Numerical Methods
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Lecture No 19
Interpolation Part IV

Welcome to the lecture series on numerical methods. In the last lecture we have discussed Newton's forward difference formula and Newton's backward difference formula, in the present lecture we will start about this error computation for Newton's forward difference formula and Newton's backward difference formula.

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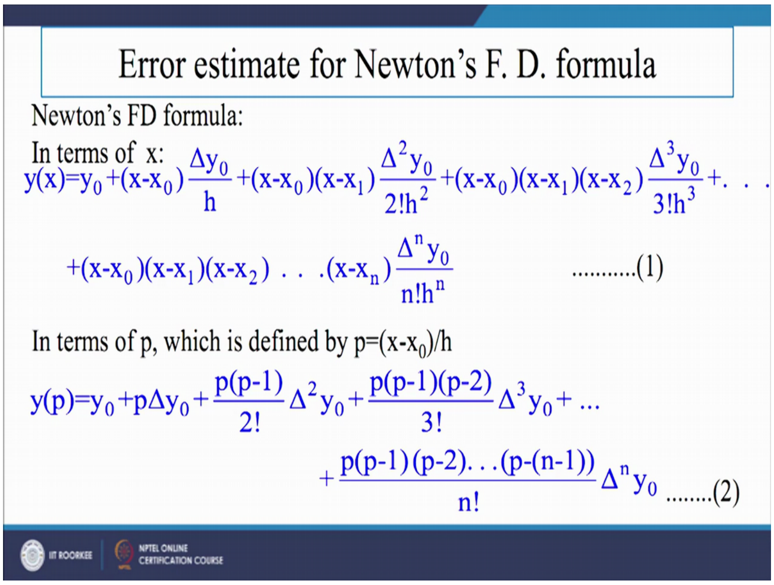

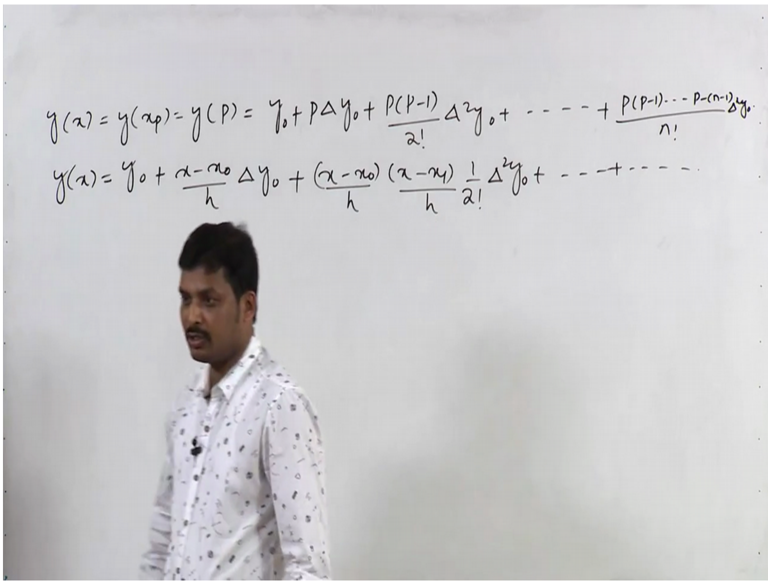

So before going to the Newton's forward difference formula error competition, first will just go for this Newton's forward difference formulas uhh representation. So how we are just representing the Newton's forward difference formula is that first we can just write this one as y of x or y of xp for a particular point usually or as yp as a function of p. We can just write this one as y 0 plus p delta of y 0, p into p minus 1 by factorial 2, del square of y 0 plus upto the term like p into p minus 1 upto p minus n minus 1 whole divided by n factorial, del to the power n of y 0.

So especially it is in the form of p, we are just writing this one and if we will just write in terms of x then we can just write y of x as y 0, so p can be expressed in the form of x that is nothing but x minus x 0 by h delta of y 0 plus x minus x 0 by h, x minus x 1 by h, 1 by 2 factorial delta square of y 0 plus upto the last term. So specifically if you just note this notice once here the first formulation we can just apply if a value is asked to compute at a particular point that where we can just determine the value of p and we can just apply that formulation to evaluate this function at that point, but if this function is asked to compute in the form of a polynomial then this second formulation we can just use there.

So if you just see the slide here that is y of x is expressed in the form of y of 0 plus x minus x 0 delta of y 0 by h plus x minus x 0, x minus x 1 del square y 0 by 2 factorial h square and the last term especially it is expressed in the form of x minus x 0, x is x 1 upto x minus x n into delta to the power n of y 0 by n factorial h to the power n here.

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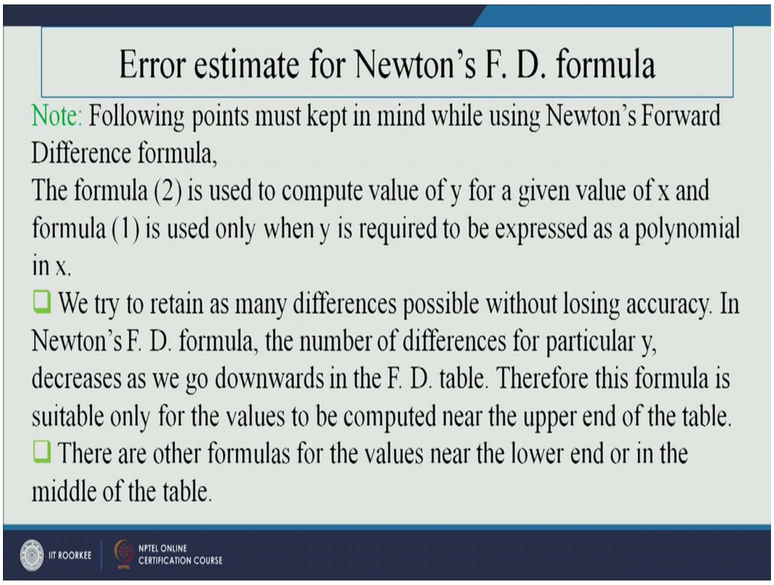

And before going to use this formula first we should have to note few points, that where you can use this Newton's forward difference formula or where we can just use Newton's backward difference formula or other forms of this finite difference operators formulas. So first if you just go for this application of this Newton's forward difference formula here, so this formula can be used to compute the value of y for a given value of x and the formula in terms of x if can be used for this representation of this function y as in the form of a polynomial here.

And then in the second form if you will just say that as many differences we can just try to retain without losing its accuracy we can just put there and in Newton's forward difference formula especially the number of difference for a particular y decreases as we go downwards the table if you just see. And if we want to find the value at beginning of the table especially this formula is the suitable formulation to evaluate this values at the beginning of the table or at the start of the table there itself or in the upper end of the table there.

So there are other formulas which can be you can just use for the computation of the values near the middle or the end of the table. And if sometimes we are just observing that this difference start behaving erratically or increasing in magnitude at any stage which would leave out those differences and higher at the differences of this formulation this means that, suppose if you just going for this computation and after suppose second-order differences if you just find that 3rd order difference values is getting increased or it is just infinitely getting increased off then we can just terminate this series upto secondary term here and we can just evaluate this value here itself.

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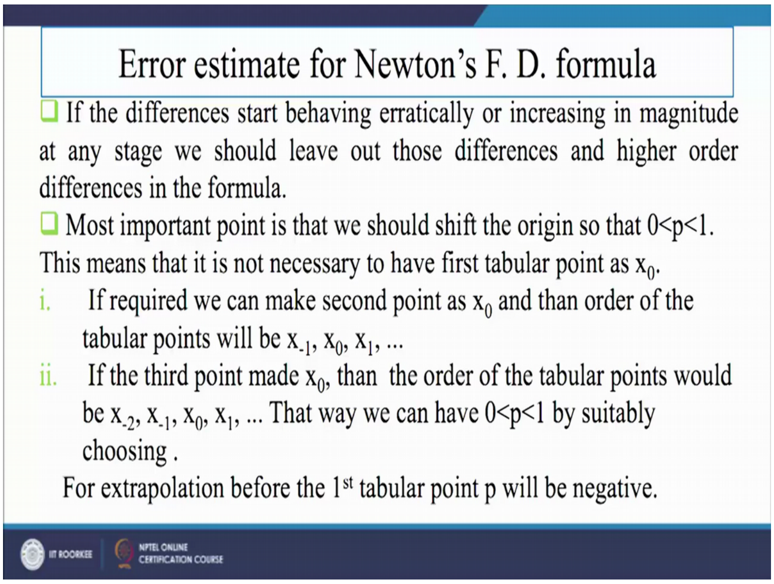

And sometimes suppose if the starting point is not the origin then we can just shift this origin to the immediate next point if you will just consider these points like your starting points are x 0, x 1, x 2, x-ray so likewise every points are existing here and suppose if the values asked to compute exactly at x 0 or within this interval x 0 to x 1 then you can just use this Newton's forward difference formula.

But sometimes if the value is lying between suppose x 1 to x 2 you can just switch this x 0 position to x 1 here then you tabular value will be shifted like x of minus 1, x 0, , x 1, x 2 upto x n minus 1 here. And sometimes also if this values is asked to compute within this interval x 1 to x 2 then again also you can just switch this point as x of minus 2, x of minus 1, x 0 to x of n minus 2 here. And if you are just shifting this origin according to your choice that where you want to evaluate this values or which interval you want to evaluate this values or the values asked to evaluate then in that interval you can just find that p should be lies between 0 and 1 here.

Since especially you are just expressing xp or x as x 0 plus ph, so that is why whatever this point if interval is lying between x 1, x 2 here and then you can just consider this point like x of like 1 and 2 within that it will just exist. So that is why you can just consider x minus x 0 by h as p there and if we are just considering this x 0 as the immediate uhh previous point to the next point and where we want to evaluate this values of p there, so that is why we can just say that p should be lies between 0 and 1 there. And Newton's forward difference formula can be applicable and sometimes for the extrapolation before the in 1st tabular point p will be the negative values.

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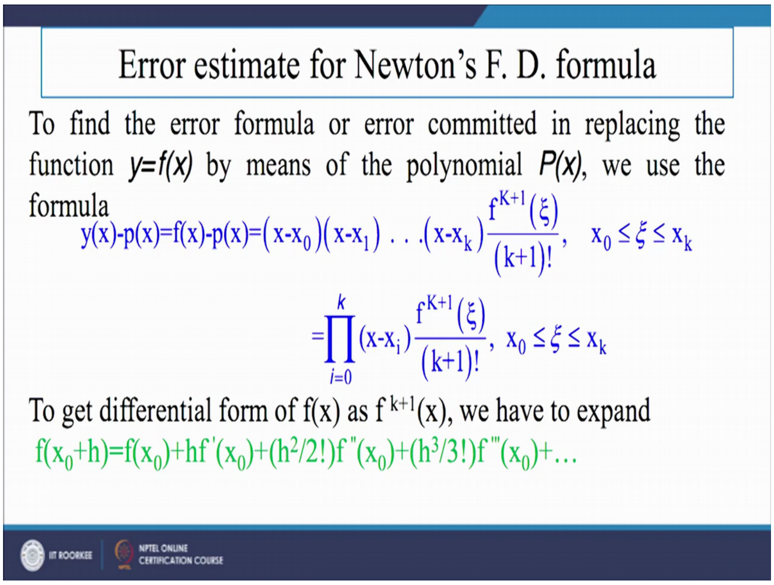

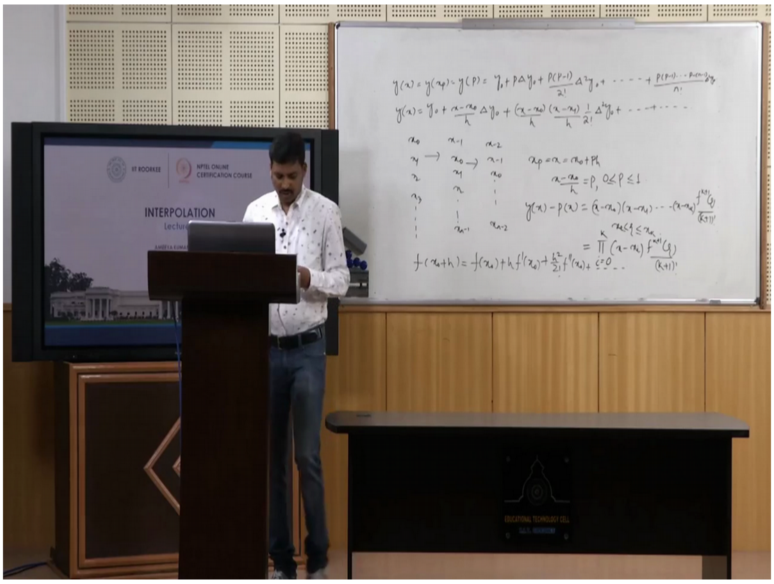

To find the error formula in a combine form or in a committed form by replacing y equals to F of x by means of a polynomial p of x, especially here what we are doing is if the tabular values is known to us and if complete function is not known to us then especially we are just evaluating this function by considering all of this tabular point.

So if you we are just approximating this function with this polynomial then the error will exist there itself and if this error is existing there then if you just take this difference like y of x minus p of x then this represents the error term at that position. So, especially this can be written in the form of like x minus x 0, x minus x 1 up to x minus x k into f to the power k plus 1 zeta by k plus 1 factorial, where zeta should be lies between x 0 to x k here.

And sometimes also in a product form if you want to express this can be written in the form of like product of i equals to or you can just say that n equals to 1 to K there, especially if we want to write in the form of I here, i equals to 0 to k and we can just write this one has x minus x i and obviously we can just write this one as f to the power k plus 1 zeta by k plus 1 factorial here, where zeta should be lies between x 0 to x k.

And to get suppose this f of x in a differential form, differential form means? How we can just express f of x in terms of f to the power k plus 1 there. Then to get that form we have to expand f at point suppose like x 0 suppose if you want to expand like f of x 0 plus h, since usually this Taylor series expansion is used if a point is given and all of its neighbourhood points if f is continuous then we can just use this Taylor series expansion that is in the form of f of x 0 plus h which can be written as f of x 0 plus h, f dash of x 0 plus h square by 2 factorial f double dash of x 0 plus all other terms.

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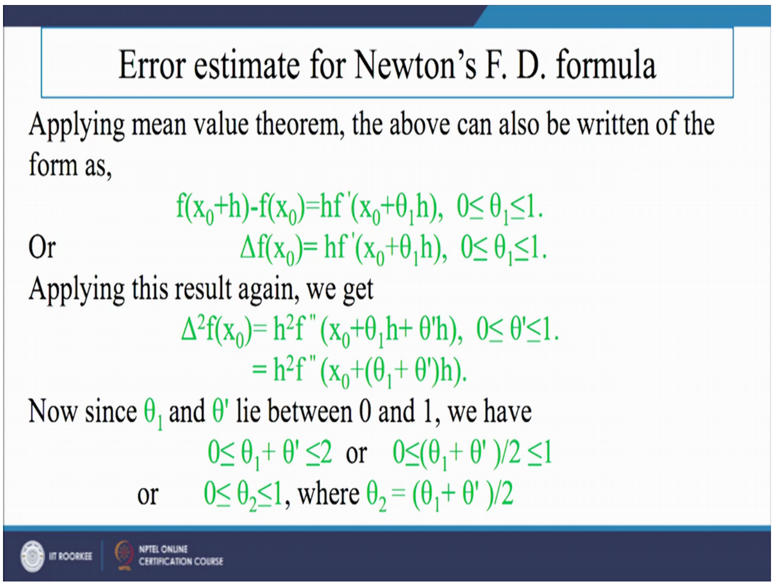

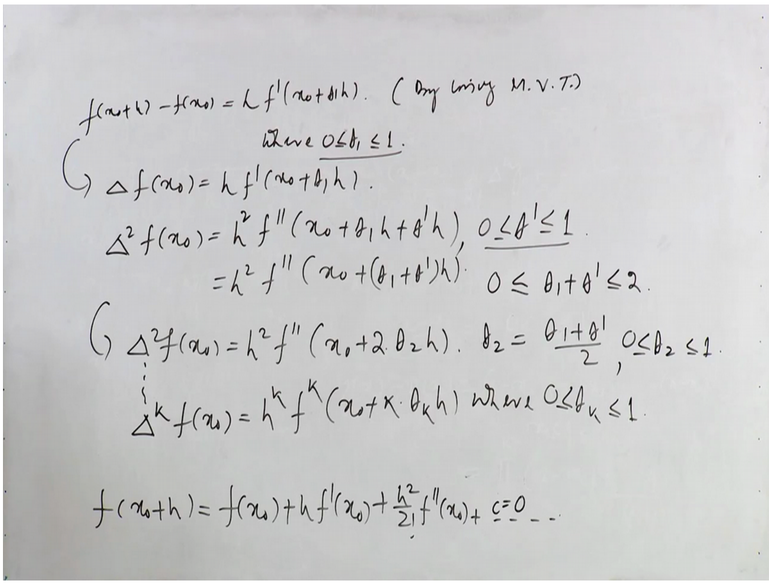

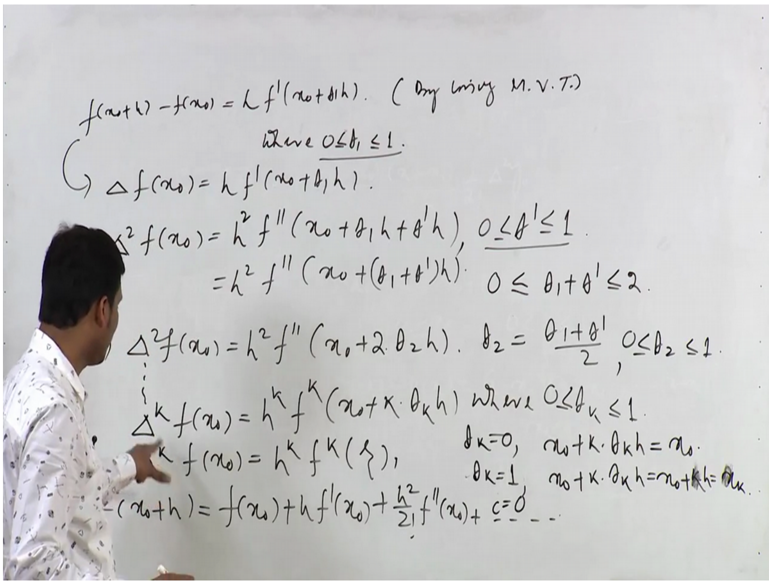

If we will just go for this computation of this uhh f of x in terms of f to the power k plus 1 term here, so then we can just use this mean value theorem for all other terms. This means that we can just write f of x 0 plus h minus f of x 0 as h f dash of x 0 plus theta 1 h here by using mean value theorem, where theta 1 should be lies between 0 and 1 there itself. So obviously we can just write this f of x 0 plus h minus f of x 0 as del f of x 0 this equals is to h f dash of x 0 plus theta 1 h here.

If we will just repeatedly apply then we can just write del square of f of x 0 as we can just write since this it is expressed in the form of h f dash of x 0 plus theta 1, so that it can be written as h square f double dash x 0 plus theta 1 h plus theta dash h here. And directly we can just say that theta dash should be lies between 0 and 1 here and we can just rewrite this formulation as h square f double dash of x 0 plus theta 1 plus theta dash into h this one, where theta 1 plus theta dash if you just see here, so theta 1 is lying between 0 and 1, theta dash is lying between 0 and 1 here.

So that is why we can just see that Theta 1 plus theta dash should be lies between 0 and 2 here and if you just write another parameters suppose theta 2 here and which can be expressed as theta 1 plus theta dash by 2 then we can just say that the theta 2 should be lies between 0 and 1 here. So obviously we can just rewrite this formulation in the form of like del square f of x 0 as h square f double dash of x 0 plus this is 2 theta 2 into h here.

And successively if you just use this formulation here repeated way then after like k-th steps we can just obtain this one repeatedly suppose this a forward difference operator we are just using yr then we can just write theta to the delta to the power k f of x 0 this equals is to h to the power k, f to the power k, x 0 plus k theta k h here, where theta k should be lies between 0 and 1 here.

Where f of k especially it is called as k-th derivative of f of x and if may be noted that since theta k is lying between 0 and 1here obviously you can just determine the exact value of like this point x 0 plus k theta k into h, exactly what it is just giving providing the value whenever theta k equals to 0 and theta k equals to 1 there itself.

So if we can just determine then we can just write it in a composite form there and if you just write in this form then we can just write del to the power k f of x 0 as h to the power k, f to the power of k zeta here. And if we want to find the range of zeta here then if you just put here like theta k equals to 0 suppose then this value like x 0 plus k into theta k h this will just give you the values as x 0 here.

And if you will just put theta k equals to 1 here then we can just say that the value of x 0 plus k into theta k h this is nothing but x 0 plus k h, that is nothing but x k here. So that is why we can just say that if we can just express delta to power k f of x 0 in the form of h to the power k, f to the power k of zeta then zeta should be lies between like x 0 to x k here. So this things you can just clearly visualise here also that if x 0 plus theta k into k h should be lies between x 0 to x k then generally it may be written as delta to the power k of f of x 0 this equals is to h the power k, f to the power k of zeta, zeta should be lies between x 0 to x k here.

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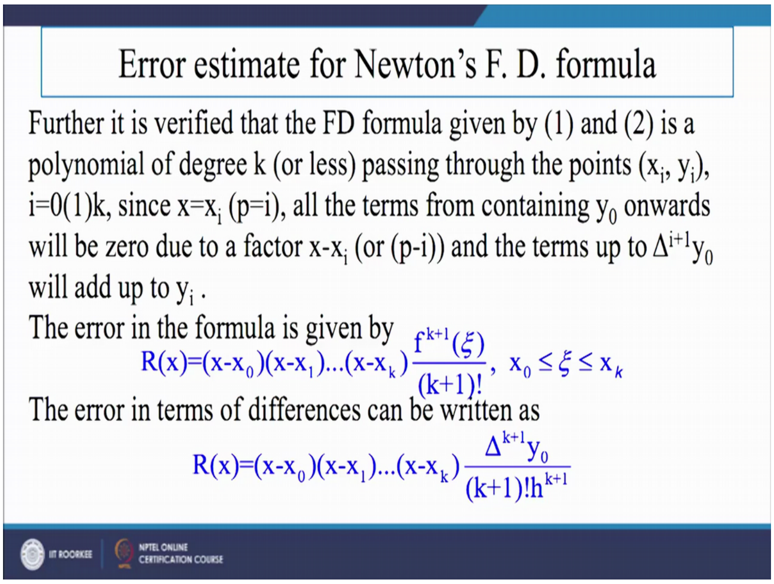

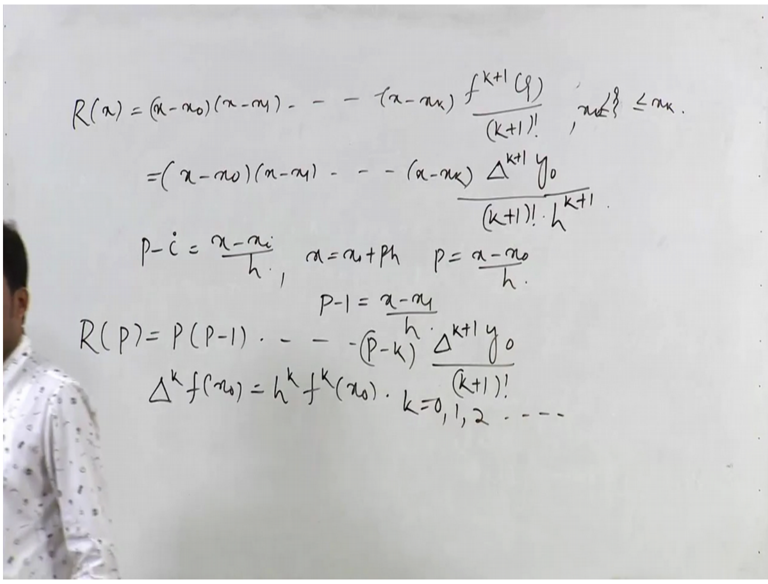

And further if we will just go for this like forward difference formula if it is just passing through this points like x i, y i, where x i are the nodal points and y i are the associated functional values then we can just express this remainder terms as if you will just see the error term R of x is usually written in form of x minus x 0, x minus x 1 upto x minus x k, f to the power k plus 1 zeta by k plus 1 factorial here, where zeta should be lies between x 0 to x k here.

And if you want to represent in terms of like Newton's forward difference operator here then we can just write this one as x minus x 0, x minus x 1 to x minus x k into del to the power k plus 1 y 0 or f of x 0 we can say by k plus 1 factorial into h to the power k plus 1 here. So this is the generalised error estimation formula for Newton's forward difference formula here. So if you just use this transformation that is a like we have already shown that if you will just write like p minus i this can be represented in the form x minus x i buy h since already we have shown that p can be represented as x minus x 0 by h that we have just express in the beginning since x equals to x 0 plus ph.

So that is why just write p equals to x minus x 0 by h and if we will just write like p minus 1, so especially it is represented in the form of x minus x 1 by h so likewise if you will just express p minus i can be expressed as x minus x i by h here. And in terms of p if you will just write this error formulation then r of p we can be written as p into p minus 1 upto p minus k, del to the power k plus 1, y 0 divided by k plus 1 factorial here.

And it may be noted that if all tabular points like x 0, x 1 upto x k approaches to x 0 suppose if h is very small then we can just see that del to the power k of f of x 0 this can be written as h to the power k, f to the power k of x 0 here, for k equals to 0, 1, 2 upto any value you can just say for a small h we can just write this formulation as in this form here. Then in that case this **(())(18:52)** error reduces to R to the R of k plus 1 x as x minus x 0 whole to the power k plus 1 by k plus 1 factorial, f to the power k plus 1 zeta, where zeta should be lies between x 0 to x k here.

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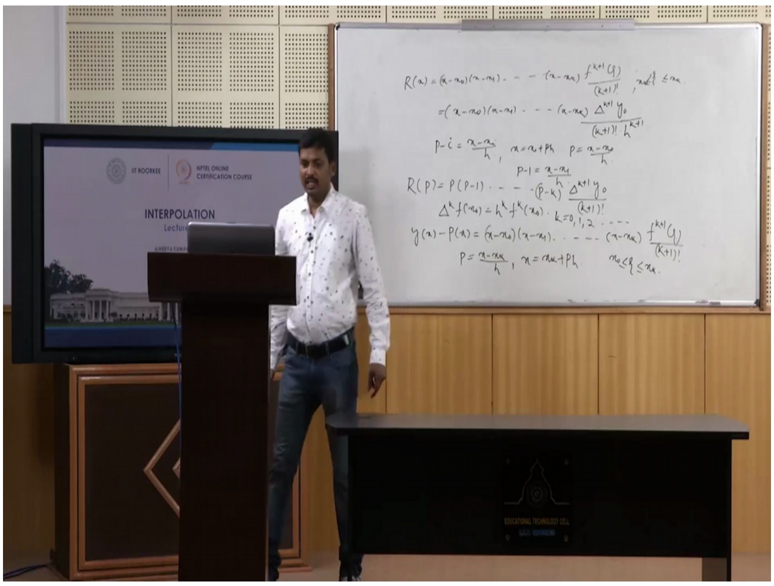

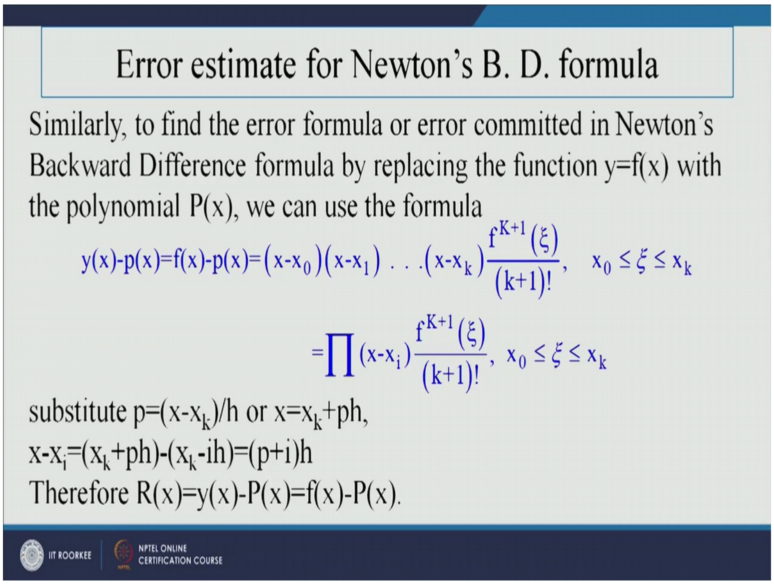

So if you will just go for this error estimation in terms of Newton's backward difference formula then this error formula or error committed is obtained by replacing this function y equal to f of x with the polynomial p of x as y of x minus p of x is x 0, x minus x 1, upto x minus x k, f to the power k plus 1 zeta by k plus 1 factorial here, where zeta should be lies between x 0 to x k there itself also. And in the product form as we have expressed in the earlier section that usually it can be expressed in the form of like product of x minus x I, f to the power k plus 1 zeta by k plus 1 factorial, where zeta should be lies between x 0 to x k here.

And if you just substitute here that is p equals to x minus x k, since we are just starting this backward difference formula at the end of the table so that is why we can just see that the last point as x k there itself. So that is why p can be written in the form of x minus x k by h here, where we can just write x equals to x k plus ph here. And your R of x or the remainder term can be written as R of x equals to y of x minus p of x as sometimes usually it is written as f of x minus px since y of x is approximated by this function f of x there.

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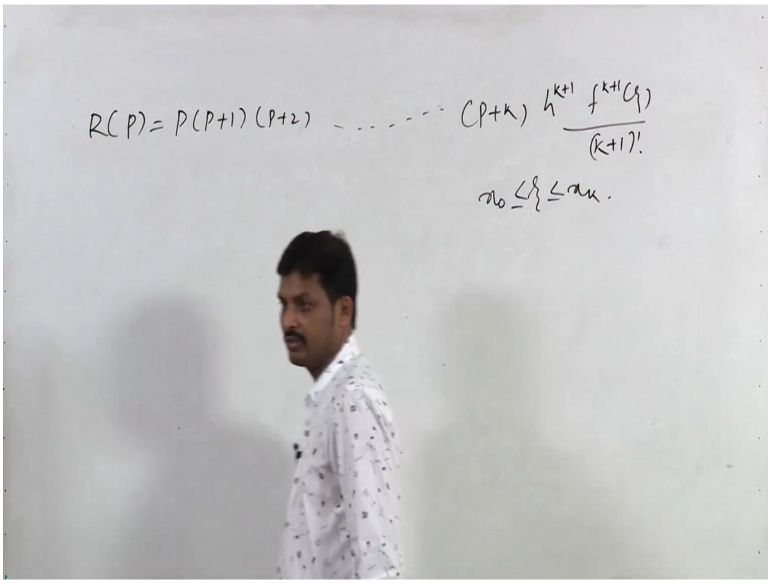

And if you just substitute all these values then we can just this Newton's backward difference formula error as R of p this is the error term for Newton's backward difference formula and it can be express in the form of like p into p plus 1 into p plus 2 upto p plus k and this can be written as h to the power k plus 1, f to the power k plus 1 zeta divided by k plus 1 factorial here, where zeta should be lies between x 0 to x k here also.

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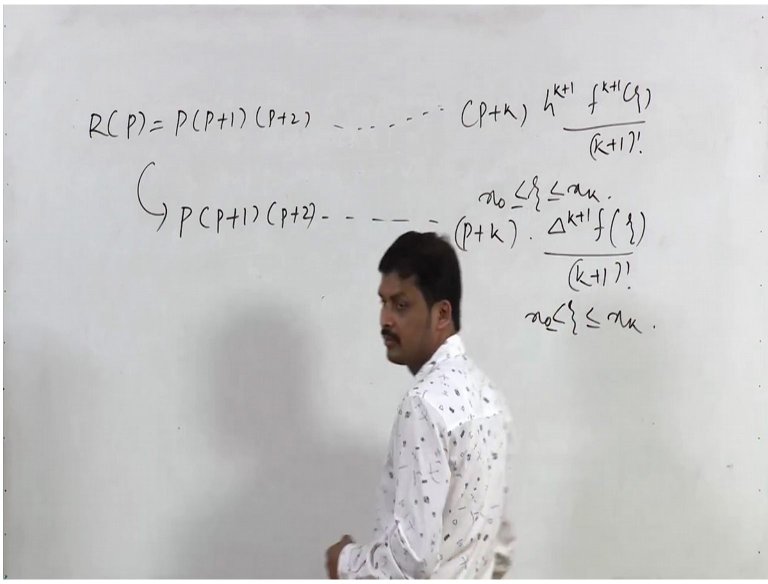

And if we want to represent it in Newton's forward difference formula form since usually this error is represented in the form of Newton's forward difference operator. So if you represent in terms of Newton's forward difference formula, so it can be written in the form of like p plus 2 upto p into p plus 1 into p plus 2 upto p plus k then the next formed it will be del to the power k plus 1, f to the power we can just say f of zeta here, divided by k plus 1 factorial and where zeta should be lies between x 0 to x k here also.

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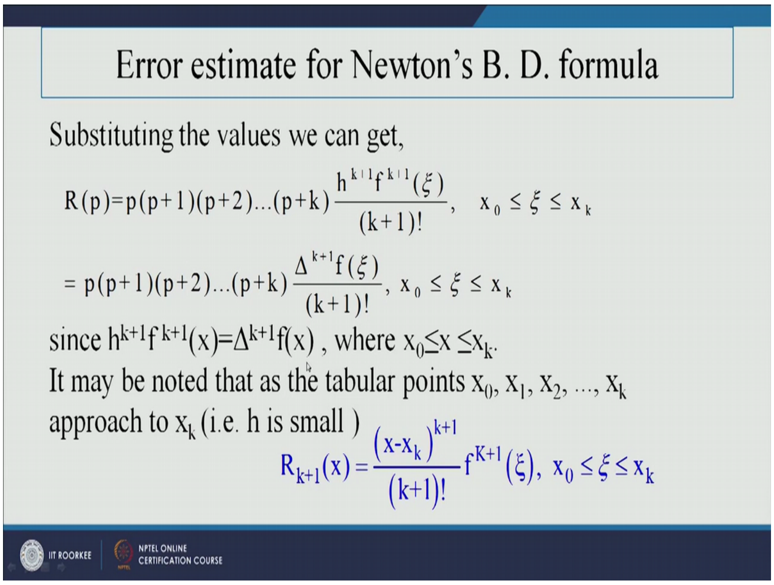

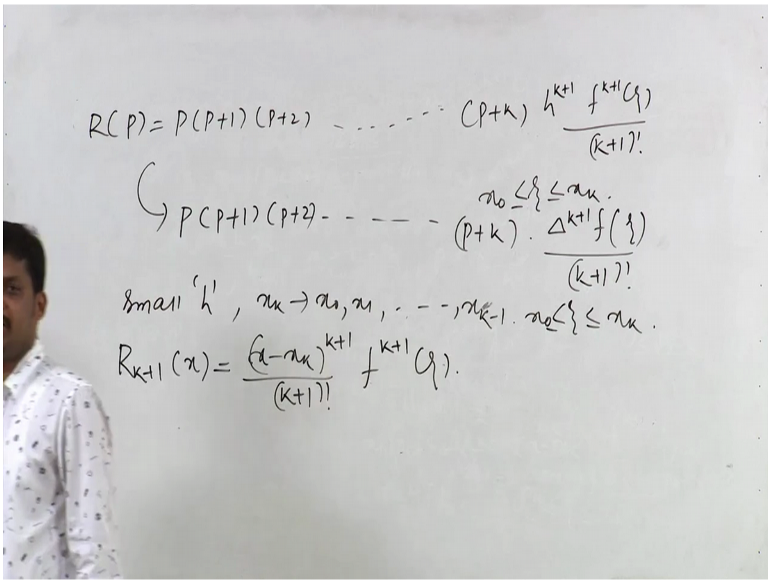

Since already we have obtained that h to the power k plus 1, f to the power k plus 1 x, this can be represented as del to the power k plus 1, f of x here, where x is lying between x 0 to x k. Then if you will just write all this tabular points which is a approximating towards the point x k for small h if x k is approaching towards x 0, x 1 to x k suppose sorry x k minus 1 here suppose then we can just say that R k plus 1 of x here this can be represented as x minus x k whole to the power k plus 1 divided by k plus 1 factorial here and f to the power k plus 1 zeta, where zeta should be lies between x 0 to x k here.

Especially if you just see here you are just saying that all of these points just extended by this point x k here this means that x k is tending towards x 0, x is tending towards x 1 then x k is tending towards x k minus 1 there. So that is why this formulation for small h can be reduced in this form here, so thank you for this listening this lecture.