into your Del to the power 4 value that is just giving you – 0.002 here. So obviously if I will just write here this can be written in the form of like absolute values of – 1 by h into 1 by 4 into Del to the power 4 of y 0 here and this is nothing but – 1 by 0.20 into since this value is just giving you like – 0.0002 hereby 4 so this can be written in this form here.

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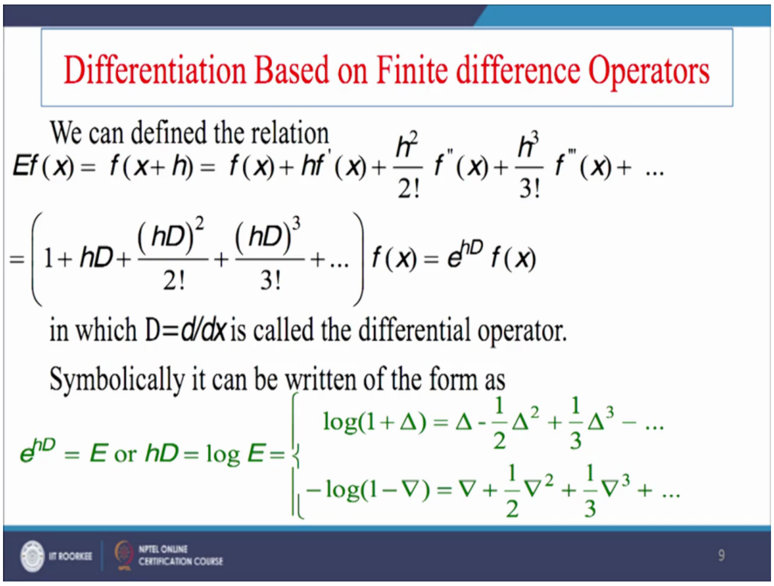


And obviously this value is just giving you 0.00025 here, so if you will just define this like differentiation using like finite difference operators, so finite difference operator means we can just consider that shift operator here, shift operator is usually expressed in the form of e of f of x here so e of f of x is usually written in the form like f of x + h here and f of x + h can be expressed in the form of Taylor series as h D of f of x, h D whole square by factorial 2 f double dash of x + rest of the terms. And usually it can be represented as 1 + h D + h D whole square by factorial 2 + this one into f of x here, and obviously it can be written in the form of e to the power h D f of x here.

And finally we are just expressing this shift operator E as e to the power h D here. That we have just expressed in the earlier lectures that d = d by d x here that is nothing but the derivative here and which can be expressed in the form of like forward difference operator, backward difference operator, centre difference operator, in any form we can just express this shift operator. So based on that we can just say that these operators can also be expressed in the differential operator form also there that means that d by dx can be expressed in the form of like Delta, Nabla or Centre difference operators like small Delta and average operators like Mu there, et cetera.

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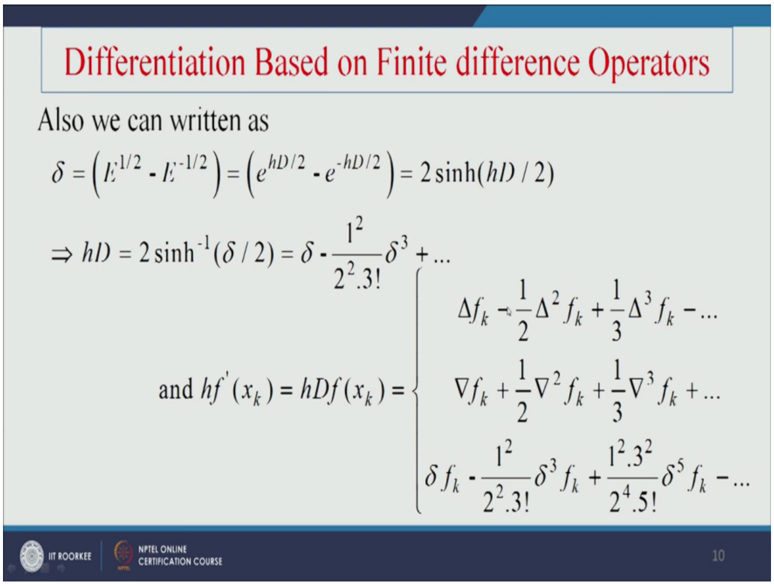


So if you just express these differential operators in the form of like forward difference operator like first difference if you just consider for this Newton's forward difference operator at a point k then we can just write h D of f of x k can be expressed in the forward difference form as Del of f of x k – half Del square of f of x k + 1 by 3 Del cube of x k – up to finite number of terms. Similarly, this divided difference sorry this differential operator can also be expressed in a backward difference form that can be expressed as Nabla of f k + half Del square of f k + 1 by the Delta cube of f k so likewise we can also express.

And also in central different approximation this can be written as Delta of f K – 1 square by 2 square 3 factorial Delta cube of f K + 1 square into 3 square by 2 the power 4 5 factorial Delta to the power 5 f k, so likewise we can just express this differential operators in various products form. So in the same way also we can just express this second order differentiation also, this means that second order differentiation we can just consider that one as h square f double dash of x k and this is nothing but h D you can just write, so h D can be written as like again one more differentiation we can just consider.

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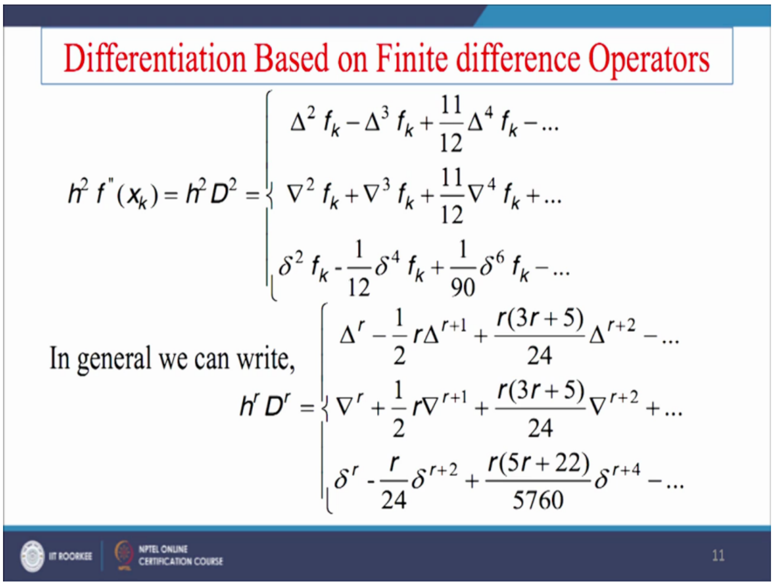


This x function of Newton's forward difference operators, backward difference operators or central different operators. So if you will just consider these operators once more here, this can be expressed as Del square of f k – Del cube of f K + 11 by 12 Delta to the power 4 of x k for forward difference operators. Similarly we can just express this one in backward difference operators that is in the form of Nabla square f k + naval cube f k + 11 by 12 Nabla cube f K + 11 by 12 Nabla by 4 of f K + all other terms.

And in general form if you will express this one as h to the power r d to the power r here, so this can be expressed as Del to the power r – half r Del to the power r + 1 + r into 3 r + 5 by 24 Del to the power r + 2 – all other terms are there. Similarly if you just take this backward difference operator, that it can be expressed as Nabla 2 by r + half r Nabla to the power r + 1 + r into 3 r + 5 by 24 Nabla to the power r + 2 + all other terms. And if you will just use this central difference operative to express this rth order derivative of any function then it can be expressed as small Delta to the power r – r by 24 small Delta to the power r + 2 + r into 5 r + 22 by 5760 small Delta to the power r + 4 – all other terms it can be expressed up to finite number of terms.

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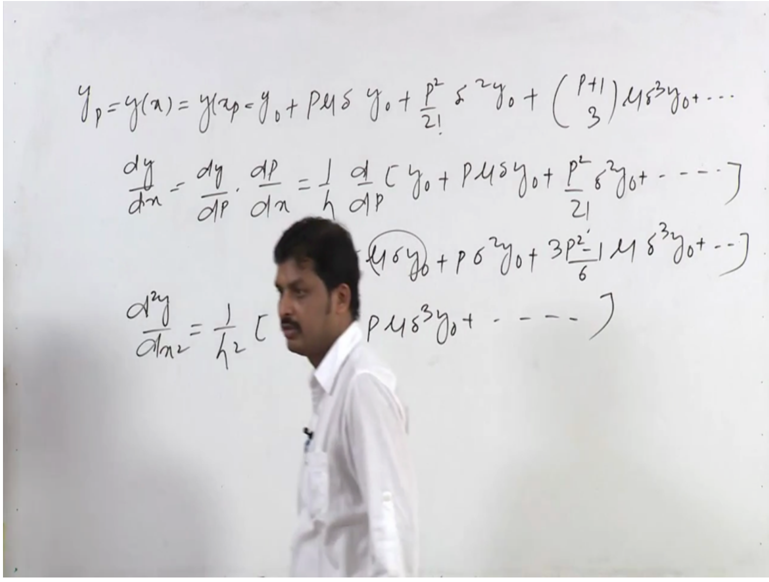


So next we will just go for central different approximation how we can eat just use this differentiation for central differential approximation like Stirling formula and Bessel's formula here. For this Stirling's formula already we have derived this expression for Stirling formula that is expressed in the form of like the central different approximation for Stirling formula usually it is written in the form of like y p or y of x or y of x p can be written as y 0 + p Mu delta of y 0 + p square by 2 factorial Delta square of y 0 p + 1 3 Mu Delta cube of y 0 + all other terms are there.

So if you just differentiate with respect to x here that can be written as dy by dx this as 1 by h into usually this expression can be written in the form like dy by dp into dp by dx so that is why dp by dx is nothing but 1 by h and d by dp of your expression y 0 + p Mu Delta of y 0 p square by 2 factorial Delta square of y 0 + all other terms are there. If you will just differentiate this one with respect to p here this can just give you 1 by h, first term it will just give you 0, second term Mu Delta of y 0 here + 2 p by 2 factorial so p Delta square of y 0 + the third factorial if you just take this one that will just give 3 p square – 1 by 6 Mu Delta cube y 0 + all other terms.

And if you just take one more differentiation of this scheme here that is d square by d x square, it can be written as 1 by h square so first term is just stated as a constant here then it will just start from this immediate next term so it is Del square of y 0 here so 3 into 2 into p here so 6 by 6 cancel it out so that is why it can be written as p Mu Delta cube of y 0 + rest of the terms here.

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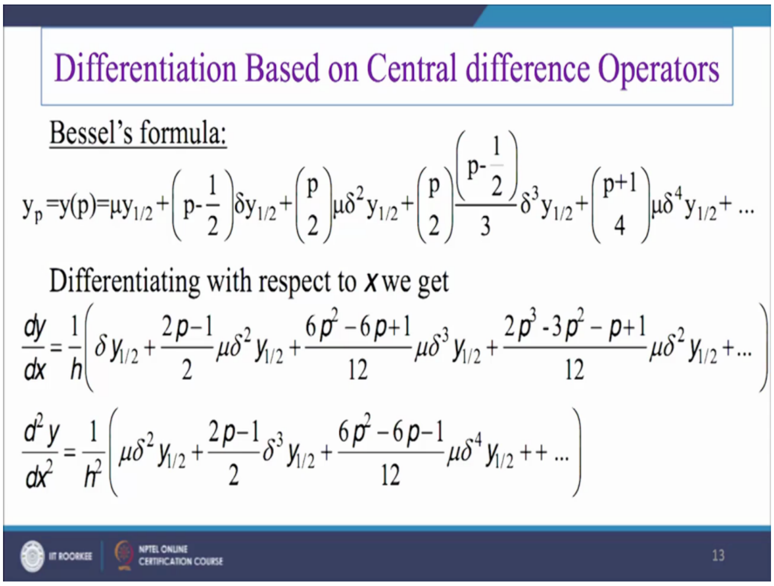


Similarly we can just use this formulation for like Bessel's formula here so Bessel's formula usually it is expressed as like y p = y of x p that as Mu y of half here so if you just write this one as y p in Bessel's formula here, this is for Stirling's formula here.

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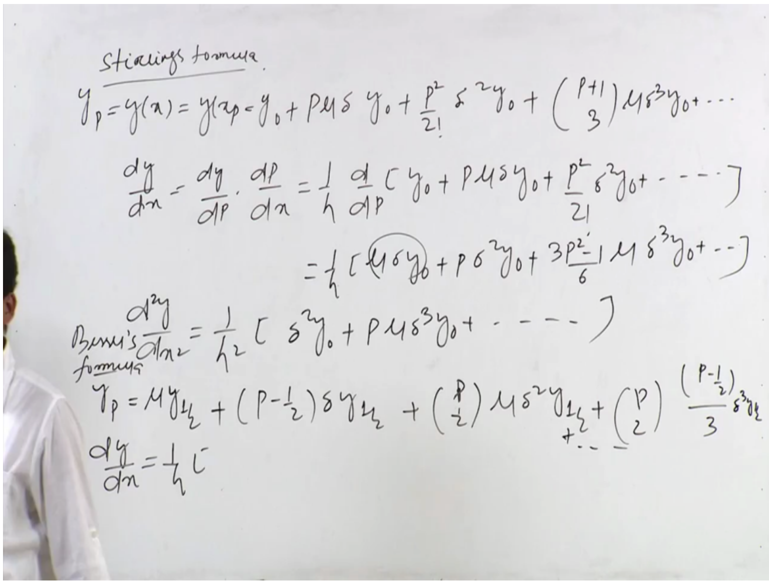


For Bessel's formula usually this y p is written in the form like Mu y of half + p – half Delta y of half here + p 2 Mu Delta square y of half + p 2 p – half by 3 Delta cube of y of half here so + all other terms. If you will just take the differentiation here, this same form we can just write dy by dx this as 1 by h, the first-term we can just take here as like Delta y of half here so then + 2 p – 1 by 2 Mu Delta square y of half then 6 p square – 6 p + 1 by 12 Mu Delta cube of y of half + all other terms there. Similarly if you just take the second differentiation here that can be written in the form of 1 by 1 by h square Mu Delta square y of half since if you will just take the differentiation of second term here since first-term is constant here.

Second term if you just take that can be represented a Mu Delta square of half here + 6 p square means 12 p – it is like 6 here so that is why 2 p – 1 by 2 Delta cube of y of half + all other terms we can just write.

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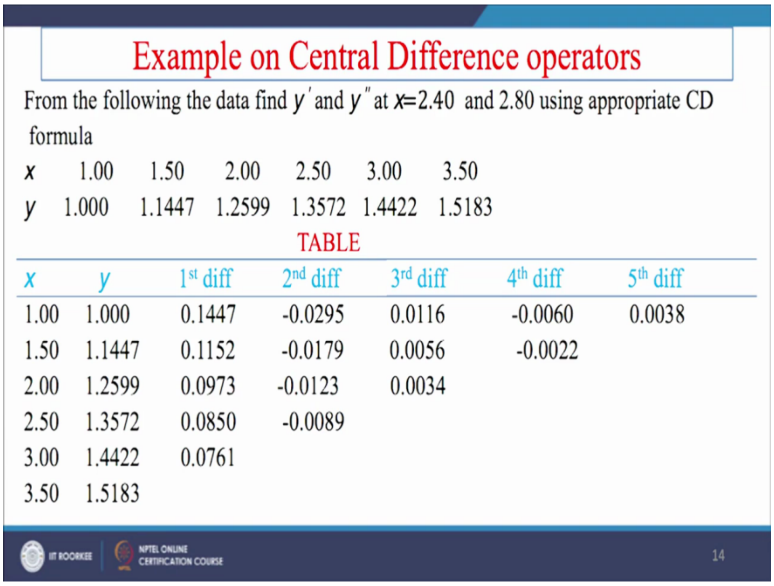


| x | y | 1st diff | 2nd diff | 3rd diff | 4th diff | 5th diff |
|---|---|---|---|---|---|---|
| 1.00 | 1.000 | 0.1447 | -0.0295 | 0.0116 | -0.0060 | 0.0038 |
| 1.50 | 1.1447 | 0.1152 | -0.0179 | 0.0056 | -0.0022 | |
| 2.00 | 1.2599 | 0.0973 | -0.0123 | 0.0034 | | |
| 2.50 | 1.3572 | 0.0850 | -0.0089 | | | |
| 3.00 | 1.4422 | 0.0761 | | | | |
| 3.50 | 1.5183 | | | | | |

So based on this if you just go for example here since central different approximation usually it can be approximated at centre of the table here where for both of these formulas we have defined this approximately parameters so that where it can be used. So in that regard if we will just consider this table as like x is defined as 1.00, 1.50, 2.00, 2.50, 3.00, 3.50, all are equally spaced points with space size 0.5 suppose with this corresponding y values as 1, 1.447. 1.2599, 1.3572, 1.4422, 1.5183 here, then if you just use these first differences for all these values we can just obtain this one as 0.1447, 0.1152, 0.0973, 0.0850, 0.0761 for first differences.

For second difference if you will just take the differences of immediate next terms; 0.1152 -0.1447 here then that will just give you – 0.0295 here. Similarly if you just take the difference of 0.0973 – 0.1152, this will just give you – 0.0179 here. And if you will take the difference of 0.0850 – 0.0973 this will just give you – 0.0123 here, if you just take the difference of 0.0761 and 0.0850 that will just give you – 0.0089. Similarly we can just define this third difference, $4^{th}$ differences and $5^{th}$ differences here since we have 6 points so that is why we can just consider the differences up to $5^{th}$ order here.

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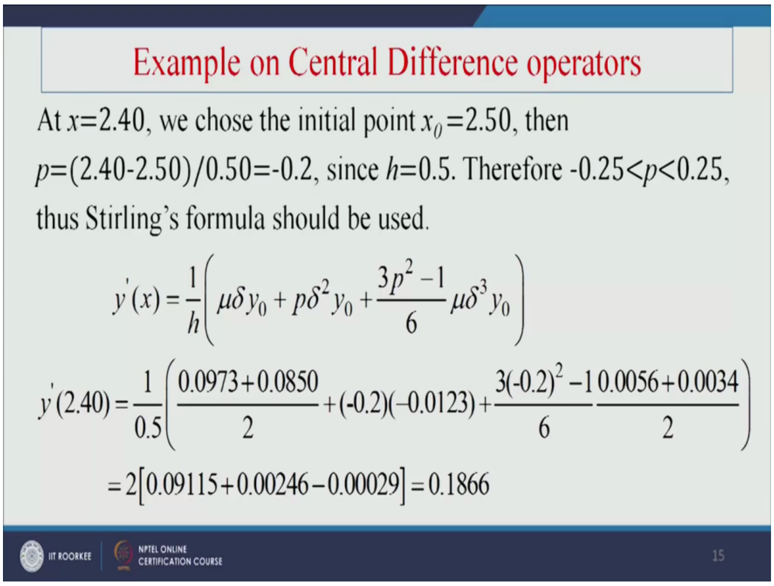


And if it is asked to use this Stirling's formula in Bessel's formula for this error approximation up to suppose third order term here so up to third order we can just consider and immediate next term we can just consider the error form here. First we have computed this derivative at that values like suppose the values asked you to compute this derivative as 2.40 here so that is why we can just put here y dash as x is defined as 1 by x Mu Delta Mu Delta of y 0 + p Delta square of y 0 + 3 p square – 1 by 6 Mu Delta cube of Y 0 here so that is why we can just use h as 0.5 here, 1 by 0.5 so average of these 2 this is like Mu Delta can be written as like your values, we can just write Delta y of 1 + Delta y 0 by 2 there so that is why it can be written as 0.0973 +0.0850 by 2 there.

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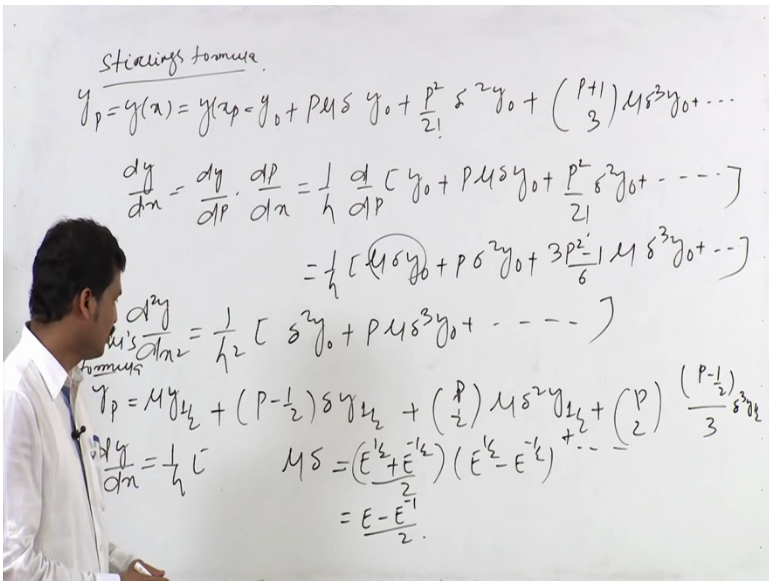

Directly you can just express in the form of like if we want to write this one as Mu Delta here so Mu Delta can be written as E to the power half + E to the power – half by 2 into E to the power half – E to the power – half here. If you just take the product here we can just obtain this one as E – E of – 1 by 2 here, so obviously we can just write this one as f of x + h – f of x – h by 2 so that is why we are just using this differentiation if you will just see as 0.0973 + 0.0850 here by 2. So using all these we can just obtain these values as 0.1866 here. The second order formula if you will just use here that is y double dash of x here so y double dash of x it is just given as 1 by h square Delta square of y 0 + p Mu Delta cube of y 0 + all other points.

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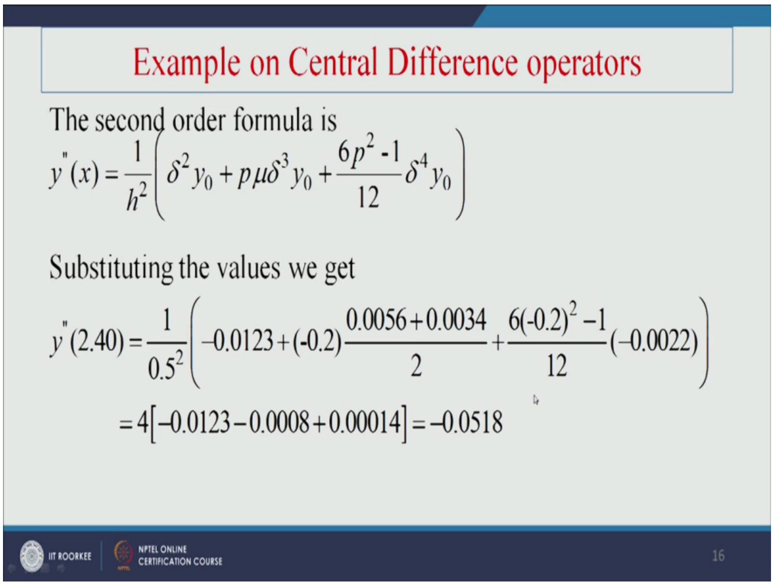


So similarly h is given as 0.5 here so that is why 1 by 0.5 square here we have considered and Delta square of y 0 that is nothing but the second order differentiation of this forward difference table also so that is why it can be considered as – 0.0123 here + p value that is given as – 0.2 into the average of Delta cube values we have just considered here and rest of the values can be defined in the same form here and finally we are just obtaining y double dash of 2.40 as – 0.0518 here. And if you just use this formula for the point 2.80 here then we can just use Bessel’s formula for that since p range is lying between 0.25 to 0.75 within that range specially we can use the Bessel’s formula.

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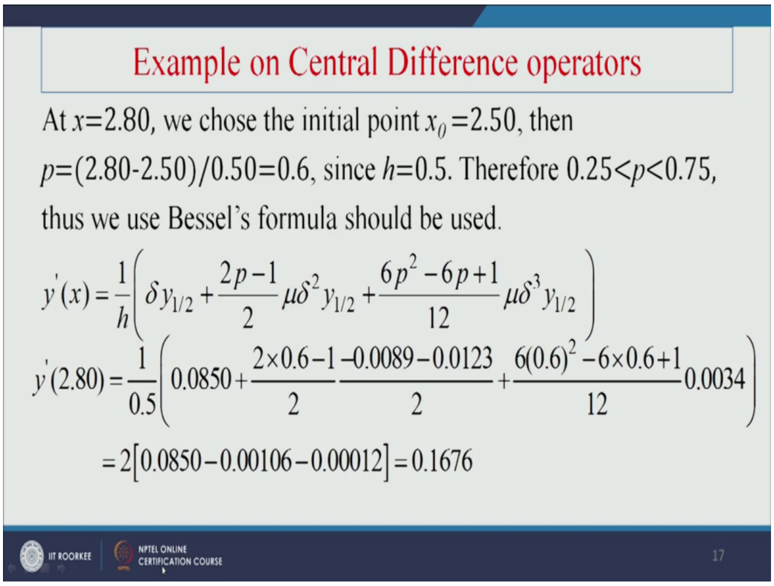


So for that if you will just find here the derivative of y of x is usually written as 1 by h Delta Y of half + to p – 1 by 2 Mu square y of half + 6 p square – 6 p + 1 by 12 Mu Delta cube y of half here. So if you just put these values like 1 by 0.5, 0.0850 + 2 into 0.6 – 1, so all other terms it can be placed accordingly. Then we can just obtain this first order derivative at 2.80 at 0.1676 here so the second order formula if you just use here that is in the form of like 1 by h square Mu Delta square Y of half + 2 p – 1 by 2 Delta cube y of half so then we can just obtain at the point 2.80 as since already we have obtained that h as 0.5 here.

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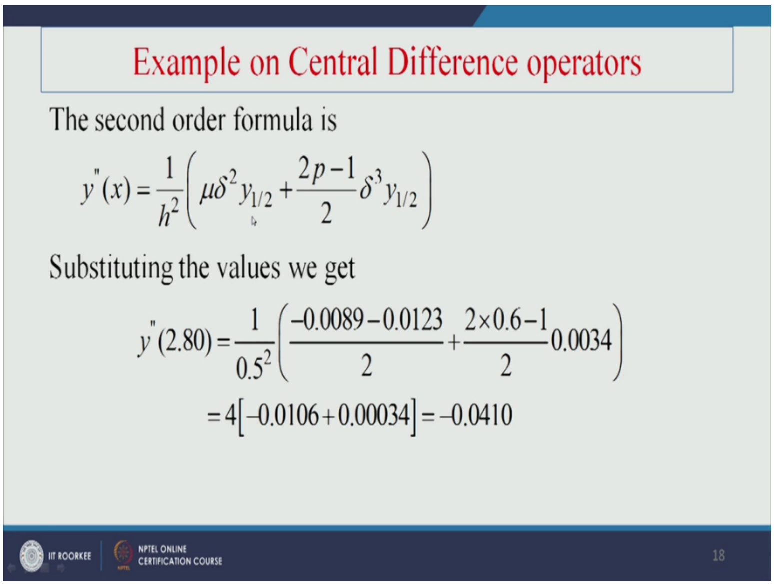


So that is why this can be written as 1 by h square means 1 by 0.5 whole square so then Mu Delta square y of half if you will just take Mu as average value here so Delta square half we can just write this one as Delta of Delta of y of half and we can obtain the values in the form of like – 0.089 – 0.0123 divided by 2 here + 2 p – 1 so p is given as 0.6 here, 2 into 0.6 – 1 by 2 into Delta cube y of half this is nothing but directly we can just obtain from this divided difference formula here sorry this forward difference formula and the final value is obtained as – 0.0410 as this one, thank you for listening this lecture.