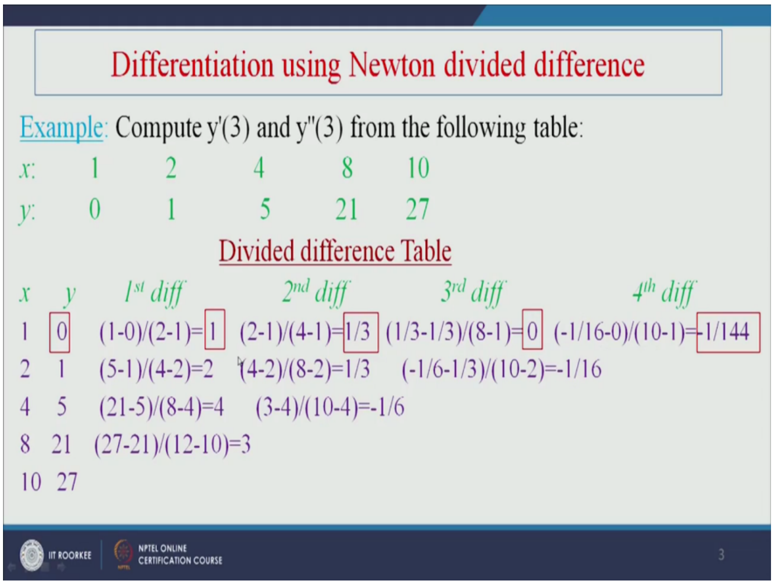
Numerical Methods
Professor Ameeya Kumar Nayak
Department of Mathematics
Indian Institute of Technology, Roorkee
Lecture 28
Numerical differentiation part-IV (Maxima minima of a tabulated function and errors)

Welcome to the lecture series on numerical methods, in the current lecture series we are discussing here numerical differentiation. In the last lecture we have started this numerical differentiation using Lagrange interpolation method and divided differences. And in the end of the last lecture I have just given one example based on this divided difference that how we can use this differentiation. And there itself I have discussed that how we can just apply the divided difference on the tabular form first and then we can just go for this derivative. And after this we will just in this lecture we will just go for this Maxima and minima of tabulated function then the error estimation in the differential equations.

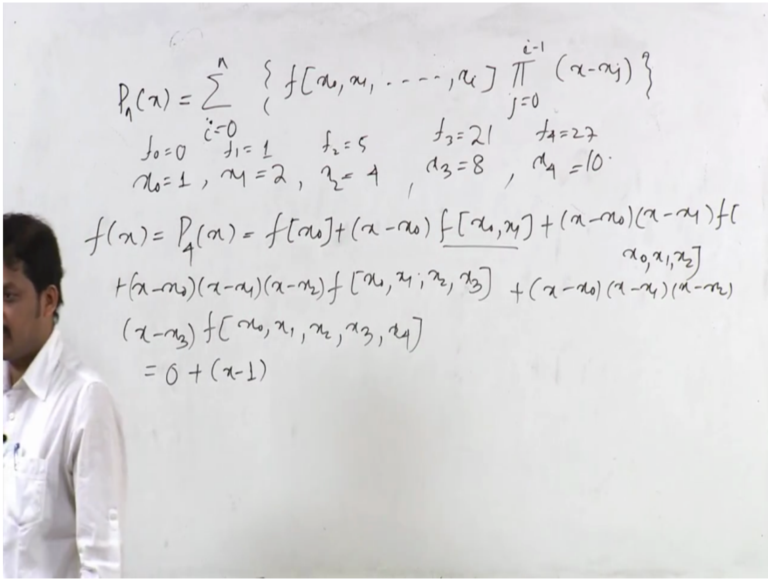
(Refer Slide Time: 1:20)



But this numerical differentiation how it can be applied for this error estimation and interpolation polynomials. So this present table already I have discussed in the last lecture that how we can just find this first order difference, second order difference, third order difference and fourth order difference. If a tabular value is given to us like x = 1, 2, 4, 8 and 10 and then the corresponding y values are given as 0, 1, 5, 21 and 27 suppose, how we can compute the derivative of this function y at point 3 suppose and double derivative also at that same point.

So first we have to go for this divided difference table, already in the last lecture I have derived this divided difference table there itself and after that once we are just using this divided difference table we will have these values like first divided difference value that is we have just obtained here as 1 here, second divided difference for the first table form, we have just obtained 1 by 3 then 0, then – 1 by 144.

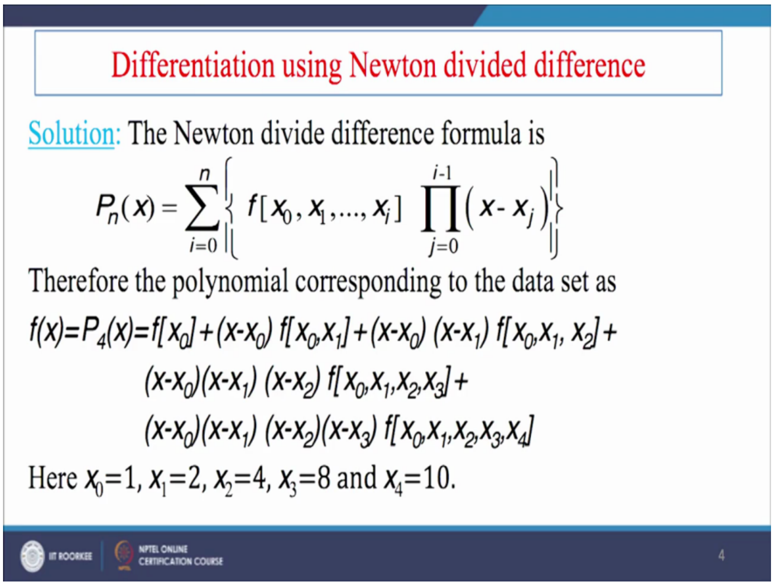
(Refer Slide Time: 2:44)



So if you will just use this divided difference formula, usually this Newton's divided difference formula is written in the form P n x, this = submission of j = 0 to n sorry this is i = 0 to n if you will just write then we can just write f of x 0, x 1 to x i then we can just write product of j = 0 to i – 1 and x – x i here, so either you can just use this one as the curl bracket we have used here. So if we are just writing this polynomial in this form then since the data it is just given to us are like x 0, x 1, x 2, x 3 and x 4 here. So if you just collectively write this data here like x 0 = 1 here, x 1 = 2 here, x 2 = 4 here, x 3 = 8 here and x 4 = 10 here.

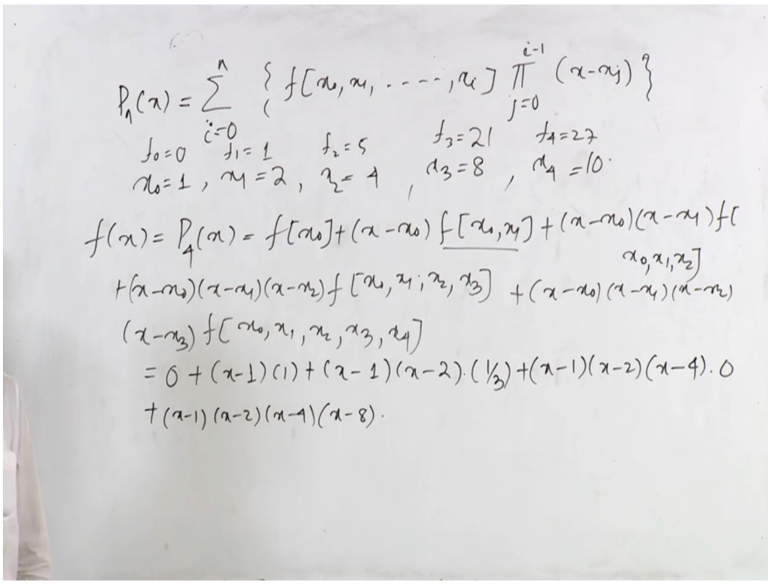
Then we will have like 5 points here, this can just generate a polynomial of degree 4 here so we can just write this function f of x it can be approximated with polynomial of degree 4 here P 4 x and that can be written in the form like f of x 0 + x – x 0 f of x 0, x 1 + x – x 0, x – x 1 f of x 0, x 1, x 2 + x – x 0, x – x 1, x – x 2 f of x 0, x 1, x 2, x 3 + x – x 0, x – x 1, x – x 2, x – x 3 f of x 0, x 1, x 2, x 3, x 4 here. So if we just put all these values here since our initial values it is in tabular form it is just given us as like 0 here.

(Refer Slide Time: 5:14)



If you just see this tabular values here, so first corresponding value of x is just given here if you just see f 0 it is given as 0, f 1 as 1 here, f 2 as 5 here, f 3 as 21 here, f 4 as 27 here. So if you just put these values, f of x 0 is 0 here so we can just write this one as 0 + x – x 0, x 0 is once here then f of x 0, x 1 this is the tabular form already we have discussed in the last lecture that is specially written as 1 here + x – x 0, x – x 1 here into the divided difference between this f of x 0, x 1 and x 2.

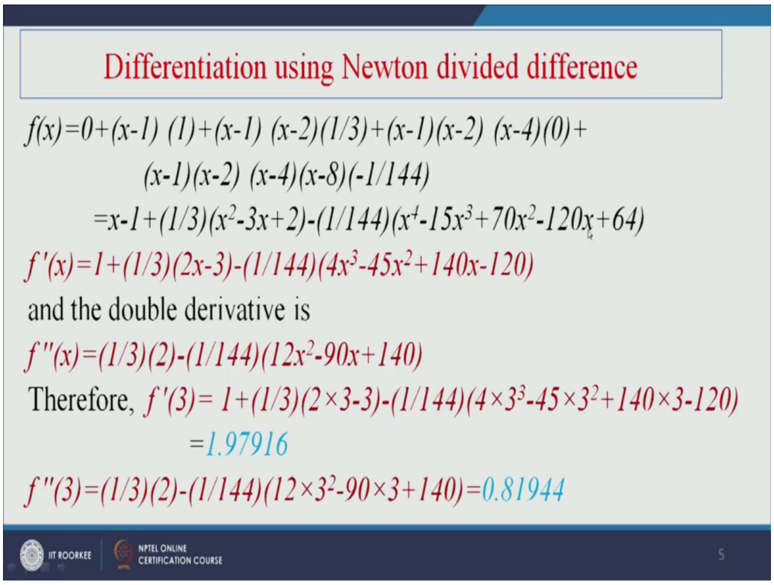
(Refer Slide Time: 6:37)



Collectively if you just see this table, so we can just we have already obtained that one the values as 1 by 3 here + x – x 0, x – x 1, x – x 2 that is just 4 here into your f of x 0, x 1, x 2, x 3 here and that value it is just giving you 0 there + x – 1, x – 2, x – 4, x – 8 into f of x 0, x 1, x

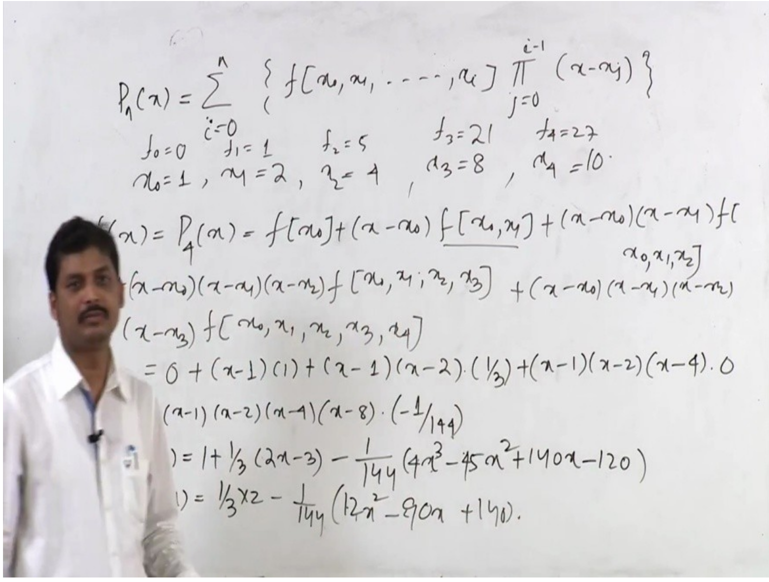
2, x 3, x 4 in arguments form and that value especially just giving us in the tabular form as – 1 by 144. So if we can just multiply all these terms here then we can just obtain this polynomial that is in the form of x only here. So obviously if this polynomial is expressed in the form of x, directly we can just apply the derivative to get this differential form of this polynomial P 4 x here.

(Refer Slide Time: 8:09)



So if we will just take all these products here then this product form can be written in the form like first-term in x – 1 here, second term if you will just see here so that can be written as 1 by 3 x square – 3 x + 2 and next immediate term is 0 there so then the last term it can be written as per – 1 by 144 x to the power 4 – 15 x cube + 70 x square – 120 x + 64 here. And if you will just take this derivative here that is in the form of f dash x, so f dash x can be written as first derivative here 1 + 1 by 3 into x square – 3 x + 2 derivative is 2 x – 3 – 1 by 144 then x to the power 4 is 4 x cube – 45 x square + 140 x – 120.

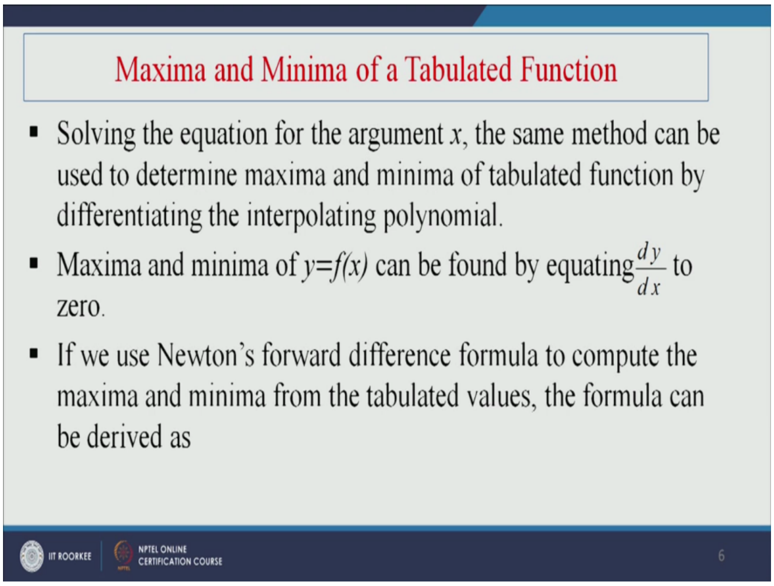
(Refer Slide Time: 9:35)



So final form we are just obtaining here f dash x as 1 + 1 by 3 into 2 x – 3 – 1 by 144 x to the power 4 sorry, I will just write this one only the derivatives that is why 1 by 3 into 2 x – 3 – 1 by 144 x to the power 4 – 15 x cube then + 70 x square, 70 x square means derivative will come as 140 x here then the last term is – 20. So if you want to find suppose this derivative at the point 3 so directly we can just replace this x by 3 here and we can just obtain this derivative at the point 3. And if we want to evaluate this second order derivative here then directly we can just differentiate once more here that is first-term is 0 here then 1 by 3 into 2 – 1 by 44 then this is 4 x cube – 45 x square + 140.

And similarly if you just put here f double dash of 3 then we can just obtain the second order differentiation of this fourth order polynomial at the point 3 there. So obviously these values if it can be calculated so that can be represented in the form of f dash 3 = 1 + 1 by 3 into 2 into 3 – 3 – 1 by 144 into 4 into 3 cube there so specially it is just it should be like 4 x cube – 45 x square if you just see so this should be 4 x cube – 45 x square if you will just take the derivative. So then it can be multiplied and we can have the clear form here then it will just change to this is like 12 x square, this should be like 90 x here and it is like 140. So if you will just directly put this f double dash of 3 here then we can just obtain this value as 0$81944 here.

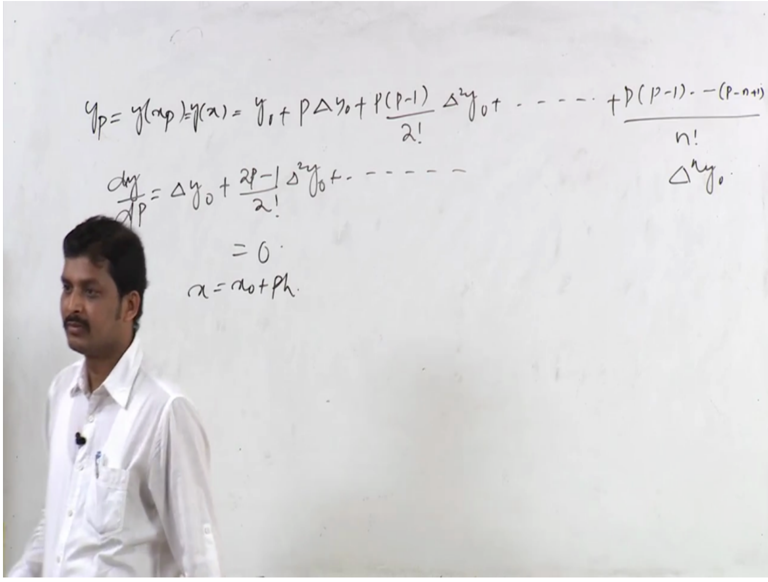
(Refer Slide Time: 11:36)



So then we will just go for this computation of Maxima and minima using this polynomial differentiation. The beauty of this method is that even if the function is not known to us then we can have this maxima or minima of this function if the prescribed tabular value is given to us. So we can just use this same arguments form of x here and we can just determine this maxima and minima for this tabulated function by differentiating this interpolating polynomial here.

Maxima and minima of y = f of x can be obtained by equating dy by dy = 0 and first if you just use Newton's forward difference formula to compute this Maxima and minima from the tabulated values, then the formula can be written in the form like the formula for this Newton's forward difference formula already we have discussed this one in previous lectures and if I will just write once more here.

(Refer Slide Time: 12:34)

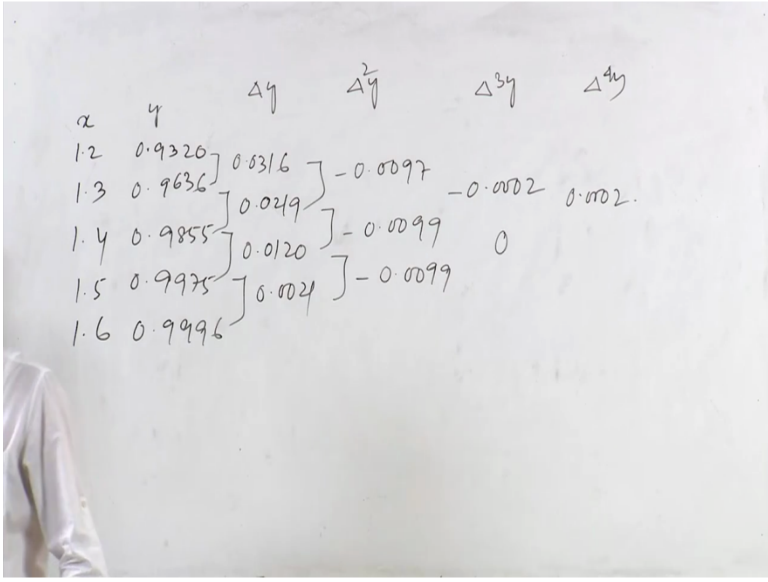


Then this Newton's forward difference formula can be written in the form of y of p or y of x P or y of x, this can be written as y 0 + the Delta y 0 + P into P – 1 by factorial 2 Delta square of y 0 + up to P into P – 1 up to P – n + 1 by n factorial Delta to the power n of y 0 here. And if you just differentiate this word with respect to P here so then we can just obtain dy by dp as Del y 0 + 2 p – 1 by 2 factorial Delta square of y 0 + all other terms there. For maxima if you just put here dy by dp = 0 suppose, so this can be written as 0 here. This implies that we can just terminate this left-hand side series expansion up to certain terms then we can just obtain this maxima or minima of a function.

Suppose if the left-hand side is just truncated after suppose third differences for our convenience suppose, then we will just obtain quadratic equation here in p and which gives 2 values of p since it is just the quadratic equation here. Corresponding to these values of p then we can have maxima or minima at that point and so when we have these values of P there then we can just obtain this value of x at that point. Since usually this x can be written as in the form of x = x 0 + P h here, or for convenience we can just write sometimes if x 0 = A is a particular value then we can just write this one as A + P h here.

And once this value of P is known to us then we can just determine the values of x and if the x is known to us then we have to see these tabular values that where this x is placed inside this table and at that point either this forward difference formula or backward difference formula can be applicable that we have to check and once we can just check that one since that depends on the values of p at that point. So once we are just obtaining the values of p at that point then we can just obtain the values of another y at that point for these values of y of x. And to obtain this Maxima or minima the usual criteria is that we have to see that for Maxima we have to put D square y by D x square is negative and we have to show that for minima the square y by D x square is positive.

(Refer Slide Time: 16:16)



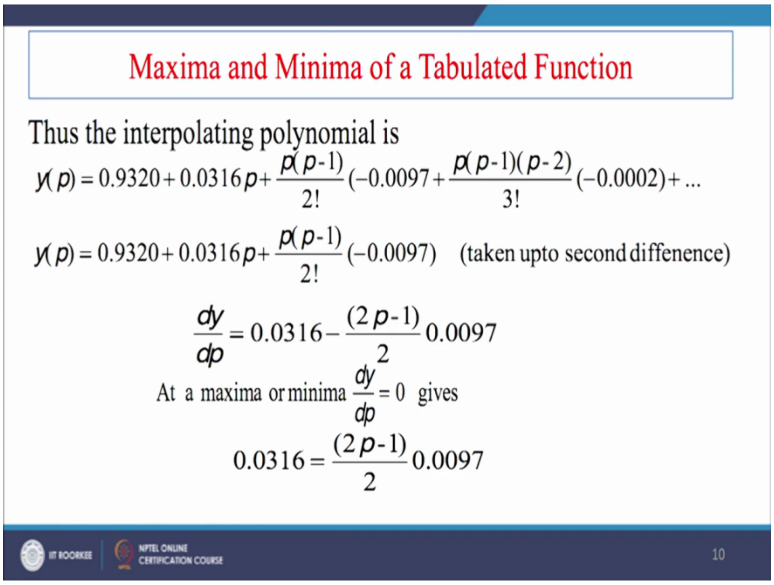
Suppose we have example here that the question is asked to find x for which y is maximum taking the difference up to second order from the following table and find maximum value of y. And if I will just consider this table here like x values are prescribed as 1.2, 1.3, 1.4, 1.5 and 1.6 with space size h = 0.1 here. Suppose the tabular values are given at x = 1.2, 1.3, 1.4, 1.5, 1.6 here and the corresponding values of y are given as 0.9320, 0.9636, 0.9855, 0.9975, and last value is given as 0.9996.

And the corresponding values for forward difference formula or forward difference table can be given as the difference of these 2 so that is just given us 0.0316 here, If we will just take the difference of these 2 here so the values can be written as 0.0219 here. If you will just take the difference of these 2 here then that will just give you 0.0120 here, if we will just take the difference of these 2 here that can be given as 0.0021. Then the second difference for these numbers can be written as – 0.0097 first one, then second one it can be written as – 0.0099, third difference it can also be written as 0.0099 here. Sorry, this is Del square then we will just go for Del cube of y here.

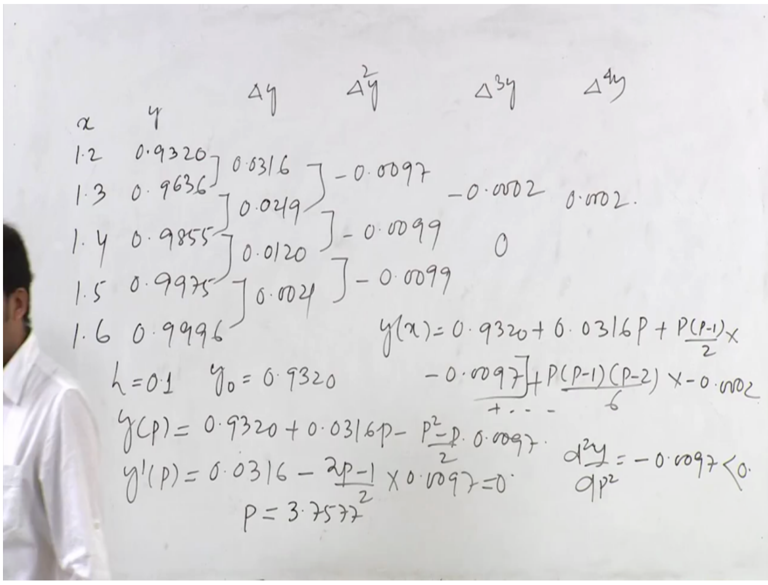
And if we will just take the difference here so the first difference this will just give you – 0.02 here and this will just 0 here and fourth order difference you can just write this one as 0.002.

And the space size if you just see, these equidistance points are there so that is why h can be written as 0.1 here and starting value for y 0 it can be written as 0.9320 here.

(Refer Slide Time: 18:39)



(Refer Slide Time: 19:17)



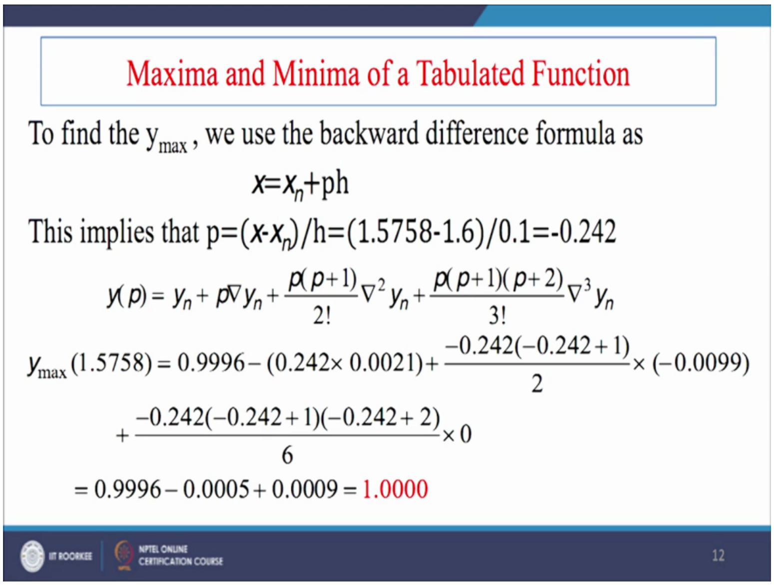
And based on this if we want to determine this interpolating polynomial, the interpolating polynomial can be written in the form like y of x = y 0, y 0 means 0.9320 here and the second point it can be written as P Delta of y 0 here so 0.0316 P here and third one if you will just write here so + P into P – 1 by 2 into the third difference if you will just write – 0.0097 here. And then again this difference if you just write P into P – 1 into P – 2 by 3 factorial into next immediate value is – 0.0002 here + all other terms we can just write here and if the question since the question is asked to compute or taken these terms up to suppose second differences here.

So we can just consider these terms of 2 this one only here, and then we can just write y of p up to second differences as 0.9320 + 0.0316 P + P square – P by 2 since this sign can come as – here 0.0097 here. And if we want to first find the first order differences here so y dash P this can be written as 0.0316 – so this will be just giving you 2 P – 1 by 2 into 0.0097 and this = 0 here. If this equates to 0 here for maxima and minima of this point then we can just obtain these values as 0.0316 this = 2 P – 1 by 2 into 0.0097 here.

And finally we will have this value like P = if you just solve here, P can be written as 3.7577 here. If you will just see here, P is just giving you this value then to find this maximum or minima at that point we have to check d square y by d square y by dp square here, so if you will just evaluate here d square y by dp square here, then this value it is just giving you directly as – 0.0097 here this is less than 0 already, so that is why we can just say that this second order derivative is minimum means or negative means we will have the maximum value at P = 3.7577 here. To obtain the values of x for that p we have to consider again x = x 0 + P h or x = A + P h there.

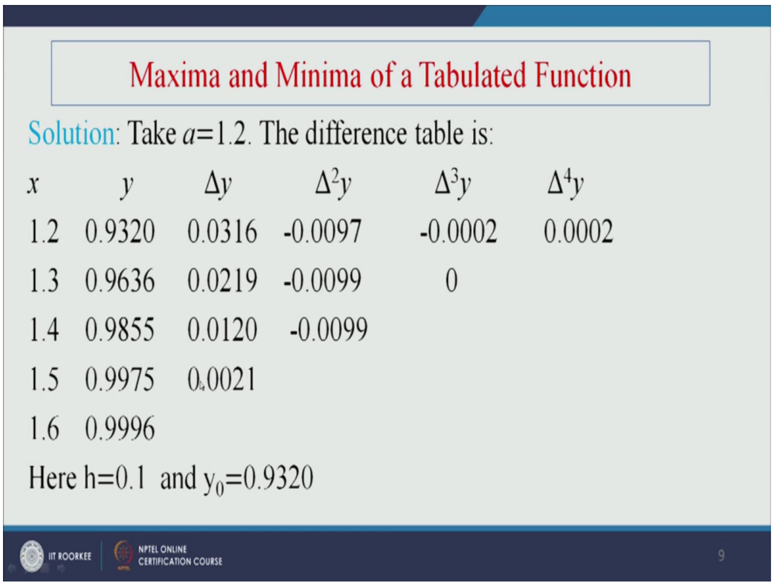
So if you just consider that to find this x value for this corresponding P value here so x can be written as x 0 + P h here or this can be written as A + P h so the final value if you just see here that initial value is taken as 1.25 here + P is 3.7577 into h is 0.1 here, so finally we can obtain these values as 1.5758 if you just see this point here, so this value of x is just observed in this point here on these tabular values. This means that we can have to use this Newton's backward difference formula to obtain this value at that point.

(Refer Slide Time: 23:19)



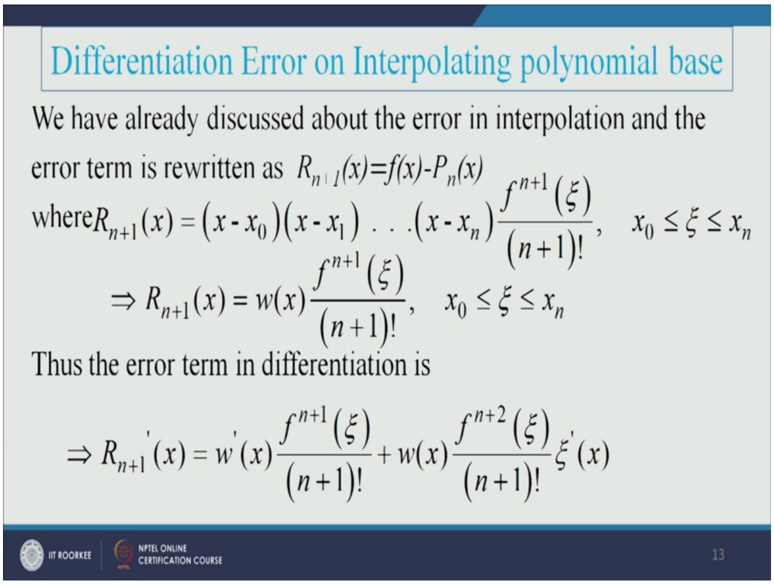
So to find this maximise value at that point especially we have to use this backward difference formula as x = x n + P H at that point. This simply satisfies that P can be written as x – x n by h there which can be written as 1.5758 – 1.6 since 1.6 is your x n value so we can just write that one by 0.1 this = – 0.242 since P is lying between like – 1 to 0 there so we can just use Newton's backward difference formula. And if we want to evaluate this polynomial at that point especially we have to consider that y n = 1.6 the particular value will be 0.9996 here, then + P naval y n that is P is just giving you here the value as – 0.242 so – 0.242 into.

(Refer Slide Time: 24:24)



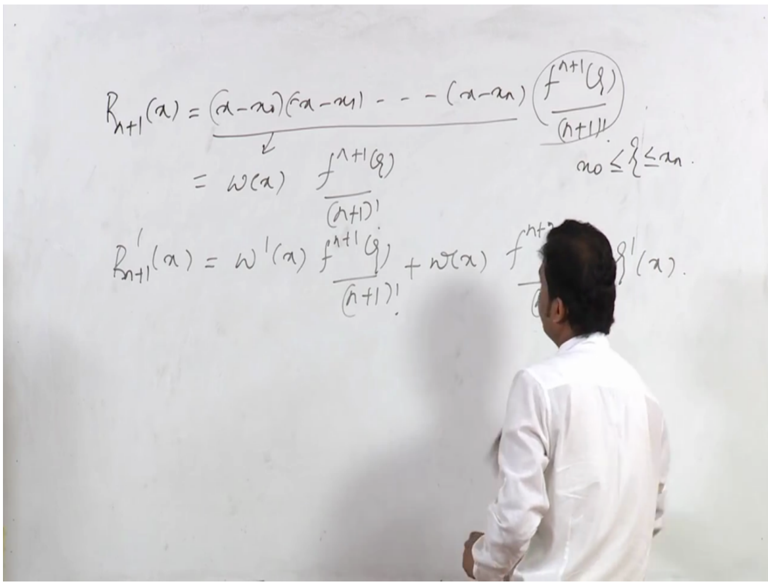
So – 0.242 into this naval of y n if you just see this backward table differences here that is just giving you 0.0021 here if you just see and + then P into P + 1 into P + 2 sorry P into P + 1 by 2 factorial del square of y n here so P is given as – 0.242 into – 0.242 + 1 by 2 into – 0.0099 here. So the final value it is just obtaining as 1.000 here up to 4 decimal places.

(Refer Slide Time: 25:01)



So then we will just go for this error approximation in the interpolating polynomial case. So for that we have to consider like Newton's forward difference formula, backward difference formula or like divided difference formula to value at this error at a different points there.

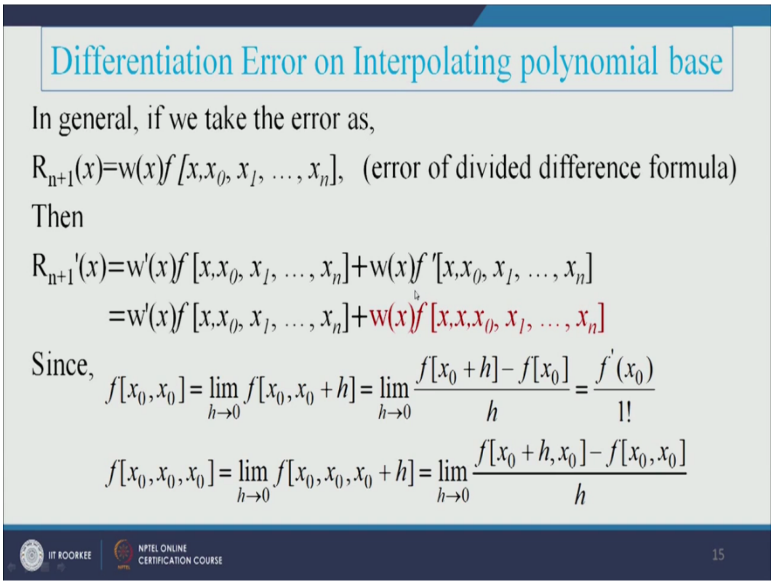
(Refer Slide Time: 25:40)



So already we have discussed that these errors that is written in the form like r n + 1 x as r n + 1 x this error term is defined as x – x 0, x – x 1 up to x – x n f to the power n + 1 zeta by n + 1 factorial. Usually this is the error term for general approximation of this interpolating polynomial so we have defined where this zeta value should lie between x 0 to x n here. And if we just go for this differentiation of this error term here then we can just differentiate this x let us see here separately and we can just take the differentiation of this zeta since zeta is a function of x here, we can just treating these 2 variables product form here.

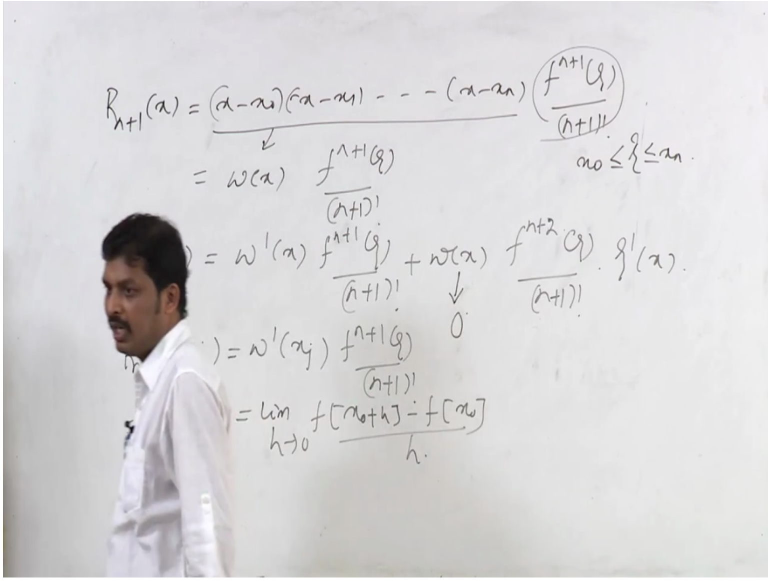
This means that if you just consider this side as Omega x here and this side as up to the power n + 1 zeta by n + 1 factorial here then this differentiation with respect to x for this error term it can be written as Omega dash x f to the power n + 1 zeta by n + 1 factorial + Omega of x f to the power n + 2 zeta by n + 1 factorial into zeta dash x here. So based on this we can just say that this error it will just occur at a different polynomial basis, this means that if you just consider exactly x = x a then obviously this Omega x will just give you a 0 value here. Then we can just write this polynomial as r n +1 dash x j exactly this = Omega dash of x j f to the power n + 1 zeta by n + 1 factorial here. Where this zeta should be lies between like x 0, x 1, to x n for maximise value within any of these intervals there and minimum value within this interval.

(Refer Slide Time: 28:16)



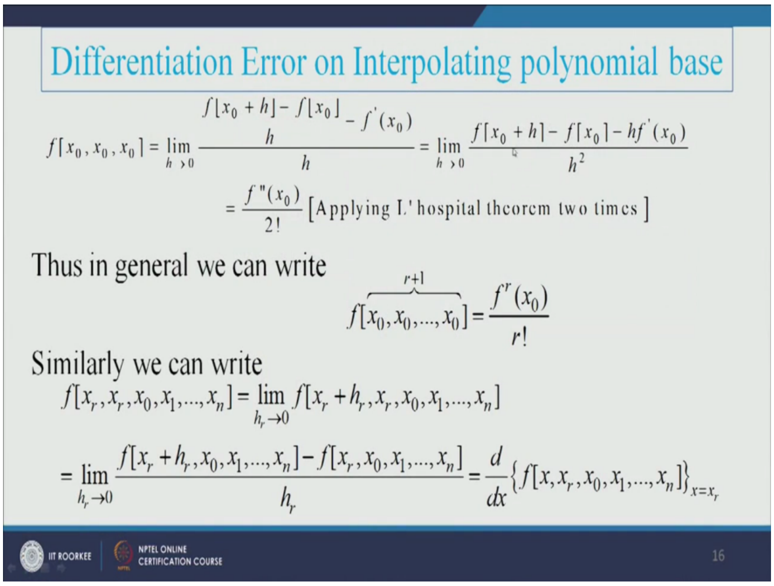
Why we are just considering this maximum and minimum value is that sometimes maybe it is just occurring this variable x the form of fractional form there so we have to consider this minimise value for minimise values of x to determine this maximum value of that function so that is why we have to consider either it is in minimise form or in the maximise form to obtain this maximum value for this error term there. So if you just consider this one in a divided difference form here so this divided difference form of this polynomial can be written as if you will just see this Newtons's divided difference error derivation from our earlier lectures, you can just find that r n 0 1 x term can be written as Omega x into f of x, x 0, x 1, up to x n there.

(Refer Slide Time: 29:07)



So obviously r n + 1 dash x can be written as Omega dash x into this first product term + Omega x into the derivative of the second product. And we can just write this one as Omega dash x into f of x, x 0 to x n + Omega x, since we are just considering the derivative of x, x 0, x 1 to x n arguments so one more x can be involved inside these arguments. Why it is just coming if you just see here, f of x 0, x 0, usually it is just written in the form like limit x tends to 0 f of x 0 + h – f of x 0 divided by h here. And obviously this can be written in the form of like f dash of x 0 divided by 1 factorial here. Similarly we can just write f of x 0, x 0, x 0, 3 arguments if it is just placed.

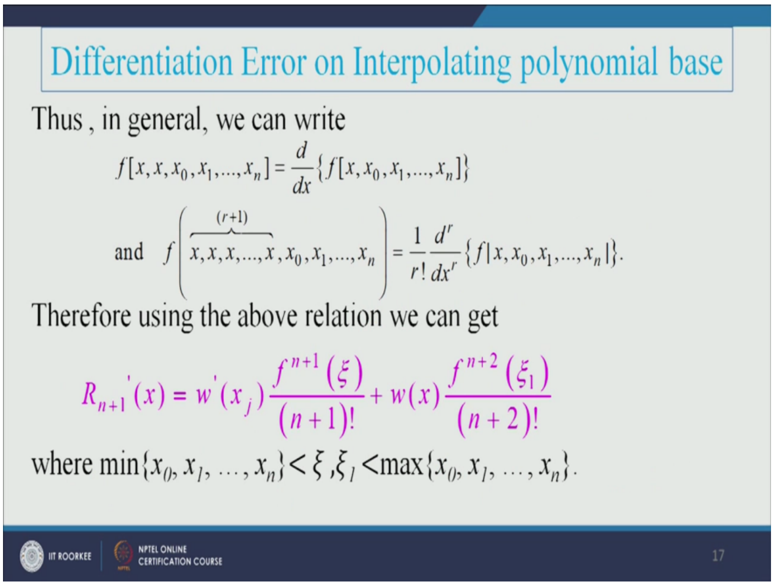
(Refer Slide Time: 30:01)



We can just write this one as limit x tends to 0 f of x 0 + h x 0 – f of x 0 by h there. So if you just write in that form then we can just apply this L hospital rule twice then we can just obtain the second differentiation of f at the point x 0 there. This means that if you will just see here that h square is just occurring if you just take f of x 0 + h – f of x 0 by h, so 1 by h product it will be there so that is why 0 by 0 form it is just occurring so that is why we have to take this L hospital's rule to obtain this limit at that position and this gives you like f double dash x 0 by 2 factorial. So if suppose r times the same argument is placed then we can just write that one as f to the power r of x 0 by r factorial.

Similarly we can write if suppose 2 arguments are equal and all arguments are defer so then we can just transfer that one as f of x r, x r, x 0 to x n limit h r tends to 0 if h r can be placed at that point then that will just represent the same expression as we have just seen from the previous one. And if we just write in a limiting form there, that can be transformed directly to the derivative form as dy by dx of f of x , x r, x 0 to x n there at x = x r.

(Refer Slide Time: 31:05)



And which can be written as derivative form with r arguments as 1 by r factorial d to the power r by dx to the power r f of x, x 0 to x n there. And if you just use this expression in the complete derivative term of this remainder term then we can just get r n + dash x as Omega dash of x j up to the power n + 1 zeta by n + 1 factorial + the arguments involved for this derivative of up to the power n + 1 zeta by n + 1 terms there, so that can be represented as Omega x up to the power n + 2 zeta 1 by n + 2 factorial. So next lecture we will just continue that how we can just obtain this differentiation for (())(31:51) differential approximation, thank you for listening the lecture.