Numerical Methods
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Lecture 26
Numerical differentiation part-II (Differentiation based on Lagrange's interpolation)

Welcome to the lecture series on numerical methods, so we now discussed Interpolation then based on Interpolation in the last lecture we have discussed about this numerical differentiation based on Newton's forward difference and Newton's backward difference operations. So today we will just discuss this numerical differentiation based on Lagrange interpolation, divided different interpolation and then from that differentiation how we can determine this maxima and minima of a function based on this polynomial approximation so that we will discuss.

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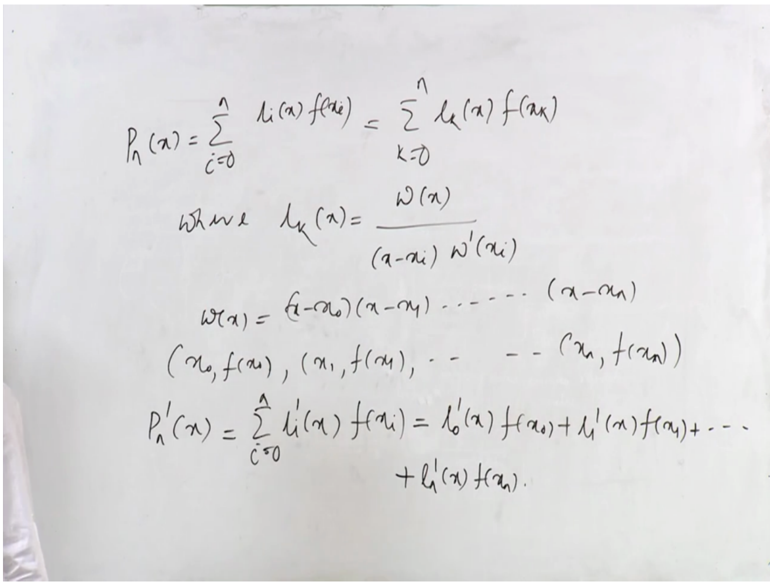

So basically we are just discussing here that whenever you will have tabular values like x i, f i at n + 1 distinct points starting from 0, 1 up to n. Then usually this Lagrange polynomial P n x is written in the form like P n x this = submission i = 0 to n L i x f of x i.

Or sometimes we also for our convenience we are just writing k = 0 to n L k x f of x k also, where L k x is called the Lagrange coefficient and if it is interpolating with a polynomial then it is called P n x here. This means we are just formulating or constructing a polynomial P n x based on this n + 1 model point considering these functional values based on this function f of x k here. So if you just write here this Lagrange coefficient then usually we are just writing this Lagrange coefficient L k x is Omega x by x – x i W dash of x i, so where usually this W of x is defined in the form like x – x 0, x – x 1 up to x – x n here.

Since we have these tabular points expressed in the form of like x 0, f of x 0, x 1, f of x 1 up to x n f of x n, specially we are just evaluating these derivatives at the position that if the function is not known us explicitly. This means that if the tabular values has been given like x 0, y 0, x 1, y 1 up to x n, y n where exactly this function is not known to us then we can just also approximate this polynomial in the form of P n x from where or from that we can just get this differentiation of this function f of x at different points, maybe at the same tabular point or sometimes maybe at different points also which is existing within any of the intervals of this tabular point.

Basically if you will just want to evaluate this derivative for this function P n dash or P n of x here, the derivative of P n x usually it is written in the form of P n dash x here and this can be written as submission i = 0 to n L i dash of x f of x i here. Since if you will just see here, this Omega x consists of these variables in the form of x here where we can just differentiate this function specially if you just see this derivative of P n x is nothing but the derivative so whatever it is just operating on L i of x here, multiplied with these functional values or the tabular values which has been given and all of these nodal points are tabular points.

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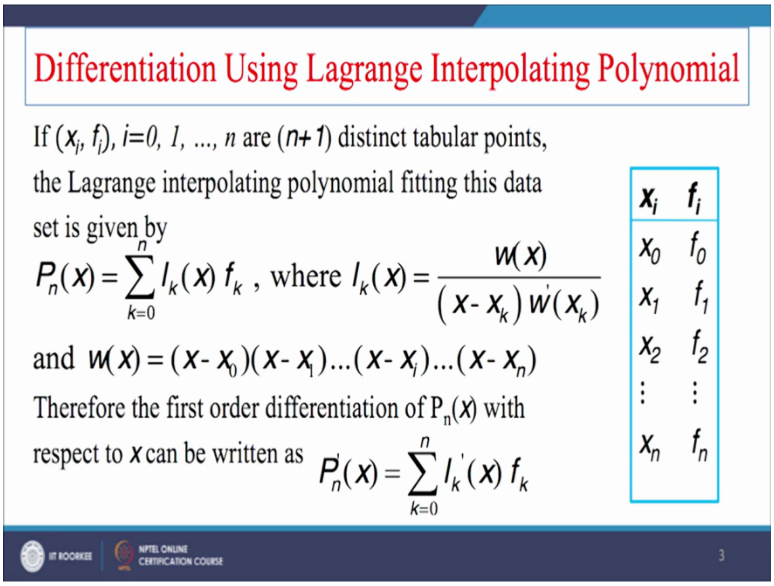

If we can just write this one in a complete form we can just write this one as L 0 dash of x f of x 0, L 1 dash of x f of x 1 since how these known values of x 0, x 1 and x 2 up to x n and where also these functional values are known to us which all are constants here. Specially it can be written of the function in the final point is here nth point so L n dash x f of x n this one, so specifically if you just see here the slides so we can just say that x i are the values which is expressed at all of these nodes points that is in the form of x 0, x 1 to x n and the functional values that expressed in the form of f of x 0, f of x 1 to f of x n here. So suppose first we will just consider Lagrange polynomial which is used 3-point suppose, since at a time if you will consider n points it is difficult to understand, so first we will just consider 3-point suppose.

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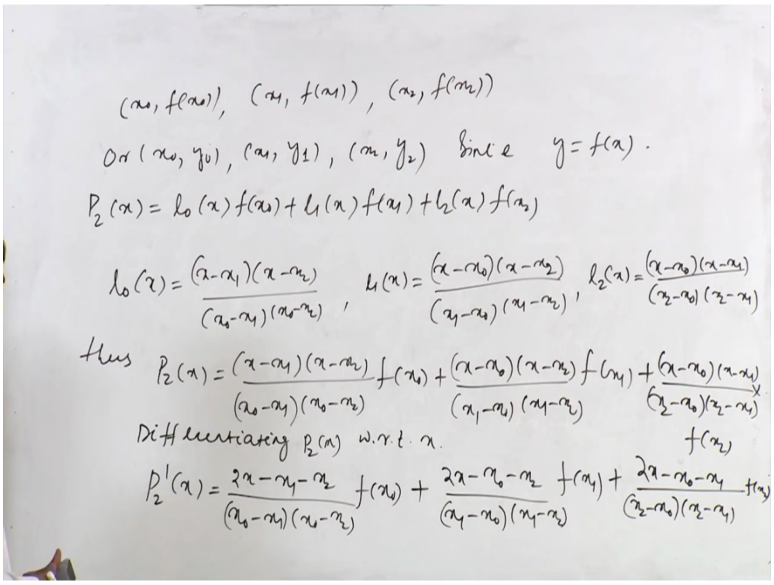

So if we assume that the function goes through suppose 3 points and the points are defined in the form like x 0 f of x 0, x 1 f of x 1, x 2 f of x 2, either you can just write in the form of f of x 0, f of x 1, f of x 2 or you can just write that one as x 0 y 0, x 1 y 1, x 2 y 2 sorry this one x 2 y 2 here, since we are just expressing here y = f of x. And if we will just consider this polynomial here so this can just generate a polynomial of degree 2 that is in the form of P 2 x here which can be written in the form of L 0 x, f of x 0 L 1 x f of x 1 + L 2 x f of x 2 here. And if we will just slide independently all the terms here, this means that L 0 x can be written in the form of like x – x 1, x – x 2 divided by x 0 – x 1, x 0 – x 2 here.

Similarly, L 1 x this can be written in the form of like x – x 0, x – x 1 sorry x – x 2 divided by x 1 – x 0, x 1 – x 2 here. Similarly L 2 x can also be written in the form of like x – x 0, x – x 1 divided by x 2 – x 0, x 2 – x 1 here.

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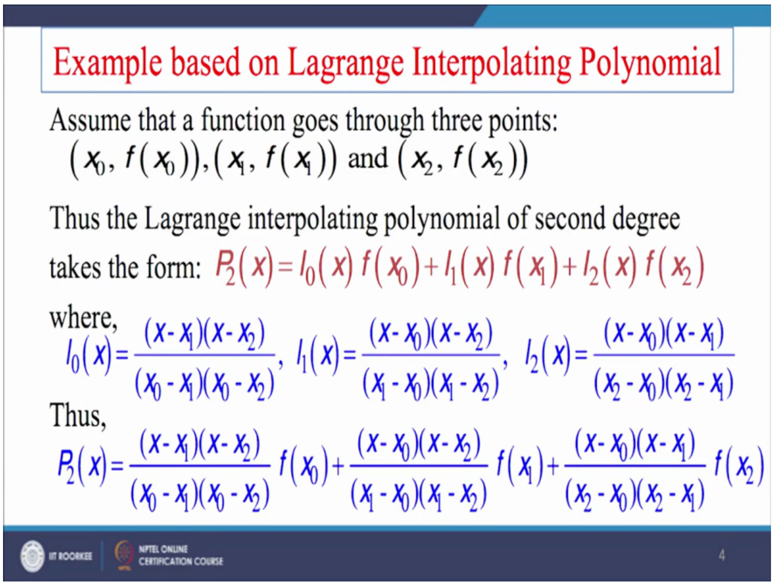

So if we will just express these 3 Lagrange polynomial coefficients L 0 x, L 1 x and L 2 x in this form here then we can just write P 2 x as thus P 2 x can be written in the form of like x – x 1, x – x 2 by x 0 – x 1, x 0 – x 2 into f of x 0 + x – x 0, x – x 2 divided by x 1 – x 0, x 1 – x 2 into f of x 1 + x – x 0 into x – x 1 divided by x 2 – x 0, x 2 – x 1 into f of x 2 here. So if you will just write this polynomial in this form then usually we can just evaluate the derivative for this function P 2 x here or the polynomial P 2 x here. So if you just differentiate this polynomial P 2 x with respect to x here, differentiating P 2 x with respect to x here, if you just differentiate this one then we can just write this differentiation as P 2 dash x here.

So P 2 dash x if you just see here, first product if you just take so x square so this means that we can just write 2 x – if you just say x is a product of here that can just gave you – x 1 here, if you just put the product of this one also here that will just give you – x 2 here and the last derivative if you just consider, this is nothing but a constant here + x 1 and x 2 and its derivatives will be 0 here, then divided by all of these points if you just see all are here constant so that is why we can just write x 0 – x 1, x 0 – x 2 f of x 0 since all are here constant only.

So next derivative point if you just consider here, the next derivative term associated with this P 2 x that can be written in the form like, 2 x – x 0 – x 2 divided by x 1 – x 0, x 1 – x 2 f of x 1 + the last term if you just differentiate it can be written in the form of 2 x – x 0 – x 1 divided by x 2 – x 0, x 2 – x 1 into f of x 2. So now if you just differentiate once more here P

2 dash x then we can just obtain P 2 double dash x also here so next if you just go for the second differentiation of this function here.

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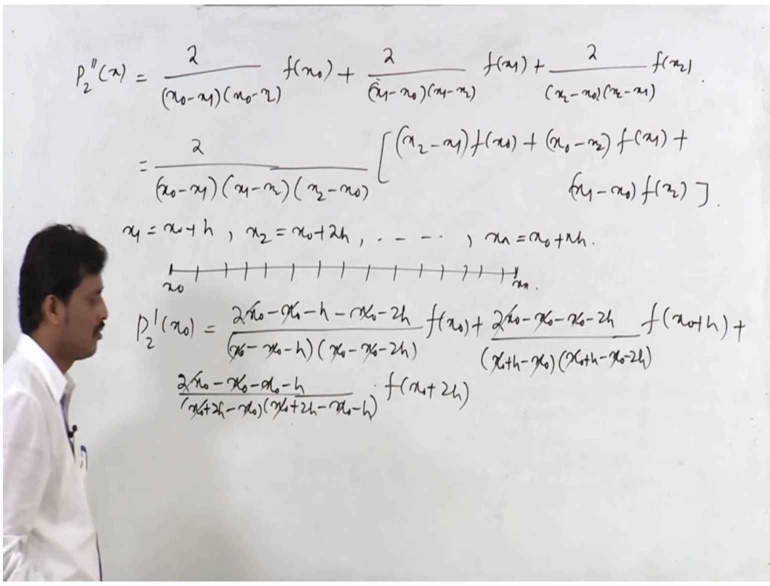

If you will just differentiate once more here, then we can just write P 2 double dash of x this and be written as, first one is the variable term only this is involved here. So 2 by x 0 – x 1, x 0 – x 2 into f of x 0 + 2 by x 1 – x 0, x 1 – x 2 into f of x 1 + the last term if you just take one more differentiation, it can just represent the term here x 2 – x 0, x 2 – x 1 into f of x 2 here. And combine if you will just write this term this can be written in the form of like 2 is common for all the terms here, 2 divided by x 0 – x 1, x 1 – x 2, x 2 – x 0 here.

And if you will just write the other part of the term here this can be written in the form of like x 2 – x 1 into f of x 0 + next term if you just say see here that is not common to this term here that is in the form of like if you will just see that is nothing but x 0 – x 2 f of x 1 + last term that is in the form of like if you will just see that is in the form of x 1 – x 0 here so x 1 – x 0 f of x 2. So this is the complete representation of this second order differentiation of P 2 x, this means that if you will just involve only 3 points then we can just represent this polynomial in this form.

So likewise we can just express the nth differentiation or if you will just consider polynomial of degree n also here then the first order differentiation, second order differentiation, third order differentiation we can just get it in the same manner here. So based on this we can just also say that if all these points suppose equally spaced, sometimes we say that the Lagrange interpolating polynomial this can be applicable both for equi-spaced points and unequi-spaced points. In this case we now just consider that all points may be unequally spaced, if we will just consider that all the points are equally spaced suppose.

Equally spaced means we can just consider x 1 = x 0 + h, x 2 = x 0 + 2 h, so likewise we can say that x n = x 0 + n h here. Obviously in the graphical representation we can say that if x 0 is the starting point here and x n is the endpoint here, if we will just divide this complete interval into n equal parts we can just say that it is equally spaced or all the points are equi spaced. So if we will just replace all of these points right this transformation then we can just obtain the first order differentiation P 2 dash of x 0 suppose. (())(15:06) we are just considering this differentiation at this point x 0 here, if you will just consider that one that can be written in the form of like if you will just see this one is the differentiation here.

So first point it can be written in the form of like 2 x 0 –, suppose this differentiation we want to determine at the point x 0 here so then x = x 0 here – x 0 – h – x 0 – 2 h divided by this is x 0 – x 0 – h, this is x 0 – x 0 – 2 h into f of x 0 + to x 0 – x 0 – x 0 – 2 h divided by x 0 + h – x 0, x 0 + h – x 0 – 2 h into f of x 0 + h + the last term if you will just write that is 2 x 0 – x 0 – x 0 – h divided by x 0 + 2 h – x 0, x 0 + 2 h – x 0 – h into f of x 0 + 2 h here.

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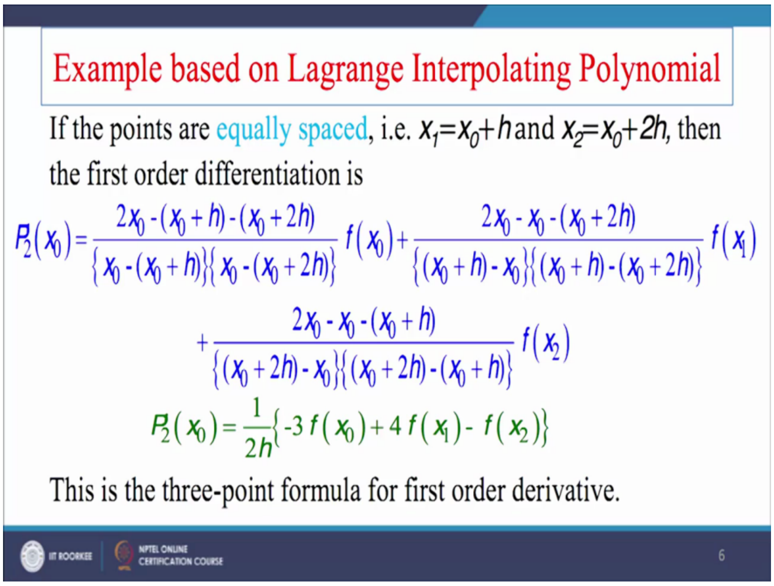

So if you just cancel all the terms here then we can just say that this x 0 x 0 we cancel it out, this x 0 we cancel it out, this x 0 be cancel it out here, this is 2 x 0 we cancel it out so this x 0 we cancel it out, this x 0 we cancel it out, this x 0 we cancel it out, this x 0 we cancel it out. So finally if you just right all the terms here, we can just write P 2 dash of x 0, first you just see here – h, – 2 h, here also – h, – 2 h so if you will just see here we can just take common 1 by 2 h from all these terms and it can be written in the form of like – 3 f of x 0 + 4 f of x 1 – f of x 2 here.

So this means that all of the terms if you just see that first term if you just see here that is in the form of like – h, – 2 h here so that is why this is – 3 h and lower one if you just see here – h so – 2 h, this one also – 3 h here then f of x 0 so next one if you just see here so the same thing we can just say that is 2 x 0 – x 0 – x 0, we will just cancel it out then 2 h is remained in the upper side here so then lower side if you just see here that is in the form of x 0 – x 0 you just cancel it out then h remains here then h – 2 h here so that will be – h will remain so – h into h that will just generate here h square.

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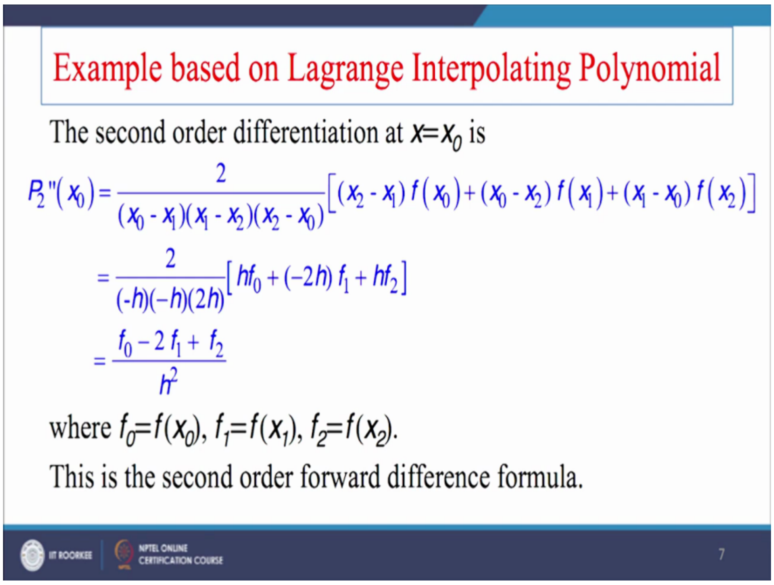

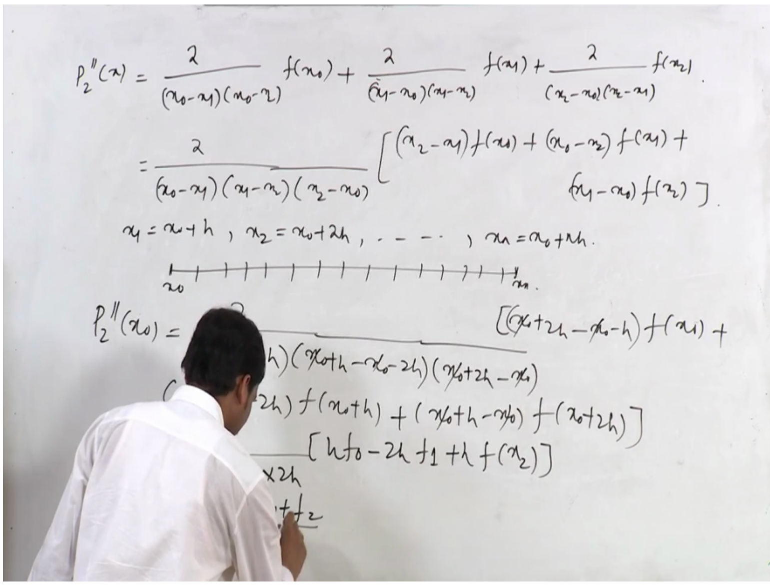

So likewise if you just solve these 3 terms, we can just obviously get this 3-point formula here that is this is called the first 3-point formula for first order derivative here. Next we will go for this second order differentiation here, so if you will just differentiate this P 2 double dash at x 0 then we can just obtain here P 2 double dash of x 0 as directly the same formulation we can just use here also that is 2 by x 0 – x 0 – h, x 0 + h – x 0 – 2 h, x 0 + 2 h – x 0 into x 0 + 2 h – x 0 – h into f of x 0 + x 0 – x 0 – 2 h into f of x 0 + x 0 + x 0 + h – x 0 into f of x 0 + 2 h. So if you just see here, this x 0 x 0 cancel it out x 0 also cancel it out here, this is also cancel it out, this is also cancelled out.

If you will just write in a complete form here then it can be written as 2 y – h, this one also here – h, last one if you just see here this is nothing but 2 h here. And in fact these functions if you just see, first one just gives h f 0, x 0 x 0 cancel it out so 2 h – h so that is why it is just giving you h here so h into f of x 0 that is nothing but h f 0 here. Second part if you just see here that is nothing but – 2 h here that we can just write – 2 h f 1 and last point if you just see here, it will be just giving you h f of x 2 and finally we can just write this as since –– it is just giving you + here this one, so f 0 since h can be taken out common from all the terms here so f 0 – 2 f 1 + f 2 divided by h square.

Since we are just writing here f 0 is nothing but f of x 0 and f 1 is nothing but f of x 1 and f 2 is nothing but f of x 2, this is called second order forward difference formulas, especially if you just see the same formula you can get it also in Taylor series expansion also, this is called second order forward difference formula. So based on this differentiation we can just go for the solution of some problems, first we will discuss these problems based on this Lagrange Interpolation where it can be exactly applied to the nodal points. We can say that either at x 0 or at x 1 directly if we just put that point whatever we have just discussed here, the same way we can just differentiate and we can obtain the solution.

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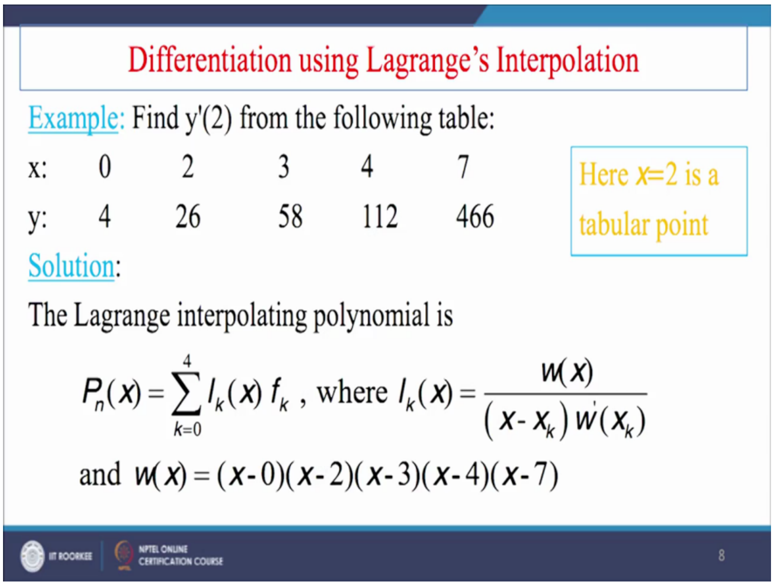

So first if you just consider this example that is in the form like x is prescribed at 0, 2, 3, 4, 7 suppose since we have considered as unequal spaced points and if the functional values are expressed as 4, 26, 58, 112, 466 suppose and the values asked to compute at x = 2. Since if you just see this tabular value here, x = 2 is a particular tabular value on this problem here and it has asked you to evaluate these values at y dash of 2 so first if you just discuss here.

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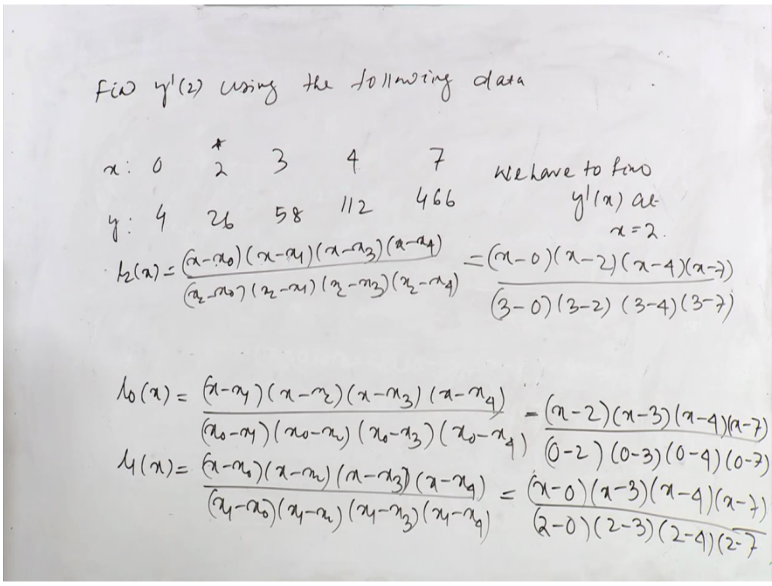

Find y dash of 2 using the following data, suppose the question is asked and the data points are given in the form like x at 0, 2, 3, 4, 7 suppose and these functional values are expressed in the form 4, 26, 58, 112 and 466, this is suppose given here. And obviously we can say that we have to find y dash of x at x = 2 here, obviously 2 is nothing but the tabular point here.

And if we will just use this Lagrange Interpolation then the formula can be written in the form of P n x = submission K = 0 to 4 since if you just see here, 5 nodal points are here so it can just generate the polynomial of degree 4 here so that is why we can just write L k x f of x k here.

And directly we can just write L K x as Omega x divided by x – x K, Omega dash of x K here. And Omega of x can be expressed in the form of x – 0, x – 2, x – 3, x – 4, x – 7 here since all the points should be included in the numerator product part here. So if we want to find this L 0 x here, so L 0 x can be written in the form like x – x 1, – x 2, x – x 3, x – x 4 divided by x 0 – x 1, x 0 – x 2, x 0 – x 3, x 0 – x 4 here. And directly if you just write these functional values in the upper part here, this can be written as x – 2, x – 3, x – 4, x – 7 divided by this all of these terms like 0 – 2, 0 – 3, 0 – 4, 0 – 7 here.

And similarly if you just write L 1 x, this can be represented in the form of like x – x 0, x – x 2, x – x 3, x – x 4 divided by x 1 – x 0, x 1 – x 2, x 1 – x 3, x 1 – x 4 and this can be written in the form of like x – 0, x – 3, x – 4, x – 7 divided by 2 – 0, 2 – 3, 2 – 4, 2 – 7 here. Similarly we can just write L 2 x as x – x 0, x – x 1, x – x 3, x – x 4 divided by x 2 – x 0, x 2 is x 1, x 2 – x 3, x 2 – x 4 and which can be represented in the data points as x – 0, x – 2, x – 4, x – 7 divided by point is here 3 – 0, 3 – 2, 3 – 4, 3 – 7 here.

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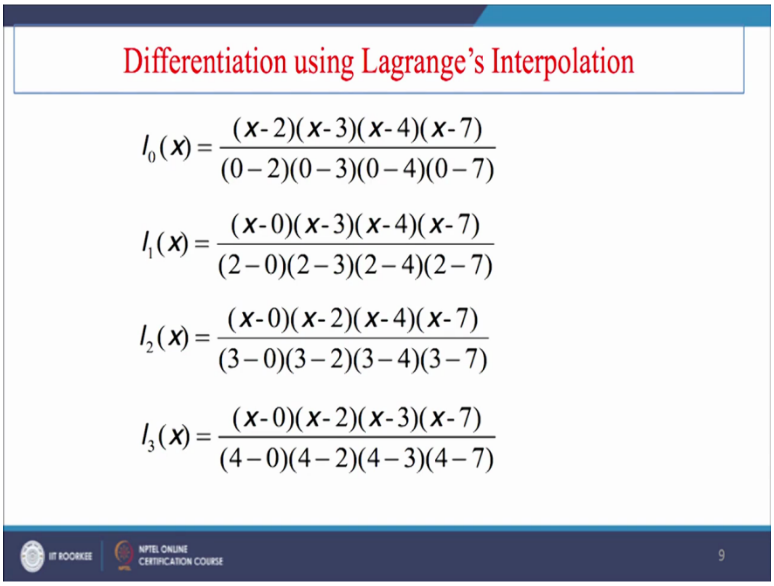

Similarly we can define L 3 x as x – 0, x 1 is 2, x – 3 into x – 7 divided by 3 – sorry this will be 4 – 0, 4 – 2, 4 – 3 into 4 – 7 here. And L 4 x can be defined in the form like x – 0, x – 2, x – 3, x – 4, divided by 7 – 0, 7 – 2, 7 – 3 into 7 – 4 so that is the representation for L 4 x here.

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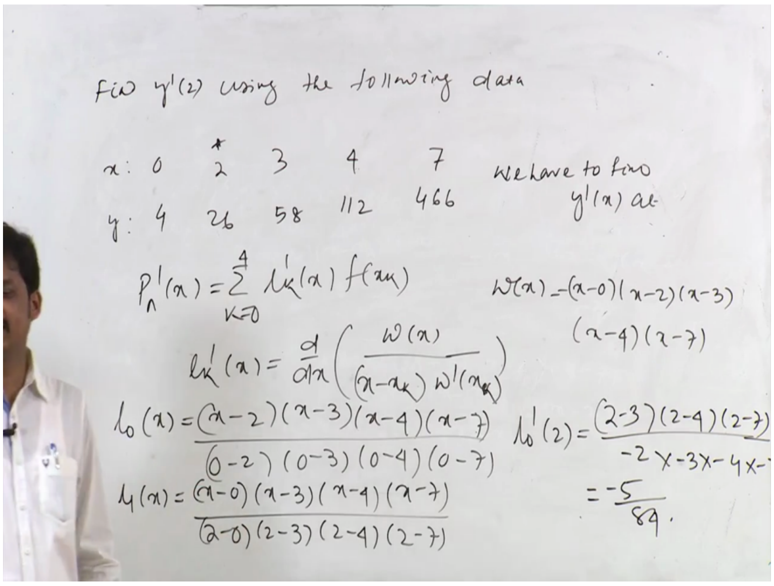

So if you just consider the first order derivative based on Lagrange Interpolation then we can just express this P n dash x as so we can just express P n dash x as submission K = 0 to 4 L K dash of x f of x K here. And if you will just define here L K x dash x as submission or directly if you just write here d by d x of Omega of x divided by x – x, Omega dash of x K then Omega x is represented as here if you just see that is the function of x only here that is in the form of like x – 0, x – 2, x – 3, x – 4 into x – 7 here, and directly we can just evaluate this derivative with respect to x for this function here and obviously if we want to write suppose at a point 2 suppose here.

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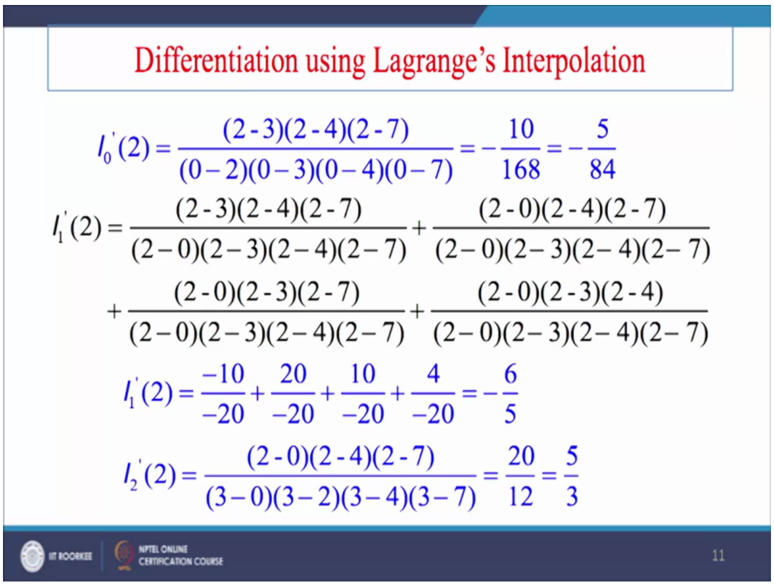

If you will just see the first function L 0 2 here so directly we can just say that since at a particular point we are just evaluating, if you just see here L 0 x which is represented in the form of like x – 2, x – 3, x – 4, x – 7 divided by your 0 – 2, 0 – 3, 0 – 4, 0 – 7 this one. So if you just take this differentiation with respect to x here that is just giving you at a point particularly 2 if you will just see here, L 0 dash of 2 this one. So then except first product if you just consider this one, this will just give you a nonzero value and everywhere else wherever this x – 2 is present for remaining all of these derivative terms that will just give a 0 term there.

So that is why we can just write this complete product form here that one as 2 – 3, 2 – 4, 2 – 7, divided by – 2 into – 3 into – 4 into – 7 here. Since if you will just see this differentiation with respect to x – 3 if you just consider, then x – 2 will be present specially x – 2 that will just give you 0 here. If you will just differentiate x – 4 with respect to x – 4 the next – 2 is also present in another term so that will just give you 0 value there. So likewise whatever these product terms if you just consider except x – 2 then all other terms will be 0 there so that is why this can be represented here in this form only and barely if you just calculate that is just giving you – 5 by 84.

Similarly if you just calculate here L 1 dash of 2 suppose, next immediate calculation is L 1 x suppose if you just here L 1 x can be written in the form of like x – 0, x – 3, x – 4, x – 7 divided by 2 – 0, 2 – 3, 2 – 4, 2 – 7 here. And if you just take the derivative with respect to x and put these functional values 2 there so all these terms will exist here if you just see, since in first fall differentiation if you take this one so we can just write this term as x – 3 into x – 4 into x – 7. Second if you just do this one then + x into x – 4 into x – 7 then next term if you just take the differentiation with respect to x – 4 here then we can just write x – 0, x – 3 into x – 7.

And last if you just differentiate so all the terms at point 2 it will just exist and in that case we can just like this one as L 1 dash 2 = – 6 by 5 here if you just calculate. Then we will just go for like L 2 dash of 2, L 2 dash of 2 if you just see again the same scenario it can happen, this means that if you will just calculate L 2 of x again, this x – 2 term will be there so that is why except one term all other terms it will just vanish after this differentiation and we can just write this differentiation as L 0 to dash of 2 there and which can be written as like 20 by 12 here so the value is coming as like L 2 dash of 2 that is just giving you like – 5 by 3 here sorry this is 5 by 3 only and L1 dash of 2 this is just giving you – 6 by 5.

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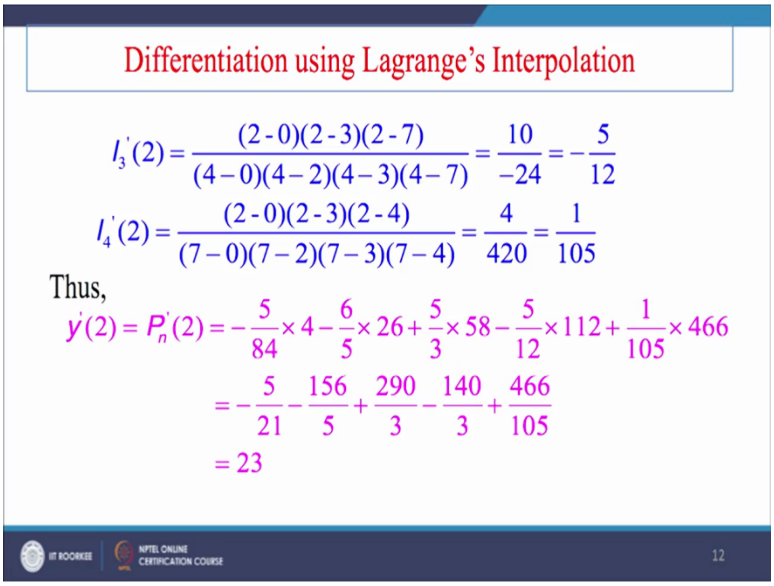

And if in the same manner if you just compute here L 3 dash of 2 that is the same way we have to treat here also that will just give you – 5 by 12. And if you just compute L 4 dash 2 that will just give you 1 by 105.

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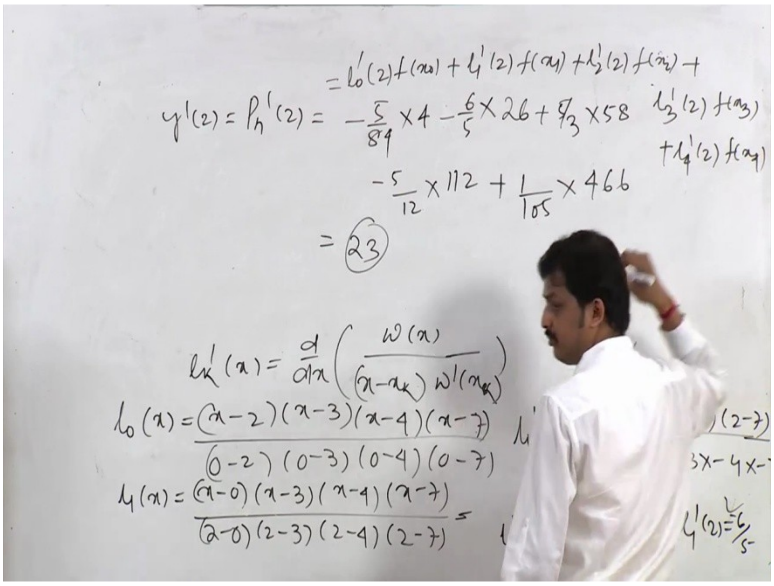

Once all of these values are known to us than we can just write these functional derivatives, this means that we can just write y dash of 2 as L 0 dash of 2 into f of x 0 + L 1 dash of 2 f of x 2 + L 1 dash of x 3 f of x 3 + L 3 dash of 2 f of x 3 and L 4 dash of 2 into f of x 4 there. So if you just write all these terms here, the derivative of like this term y dash of 2. It can be written in the form like P dash of 2 as – 5 by 84 that is the first term into f of x 0 that is nothing but 4 here + 6 by 5 into sorry this is – 6 by 5 here the value into 26, so next term is like 5 by 3 into 58 and immediate next term is like – 5 by 12 into 112 + 1 by 105 into 466 here and the computed values that is just giving you here 23.

If you just see here, this means that we are just writing these values that is in the form like L 0 of 2 or L 0 dash of 2 into f of x 0 + L 1 dash of 2 f of x 1, L 2 dash of 2 f of x 2 + L 3 dash of 2 f of x 3 + L 4 dash of 2 f of x 4 here. Or completely free just write all these terms then we can just evaluate this derivative or first order derivative for this function f of x by constructing a polynomial P n of x here. So obviously based on that we can just evaluate this derivative for all of these unequi-space points or equi-spaced points that is based on this Lagrange Interpolation formula. So next lecture we will just consider this unequi-spaced points or some other examples based on this Lagrange interpolating polynomial, thank you for listening this lecture.