Numerical Methods
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Lecture 22
Interpolation Part VII (Lagrange Interpolation Formula with Examples)

Welcome to lecture series on numerical methods and currently we are discussing interpolation. In the interpolation section we have covered this finite difference approximations like Newton's forward difference approximations, backward difference approximations and all of these finite difference operators. Today we will discuss about Lagrange interpolation method and Newton's divided difference interpolation. So first we will start about Newton's sorry Lagrange interpolation formula.

In the Lagrange interpolation formula is applicable for both these uniformly and non-uniformly spaced grid sizes. Especially in interpolation formula we will have this tabular values like x 0 y 0, x 1 y 1, x 2 y 2, x 3 y 3, so likewise. So earlier whatever this formula we have discussed or derived basically all these lectures they are discussed for this finite difference operators which are used for equispaced points.

Sometimes if you just see these points which are placed here like if you will have the tabular points like x i, y i where i is varying from 0 to n here then these points will be placed like x 0 y 0, x 1 y 1, x 2 y 2 up to x n y n here.

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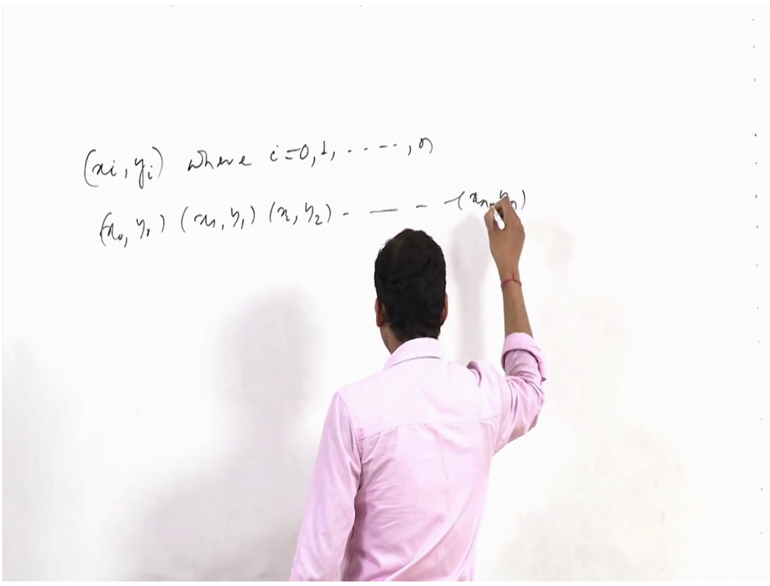

So if you will just consider these tabular points like x 0 y 0 will be placed like this point here. X 1 y 1 will be placed here and suppose the next tabular point x 2 y 2 is placed here. Next tabular point will be placed as x 3 y 3 here. So likewise suppose the points will be placed. But if you just see the distance between these two points which are unequally. Since this is not equals to this one or this is not equals to this one here. This means that we can just say that x 1 minus x 0 is not equal to x 2 minus x 1 here or this is not equals to x 3 minus x 2 here.

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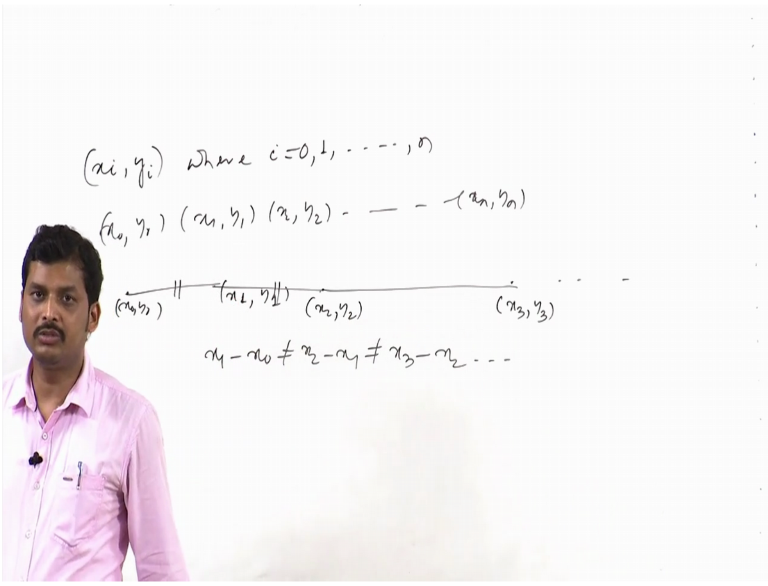

So in this case we cannot use like Newton's forward formula or Newton’s backward difference formula or any of these forward difference approximations. So for that we need a separate formula which can bill these unequally spaced points here. So basically these unequally spaced points here we will just consider three different interpolation methods to approximate this functions with a polynomial.

That is first one is Lagrange interpolation method, second one is a Newton’s divided difference formula and third one is Hermite’s interpolation method. And in this method if you will just see this method simply suggest to represent the given data like x i y i, I equals to 0 1 2 up to n which can approximated in the form of y i equals to p of x i here.

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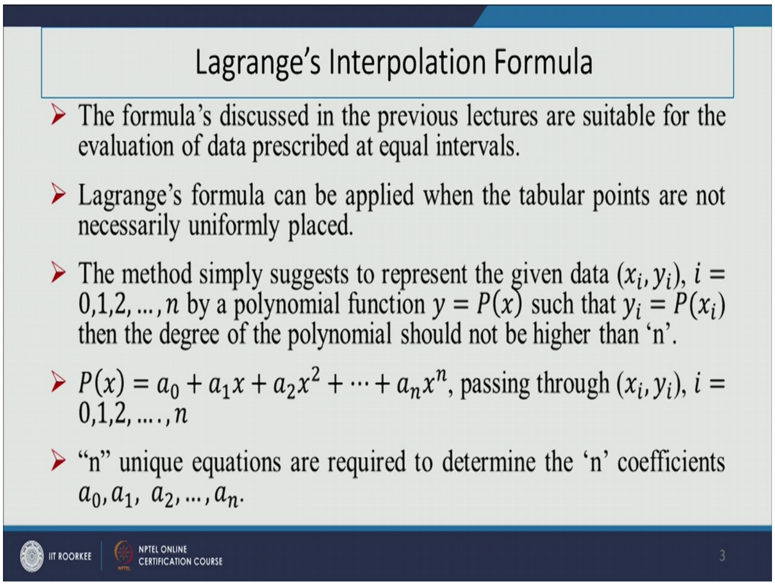

Here p of x can be written in the form of a 0, a 1 x, a 2 x square up to a n x to the power n where these coefficients a 0, a 1, a 2 up to a n are to be determined. And to determine these n plus 1 coefficients here since it is a polynomial of degree n so we need like n plus 1 equations to evaluate these coefficients there. Since all of these constants here so one value like a 0 can be taken to the right hand side.

All of these variables associated here a 1 to a n can be evaluated by considering n equations here. So for that suppose we have to consider a polynomial of degree one for the first case. If you will just consider the polynomial of degree n equals to 1 here, we can just write here the points like x 0 and x 1 with a linear polynomial approximation or a polynomial of degree 1 here or polynomial of degree 1 here. That is p of x equals to f of x here which can be written in the form like a 1 x plus a 0.

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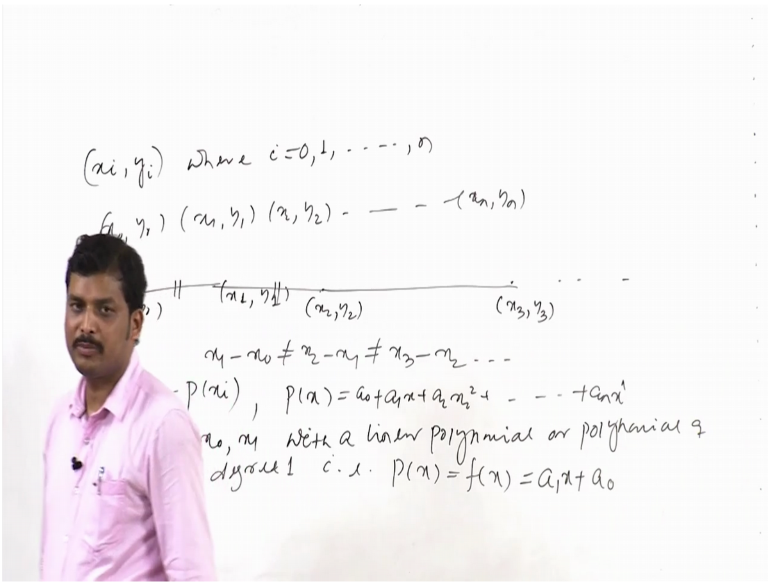

And if we will just approximate this function with a polynomial to determine these coefficients a 0 and a 1 from this equation we have to consider three equations that is in the form of like p of x. Then if you will just put that x equals to x 0 where p of x is exactly equals to f of x or x equals to x 1 suppose, p of x is exactly equals to f of x here then we can just eliminate these coefficients a 0 and a 1 in a polynomial form here.

So if you will just do that things we can just write p of x equals to a 1 x plus a 0 where a 0 and a 1 are the arbitrary constants which satisfies the interpolating conditions that as f of x 0 equals to p of x 0 as a 1 x 0 plus a 0 here. Similarly we can just write f of x 1 that as p of x 1 this equals to a 1 x 1 plus a 0. And our original form of the equation that is especially written in the form of p of x equals to a 1 x plus a 0 here.

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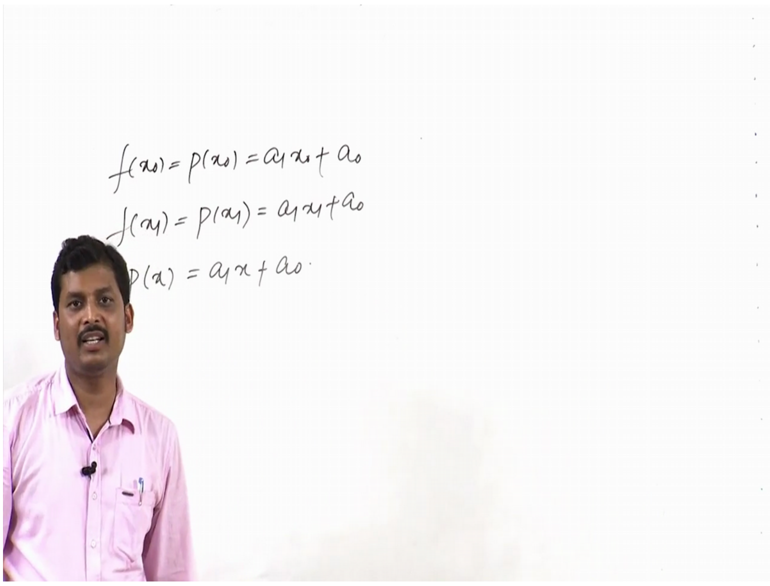

So if we want to eliminate a 0 and a 1 from these three equations we can just write these equations in the form like p of x, x 1, then f of x 0, x 0 1, f of x 1, x 1 1, this equals to 0. If you will just expand this determinant here we can just obtain that p of x into x 0 minus x 1 minus f of x 0 into, if you will just see that is x minus x 1 plus f of x 1 into x minus x 0 here. This equals to 0.

If we want to find p of x or the polynomial to be determined here so then it can be written as p of x equals to x minus x 1 divided by x 0 minus x 1, f of x 0 minus x minus x 0 divided by x 0 minus x 1, f of x 1 which can be written as x minus x 1 by x 0 minus x 1, f of x 0 plus x minus x 0 by x 1 minus x 0, f of x 1 here.

Since I just minus sign is there so I have just taken common minus from this denominator side. So that can be written in the form of x minus x 0 by x 1 minus x 0 into f of x 1. Where, I can just write this one as L 0 of x, f of x 0 plus L 1 of x, f of x 1 here.

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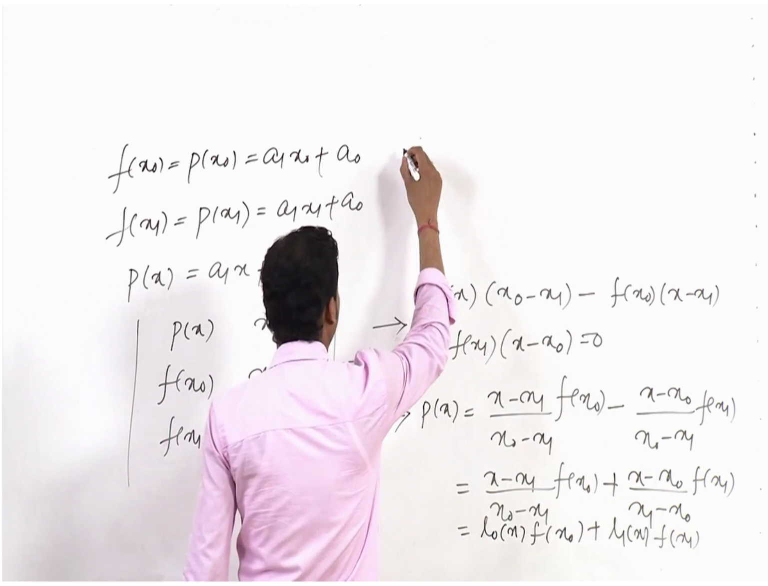

Where, I can just write L 0 x equals to x minus x 1 divided by x 0 minus x 1. And L 1 x, that can be written as x minus x 0 by x 1 minus x 0 here.

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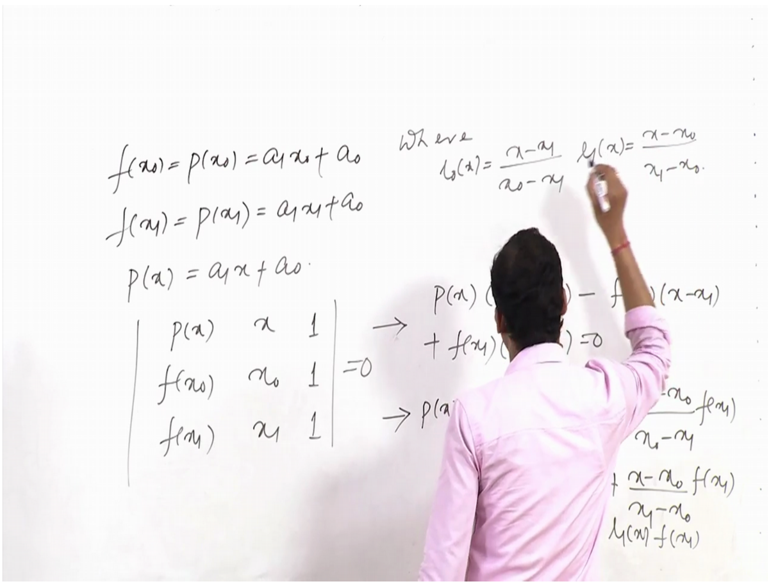

So if we will just see here that these are called L 0, x and L 1 x are called Lagrange's fundamental polynomials which satisfies. If you will just add both these terms here like L 0 x plus L 1 x here which can be written as x minus x 1 by x 0 minus x 1 plus x minus x 0 by x 1 minus x 0 here. So this total value can give you the values as 1 here.

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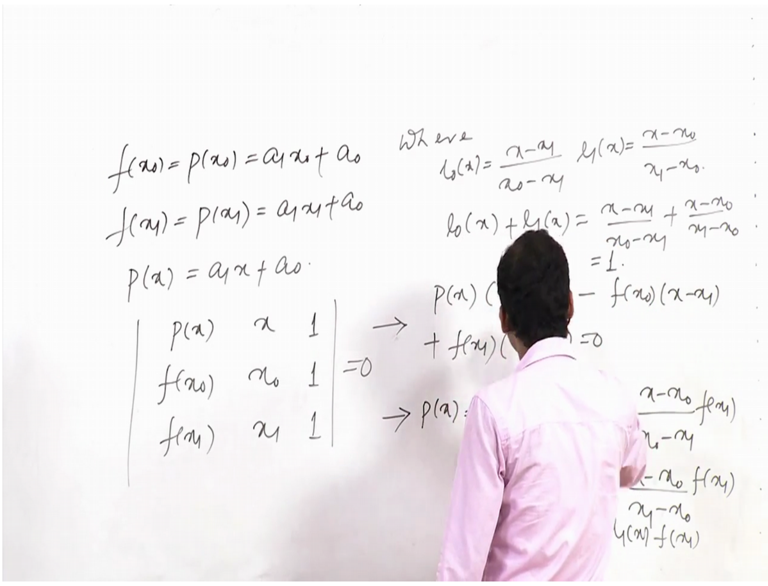

Since if we can just put here that is L 0 of x 0 that is nothing but 1 here since x 0 can be replaced here so x 0 minus x 1 by x 0 minus x 1 that we will just give you 1 here. If I will just replace x by x 1 here this means that L 1 of x 1 that will just give you x 1 minus x 0 by x1 minus x 0 as the value as 1 here. This means that L i of x j this equals to 1 whenever i equals to j and this equals to 0 whenever i is not equals to j there.

Then we can just write L i of x j this equals to delta that is chronicle delta of i j, this equals to 1 if i equals to j and j 0 if i is not equals to j there.

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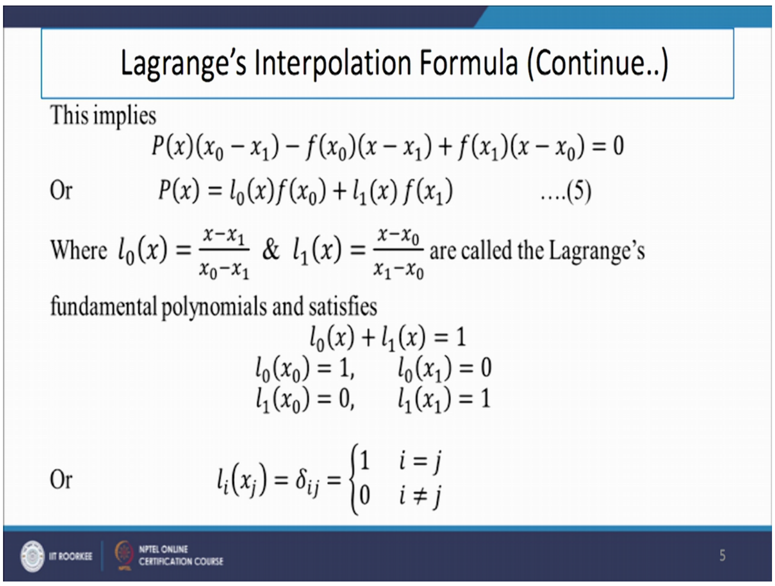

So if you just extend this linear polynomial to nth order polynomial here so we can just write this complete polynomial as p of x equals to L 0 x, f of x 0 plus L 1 x, f of x 1 plus up to L n x, f of x n where this coefficient L i x can be written as x minus x i sorry x minus x 0, x minus x 1 to x minus x i minus 1, x minus x i plus 1 up to x minus x n divided by x i minus x 0, x i minus x 1upto x i minus x i minus 1, x i minus x i plus 1 up to x i minus x n here.

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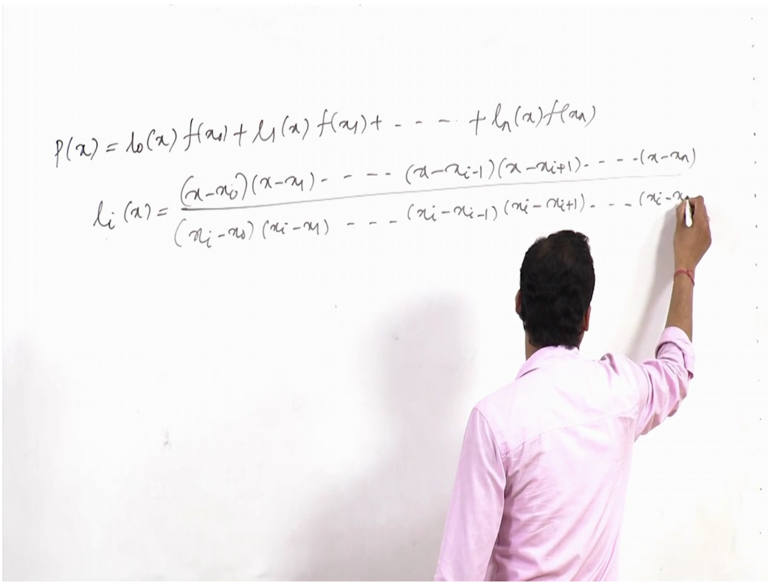

And it can be written in a combined form that as p of x this equals to summation of i equals to 0 to n L i x, f of x i here. And obviously this L i x will satisfy the property that is L i of x j this equals to 1 when i equals to j and 0 when i is not equals to j here. If I will just take the product of the term like x minus x i in the upper side here so I can just write that one as product of summation i equals to 0 to n x minus x i divided by x minus x i here into your products like x i minus x 0, x i minus x i minus 1, x i minus x i plus 1 up to x i minus x n here.

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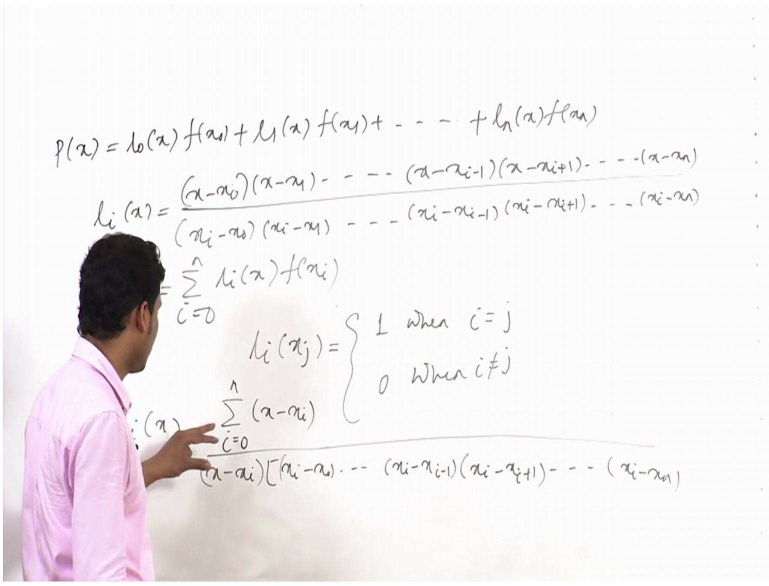

And also sometimes people are just using since this is a product form here I can just write this one as product of i equals to 0 to n here that is x minus x i divided by x minus x i into this term I can just replace as pie dash of i equals to 0 to n, x i minus or I can just write this one since these terms are written as x i minus x 0, x i minus x 1. So if I will just take the derivative of this term with respect to x i here then I can just write pie dash of x i here, that one.

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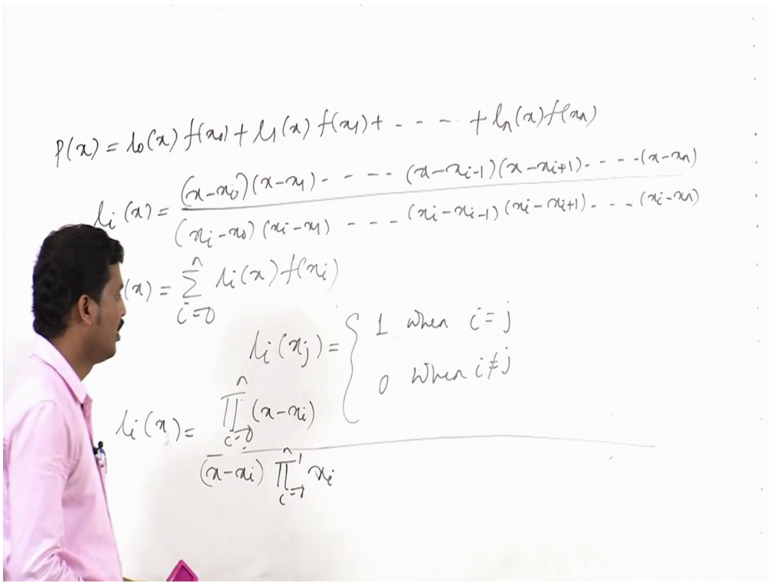

So we can just denote in our like convenient form that is in a derivative form if you will just take the expansion here and with respect to x i if I will just take the derivatives then except this x i term all other terms will be 0.

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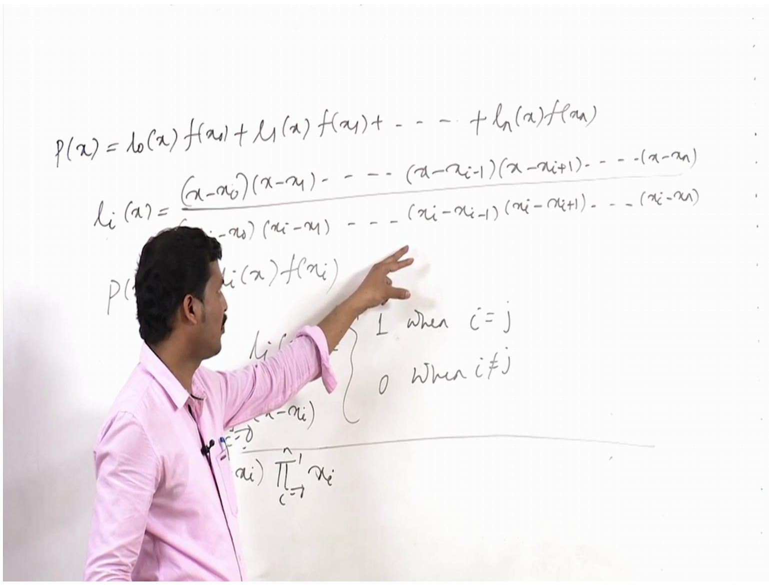

So in that way also I can just write this expansion that is in the form of like x i minus x j if I will just write or I can just write this one as x minus x i also. So it is very convenient to express this L i x term in different senses. Since all of these terms are occurring in the form of product comes here so that can be expressed.

To go for this polynomial what is just created in the form of like Lagrange interpolation so we will just discuss about an example that using suppose the data like x equals to 0, 1, 4, 5 and the corresponding data of y equals to 8, 11, 68, 123, how we can determine the value of y or the function f of x at the point 2? That we will discuss using Lagrange interpolation polynomial here.

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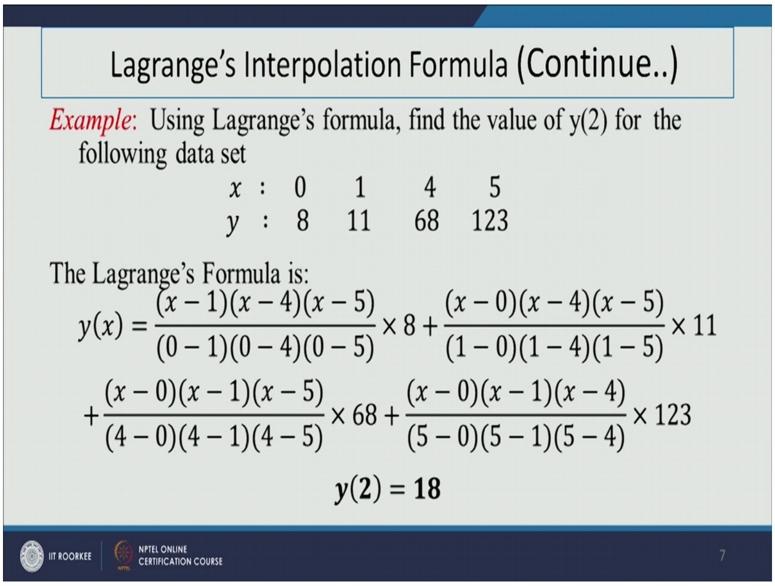

Since the data points whatever it is just given, it is given in the form of like x as 0, 1, 4, 5 here and corresponding y data that is given as 8, 11, 68, 123. And since we will have this points like 0, 1, 2, 3 so we can just go up to a polynomial that is of degree like 3 here. So that is why we can just write this one as or the Lagrange polynomial expression is like L i of x.

I can just write x minus x i. Sorry this is x minus x 0, x minus x 1 up to x minus x n divided by x i minus x 0, x i minus x 1 up to x i minus x n here. And the complete polynomial p of x can be written as L 0 x f of x 0, L 1 x f of x 1, L 2 x f of x 2, L 3 x f of x 3.

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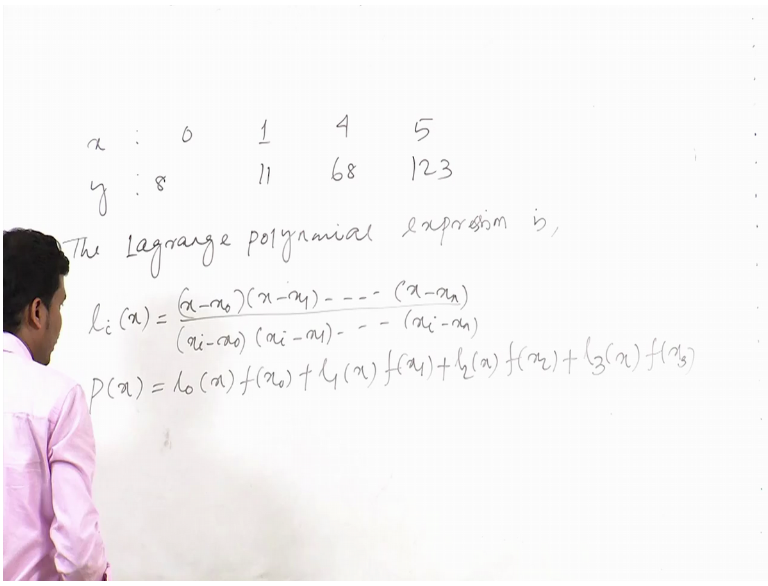

So if I will just go for the computation of L 0 x here, so L 0 x can be written as x minus x 1, x minus x 2, x minus x 3 divided by x 0 minus x 1, x 0 minus x 2, x 0 minus x 3 here. Since we have here 4 data points like the 4 data points can be signified as x 0 equals to 0 here, x 1 equals to 1 here, x 2 equals to 4 here and x 3 can be written as 5 here. So if you will just put all these points here that can be written in the form like x minus 1, x minus 4, x minus 5 divided by 0 minus 1, 0 minus 4, 0 minus 5 here.

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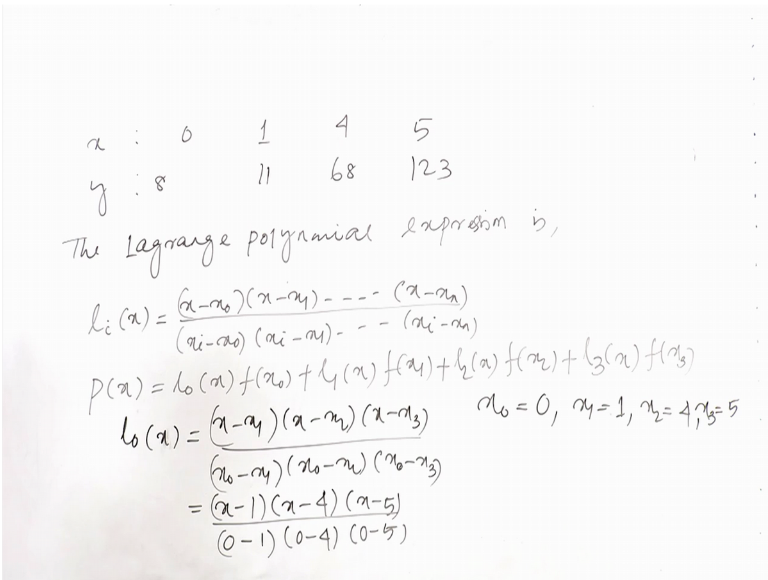

So if we want to evaluate L 0 x at the point 2, since the question is asked to evaluate the functional value at point 2 so directly we can just replace here L 0 at the point 2 and putting this 2 at the position of x in each of these expression here. So if you will just replace here like L 0 of 2 so we can just write this one as 2 minus 1, 2 minus 4, 2 minus 5 divided by 0 minus 1 that is minus 1, then minus 4, then minus 5 here. And similarly we can just write L 1 x also and L 2 x, then L 3 x we can just write.

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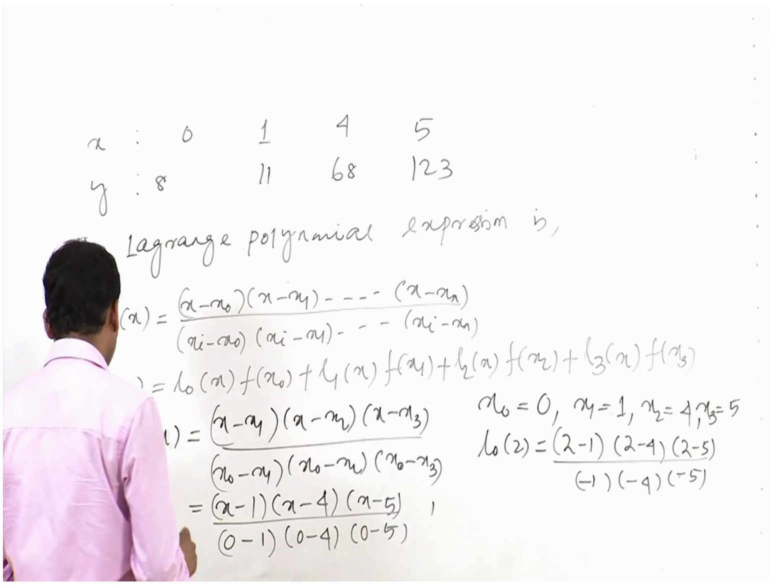

And if you just put all these values with the functional values here we can just obtain the total expression as your y of x or p of x can be expressed as x minus x 1, x minus x 2, x minus x 3 divided by your value that is 0 minus 1, 0 minus 4, 0 minus 5 into first value that is your functional value.

I can just write that one since it is just given as 8 here plus x minus 0, x minus 4, x minus 5 divided by 1 minus 0, 1 minus 4, 1 minus 5 into 11 plus x minus 0, x minus 1, x minus 5 divided by 4 minus 0, 4 minus 1, 4 minus 5 into 68 plus x minus 0, x minus 1, x minus 4 divided by 5 minus 0, 5 minus 1, 5 minus 4 into 123.

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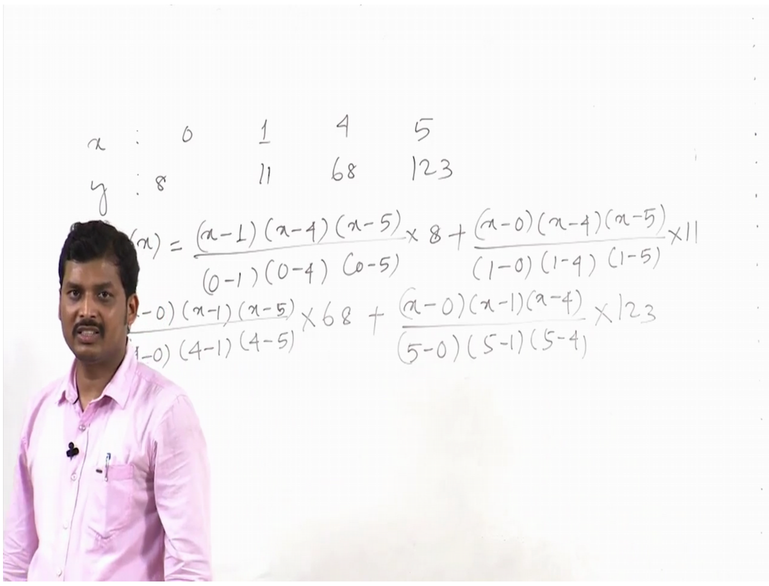

Since we want to evaluate this y of x or p of x value at exactly x equals to 2 here so we can just put here y of 2 and this total value that can become as or we can just obtain that value as 18 here.

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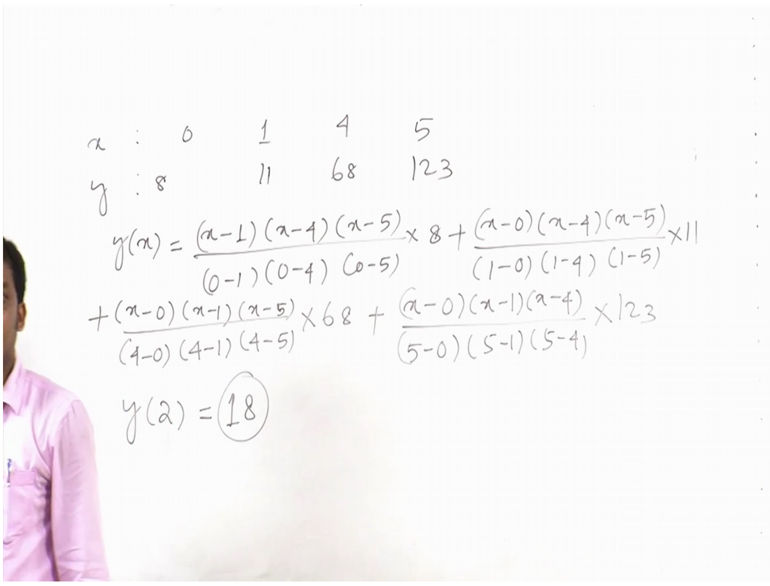

So since we are just using this Lagrange interpolation polynomial if you will just see the data points here, first point is 0, then second point is 1, then third point is 4 here. If you just see this difference or this difference both are unequal here.

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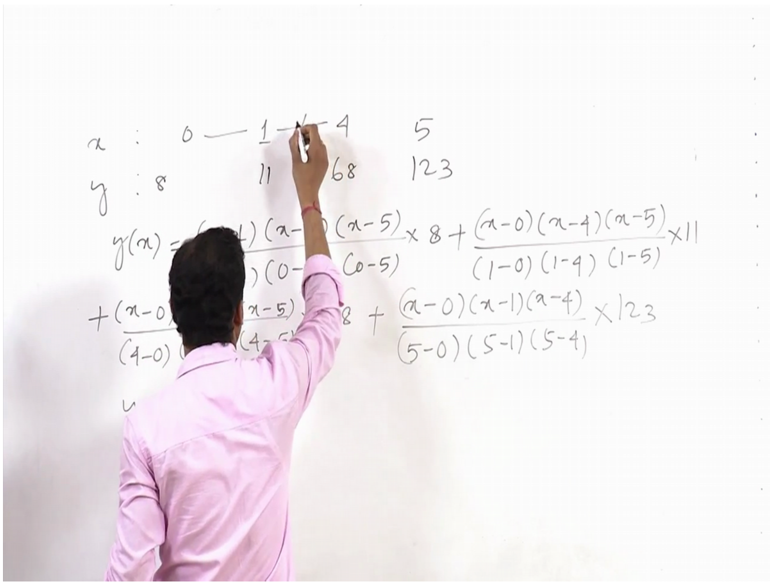

And again this 4 and 5 this difference is 1 here, this difference is 3 and this is 1 again. So that is why you cannot use any class of like finite difference operators to find this value at x equals to 2 here. So Lagrange method this can also be applicable if we will have this data points is given and if it is asked to evaluate this functional values within (cla) any of these functional values here.

And for that what we will do is, we have to fit another polynomial suppose q of y such that the function is defined as x as a function of y there. This means that always we are just expressing y as a function of x. If it is asked to evaluate this functional value we can just use it in a reverse form.

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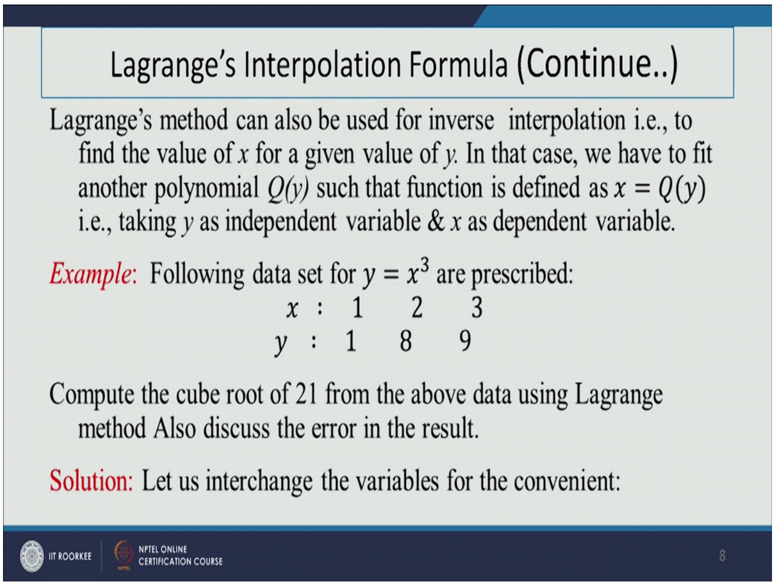

That means that we can just use that one as x equals to q of y there. Since y as the independent variable and x as the dependent variable. So for that we can just consider another example suppose y equals to f of x then usually we can just express in inverse form here x equals to g of y here to find these values at the tabular points if a function is prescribed to us.

The following data are prescribed here like y equals to x cube is given here, x at the points like 1, 2, 3 and y values are given as 1, 8 and 27 here. And the question is asked to compute the root of 21 from the above data using Lagrange's method and also discuss the error associated in that formulation.

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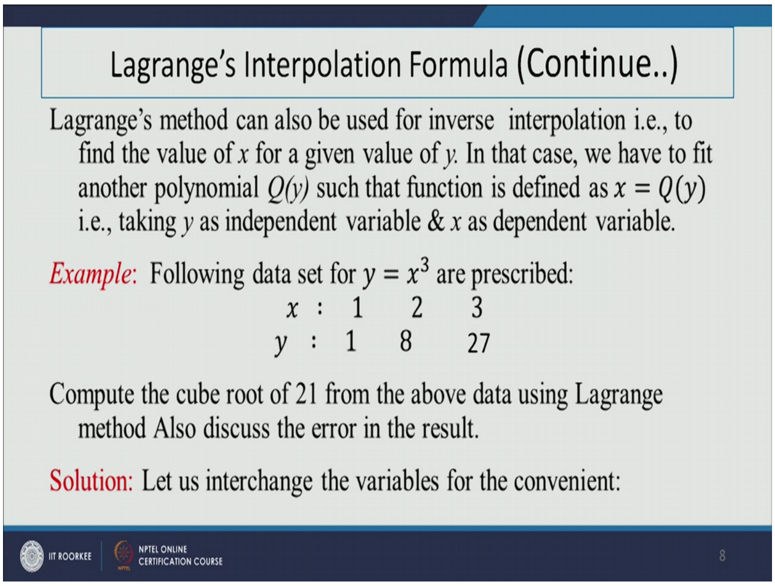

So for that what we will do is we can just express this function in an inverse way as like x is given here as 1, 2, 3 and corresponding y values that are written as here 1, 8, 27 here. And so if the question is asked to find the value of x where y value is 21 here, we can just write this formulation in a reverse form that x as a function of y here. Especially if you will just see this part here, x is expressed as a function of y here.

So that is why we can just write this formulation x as a function of y which can be expressed as y minus y 1, y minus y 2 divided by we can just write as y 0 minus y 1, y 0 minus y 2. And corresponding value it will just take these values as x 0 here plus we can just write y minus y 0, y minus y 2 divided by your values like y 1 minus y 0, y 1 minus y 2 into these functional values which can be in a inverse form it can be written as x 1 here.

And the last part you can just write this one as y minus y 0, y minus y 1 divided by y 2 minus y 0, y 2 minus y 1 into x 2 here.

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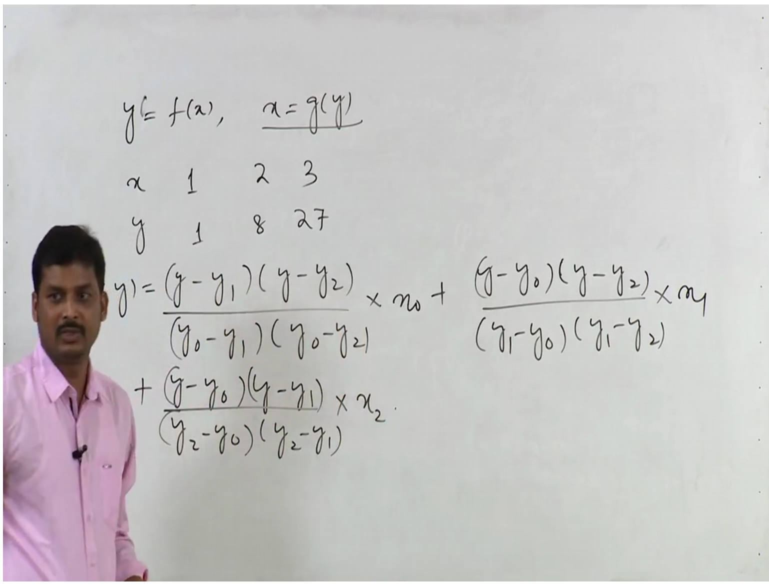

So if you will just represent this Lagrange polynomial in this form here so especially we are just writing in this form here x 0 as 1, x 1 as 2, x 2 as 3 here and corresponding y values these are expressed as y 0 equals to 1 here, y 1 as 8 here, y 2 as 27 here.

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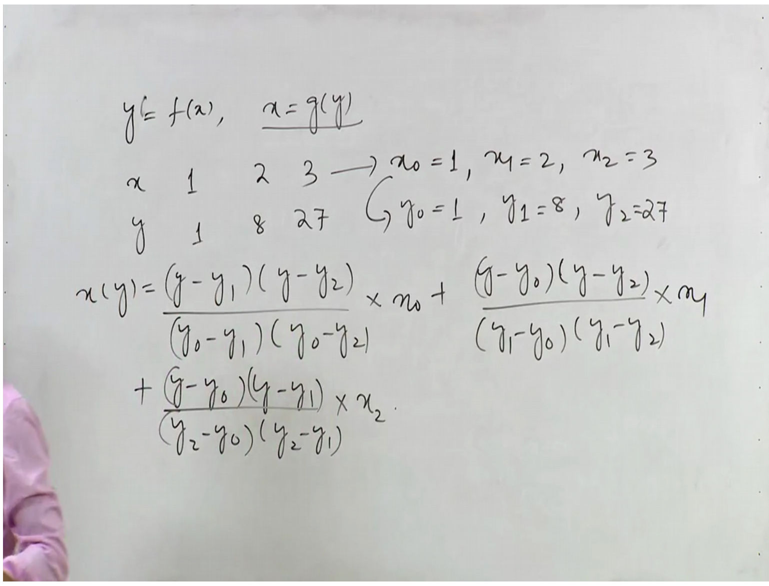

And if you will just put these values here then we can just find these functional values as like x of 21. Since the y value it is asked to compute the x position that where we can just find this cube root of 21 there. Since especially the function is defined in the form of like y equals x cube here. So that is why if in a inverse form if you will just write x can be expressed as y to the power 1 by 3 here.

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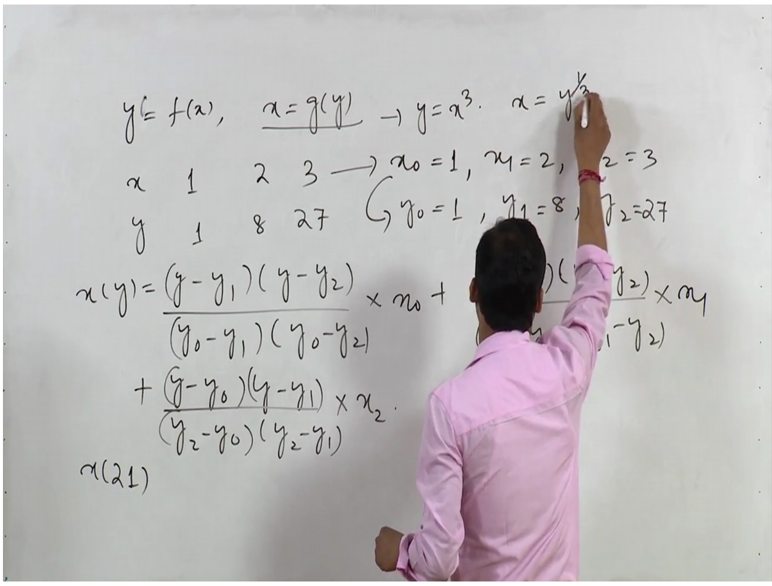

So that is why if it is asked to compute this one x of 21 here so then we can just write this one as 12 sorry this is like 21 minus your y1 that as here as like 8, 21 minus 27 divided by the corresponding values of y 0 here as 1 minus 8, 1 minus 27 into x 0 value. X 0 value is especially 1 here plus if you will just write again like 21 minus y 0. So y 0 is 1 here then into 21 minus y 2 here. So y 2 is obviously sorry this is y 2. Y 2 is 27 here.

This divided by like if you will just write here or value y 1. Y 1 means this is 8 minus 1, then 8 minus 27 and the second value if you will just write this is 2 here. Plus the third value if you will just write here that is in the form of like if you will just see, so last value we can just right 21 minus y 0, 1 21 minus 8 divided by, if you will just see here so specially y 2 means this is 27 minus 1, 27 minus 8 into the last value that is given as 3 here.

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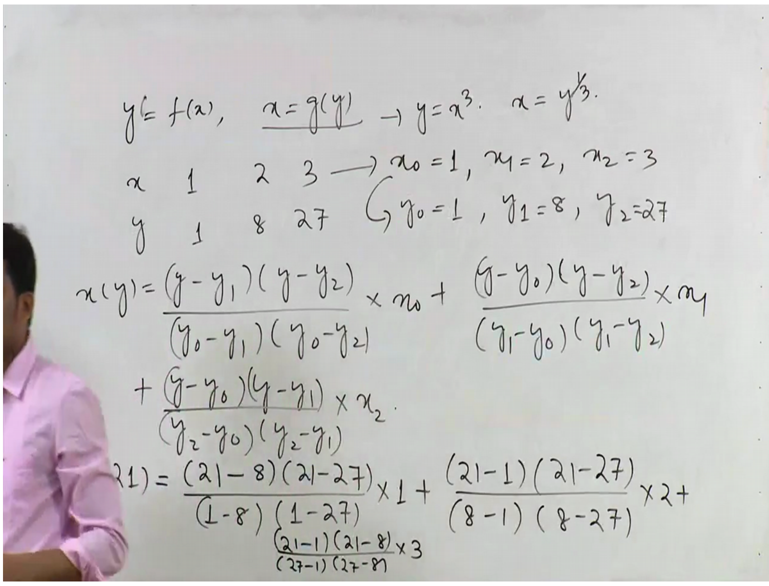

If you will just go for this computation so then we can just find this value in a particular form and we can just obtain this value of x of 21 there itself. So if we will go for this error computation here we can just find the exact cube root of 21 as 2 point 7589.

And if we want to go for the computation of error terms here in case of Lagrange interpolation polynomial that is the generalized interpolation polynomial errors, what is occurring in our finite difference operator, the same we can just find this error term that in a generalized form we have just obtained that one as r of x equals to x minus x 0, x minus x 1 to x minus x n, f to the power n plus zeta by n plus 1 factorial where zeta should be lies between x 0 to x n.

So the same approximation we can just use over here also. For n terms usually we are just writing this error term r of x as x minus x 0, x minus x 1 up to x n, f to the power n plus 1 zeta by n plus 1 factorial where zeta should be lies between x 0, x n here.

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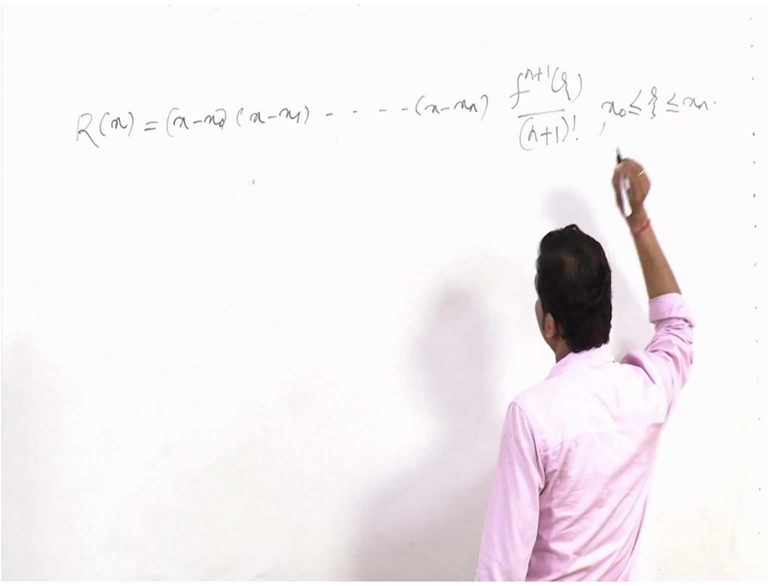

So the same approximation we can just use for the computation of error here also and in this case if you will just go for this error computation I can just write r of x equals to x minus 1, x minus 8, x minus 27 into f triple dash of zeta by 3 factorial here.

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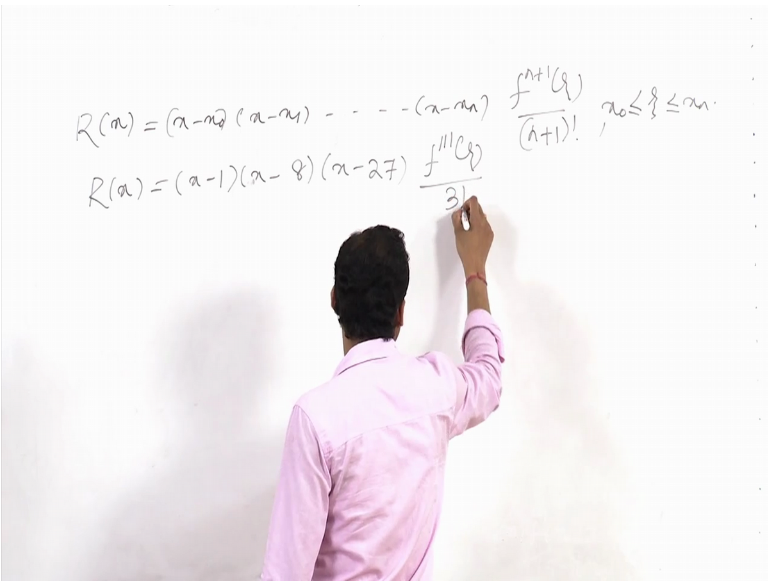

Since we will have here like 3 points like n equals to 2 here. So 3 points that is why we are just considering 1, 8, 27 so it can just generate a polynomial of degree 2 and that is why 3 points mean we will have a polynomial of degree. Suppose if we are just considering a polynomial of a degree n there then we will have exactly n plus 1 terms.

So that is why here we will have like 3 point 1, 2 and 3. We are just considering that is why this degree of the polynomial will be 2 here. And if n equals to 2 here obviously we can just write f to the power n plus 1 that is 2 plus 1 is 3 here. So this represents the third order derivative here and this is 3 factorial.

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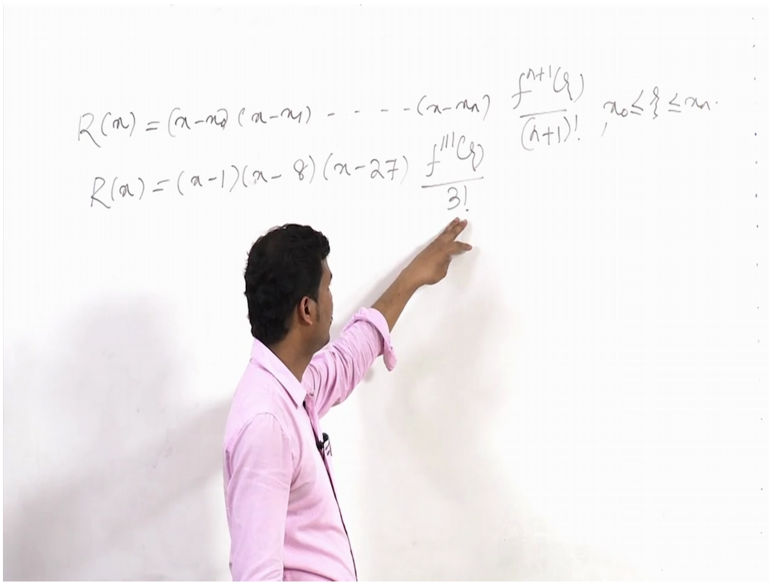

So if you will just go for the computation of this error term here so y can be expressed as in the form of like x to the power 1 by 3 here. And if you will just go for this triple order derivative f triple dash of x here, I can just express this one as 10 by 27 x to the power minus 8 by 3. And the maximum value of f triple dash x at x equals to 1, it can just obtain.

And hence your error term that is r of 21 can be written as 21 minus 1 into 21 minus 8 into 21 minus 27. And the maximum value that is occurring at 10 by 27 so that is why 3 factorial I can just write this one.

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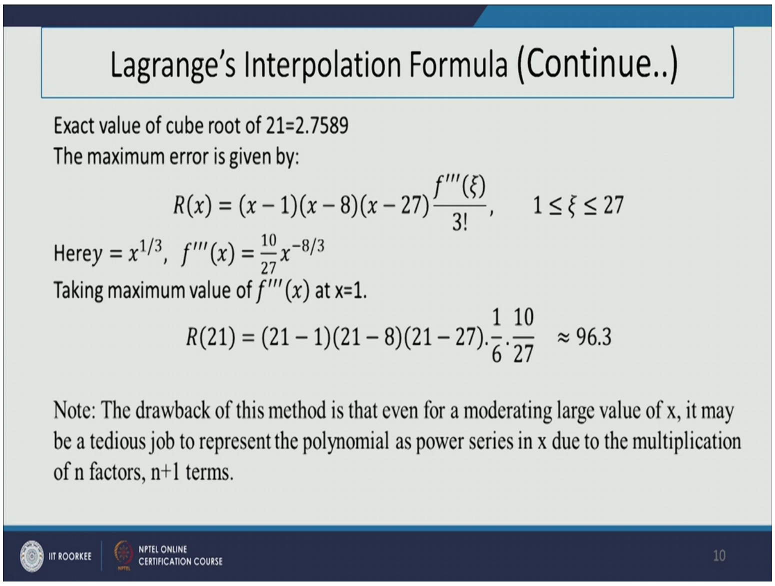

That is as r of 21 as 21 minus 1, 21 minus 8, 21 minus 27 into this one as 10 by 27 into 1 by 6 here. So this total value it is just coming as 96 point 3 here or 96 point 29 something it is just coming over that. So the drawback of this method that is if we want to add suppose any other term we have to go for this computation of n plus 1 terms here.

This means that whenever we have an extra point here usually we are just writing p of x as f of x or we are just writing this one as y of x here. That is represented in the form of L 0 x f of x 0, L 1 x f of x 1. So likewise we are just writing L n x f of x n here.

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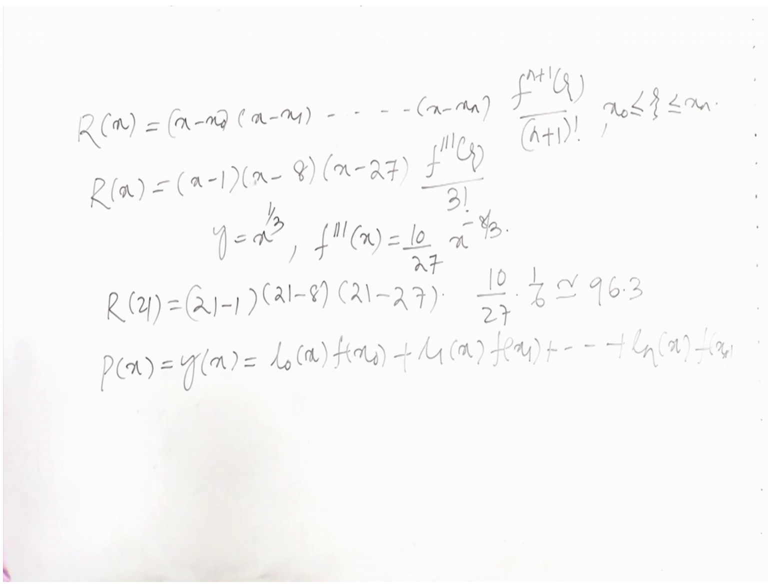

But suddenly if suppose one extra point we want to add it up here then we have to consider that this n plus 1 point in each of these products here, since L 0 x is usually expressed in the form of x minus x 1 to x minus x n by x 0 minus x 1 to x 0 minus x n there. So if another extra point if it will be added so in upper side also we have to consider like x minus x n plus 1, lower side also we will have to consider x 0 minus x n plus 1 also.

So in each of the terms if it is multiplied so a large multiplier series required to go for this solution process. In this Lagrangian method usually if we want to add suppose any term we can just find that all of the terms like L 0 x if it is written, so it can be written in the form of like x minus x 1 to x minus x n divided by like x 0 minus x 1 to x 0 minus x n here.

So if we want to add extra more term or one more term there then we have to do all of the computations again. So that is why we can just go for like a Newton's divided difference interpolation formula for the better computation or it requires like less computation for an extra addition of these points here. Thank you for listening this lecture.