Numerical Methods
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Lecture 14
Power Method

Hello everyone, uhh so welcome to the 4th lecture of this module and today we are going to discuss one more method for finding eigenvalues and eigenvectors of a matrix. So in the last lecture we have discussed Jacobi method for finding eigenvalues for a symmetric matrix and there we have uhh we have use the similarity transformation in such a way the a similar matrix has uhh we a similar matrix found corresponding to the given matrix and that similar matrix is just like uhh just a diagonal matrix where the eigenvalues are the diagonal elements, like Jacobi method was restrict up to symmetric matrix only, today we are going to discuss a method which is applicable to any square matrix; however again we are having some conditions to apply power method, which is we are going to discuss today and we will discuss about those conditions.

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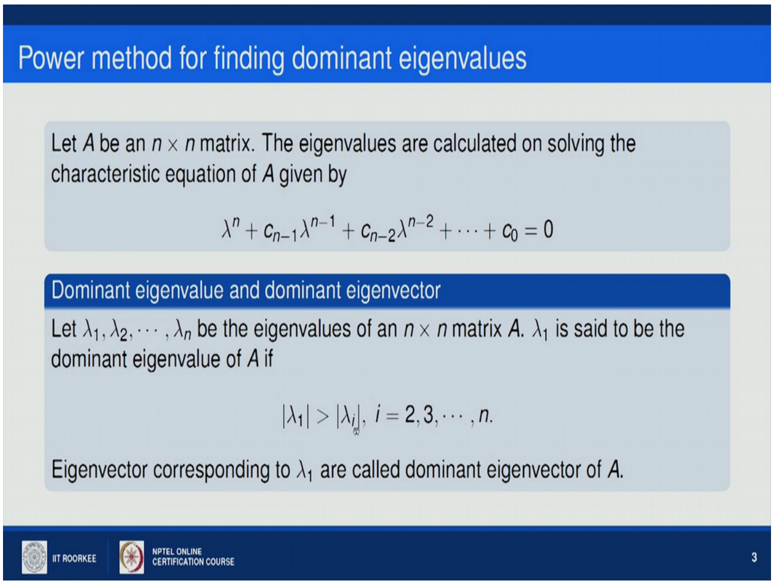

So first of all, let A be a square matrix of order n. the eigenvalues are calculated just by solving the characteristic equation, which is given as lambda raze to power n plus some constant Cn minus 1 lambda race to power n minus 1 and so on. So it is a n degree polynomial in lambda and roots zeros of this polynomial will give you the eigenvalues. Let us say eigenvalues are lambda1, lambda2, lambda n, where some of them may be equal or repeated we will say that lambda1 is a dominant eigenvalue of A, if this condition is satisfy means, the absolute value of lambda1 is greater than rest of the absolute values of rest of the eigenvalues that is mod of lambda1 greater than mod of lambda i for i equals to 2, 3, up to n and if this condition old lambda1 is the dominant eigenvalue and the corresponding eigenvector is called dominant eigenvector of A.

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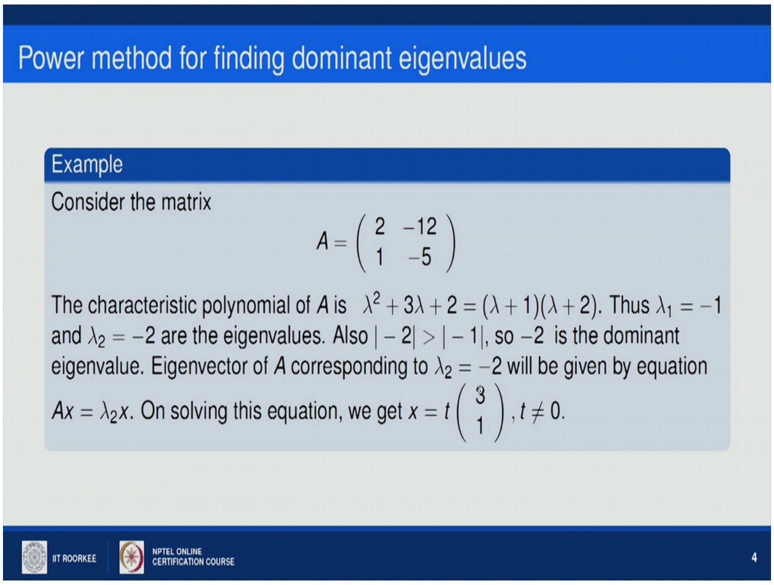

So for example, if I take this 2 by 2 matrix so the characteristic polynomial of this matrix is lambda square plus 3 lambda plus 2, which is having zeros as lambda1 equals to minus 1 and lambda2 equals to minus 2. So the eigenvalues are -1 and -2 for this matrix. Now if we see that absolute value of -2 is greater than absolute value of minus 1. Hence, -2 is the dominant eigenvalue of this matrix and corresponding eigenvector is (3, 1).

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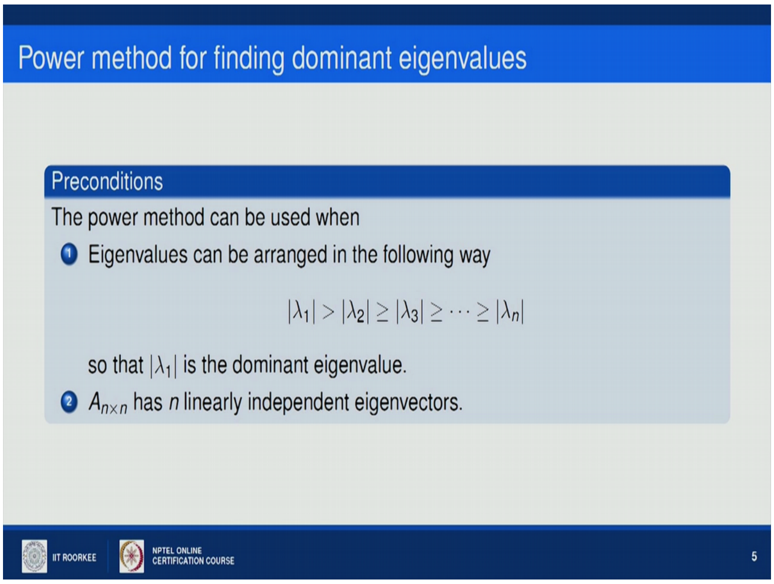

Now what are the conditions to apply power method? First of all, eigenvalues can be arranged in the following way that is, the dominant eigenvalue should be there for the matrix and there should not be any repetition of dominant eigenvalue, for example, if we are having a 3 by 3 matrix, there should be 1 dominant eigenvalue which is not equals to others, for example if A is 3 by 3 and we are having eigenvalues minus 5, 3 and 2. Then here minus 5 is clearly dominant eigenvalue, but if we are having eigenvalues like -5, 5 and 4. Here, we cannot apply power method as such and we will not be able to find out the eigenvalue, because here the dominant eigenvalue is -5 as well as 5 and it is repeated. So here lambda1 is not clear strictly greater than the rest of the eigenvalue in terms of absolute value. The second condition is which is again very important that the matrix A should have a linearly independent eigenvectors. It means A should be similar to A diagonal matrix if we talk in terms of similarity transformation.

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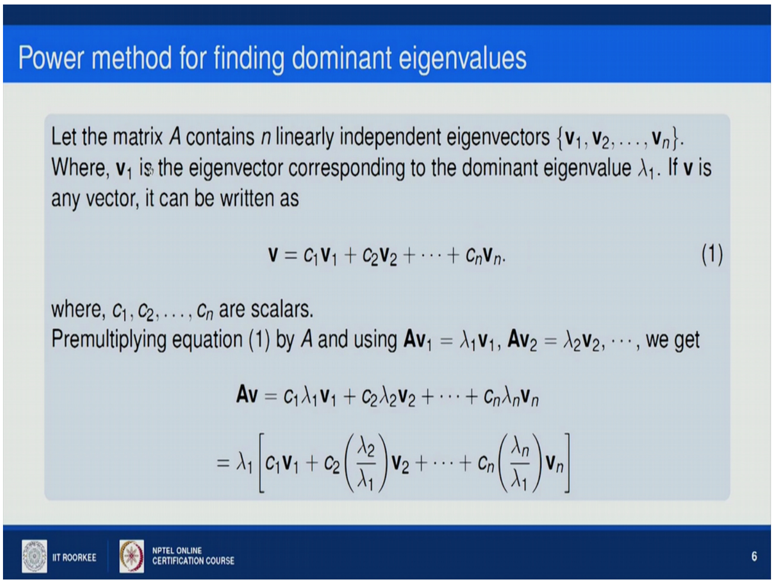

So with these two conditions let us derive the power method.

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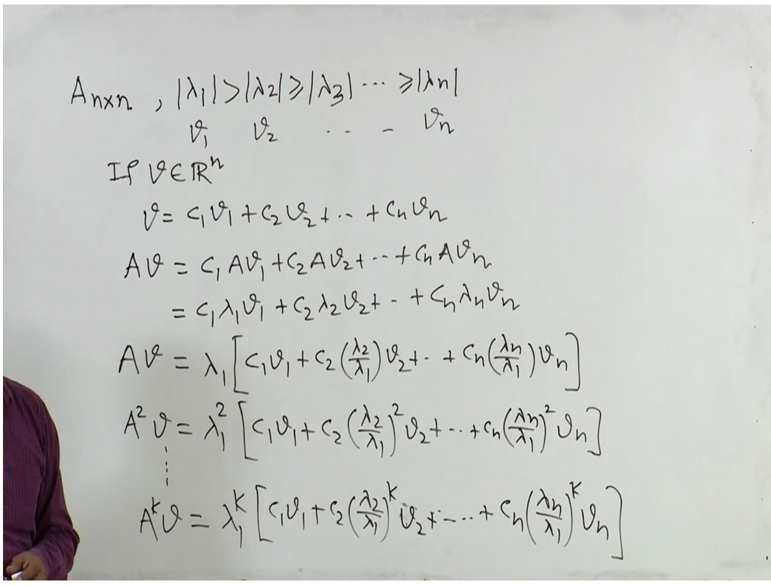

So let A is n by n matrix and it is having eigenvalues lambda1 which is strictly greater than in terms of absolute value to rest of the eigenvalues, moreover we are having eigenvector as V1 corresponding to lambda1, V2 corresponding to lambda2 and Vn corresponding to lambda n and here, we are assuming that V1, V2, Vn are linearly independent. So if these vectors are linearly independent, then any vector V from the vector space Rn can be written as the linear combination of these eigenvectors. So if V belongs to Rn or if matrix is from the complex number and from the Cn so if we form Rn then we can write V equals to C1V1 plus C2V2 plus Cn Vn. Here, C1, C2, Cn are scalars.

Now if I multiply by matrix A, in this equation I will get in the left hand side A into V and then C1A into V1 plus C2A into V2 plus Cn A into Vn, as we know that V1 is an eigenvector corresponding eigenvalue lambda1 for the matrix A. So I can write C1 lambda1V1. Similarly, this term I can write lambda2 V2 plus Cn lambda n Vn or if I take lambda1 common from the right hand side, I can write C1V1 plus C2 into lambda2 upon lambda1 into V2. So this is equals to AV.

If I multiply one more time by the matrix A, in this equation I will get A square V in the left hand side and this will become, so C1V1 plus C2 and the square of lambda2 upon lambda1 into V2 plus Cn square of this ratio term into Vn or if I continue by multiplying A again A again, let us say I multiply k time it will become A race to power k V equals to lambda1 race to power k into C1V1 plus C2 lambda2 upon lambda1 race to power k and finally the last term will become

Now look at this equation, here we are having in the this term lambda2 upon lambda1. Similarly, in the next term we will be having lambda3 upon lambda1 and so on. Here our assumption is that lambda1 is the dominant eigenvalue. It means that this particular term lambda2 upon lambda1 will be less than 1. Similarly lambda3 upon lambda1 will be less than 1 and up to lambda n upon lambda1 which is again place then 1.

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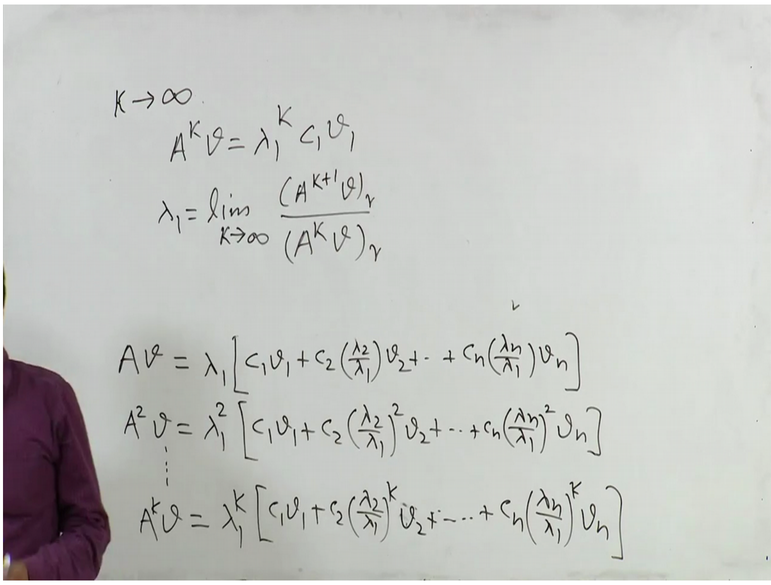

So when k is tending to infinity my Ak V will become lambda1 race to power k into C1V1 it means I can find out lambda1 as limit k tending to infinity by this ratio and here R is the component of the vector this particular vectors having the highest value in magnitude. So this will give me dominant eigenvalue and since, we are using various powers of A that is why we are saying it the power method and from this equation, it is clear that if lambda1 is the eigenvalue that is the dominant eigenvalue then the corresponding eigenvector will be V1. So with this I can talk about the conversions of this power method.

So if lambda2 upon lambda11 and absolute of this ratio term is less than 1, the rate of conversion is fast, moreover whatever will be the means if it is quite small then 1 the method we converse faster if it is close to 1, this ratio term, the method will converge slowly, as we know that power method is an iterative method, because each time we are making we are starting with the initial solution V0, then we are finding V1 as A of V1, then I V2 will be A of V2, V3 will be A of V2 and so on until unless it will not converge.

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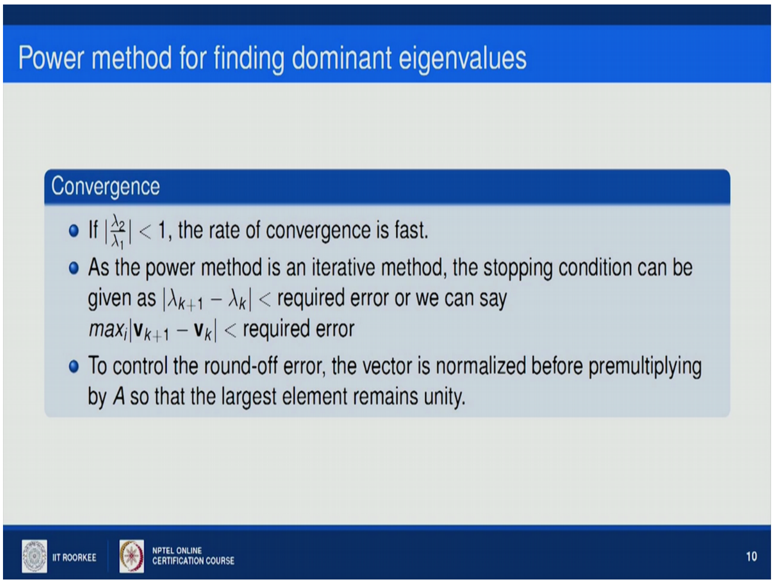

So there should be some stopping criteria and the stopping condition is if in the two successive iterations lambda is less than a given threshold, okay, in terms of 10 race to power -3 or -5 whatever accuracy you want in your method or the maximum component in 2 successive vector Vk plus 1 and Vk and difference of this, which is having the maximum value is less than a given threshold. Moreover to control the round off error the vector is normalize before pre-multiplying by A so that the largest element remains unity, for example, if you start with 1,1,1,1 and after multiplying with A you are getting let us say some vector 2, 3,4. So what I will do I will normalize it, in such a way that the biggest component of this V2 vector that is 4 should become 1 so that vector will become 2 by 4, 3 by 4 and 1.

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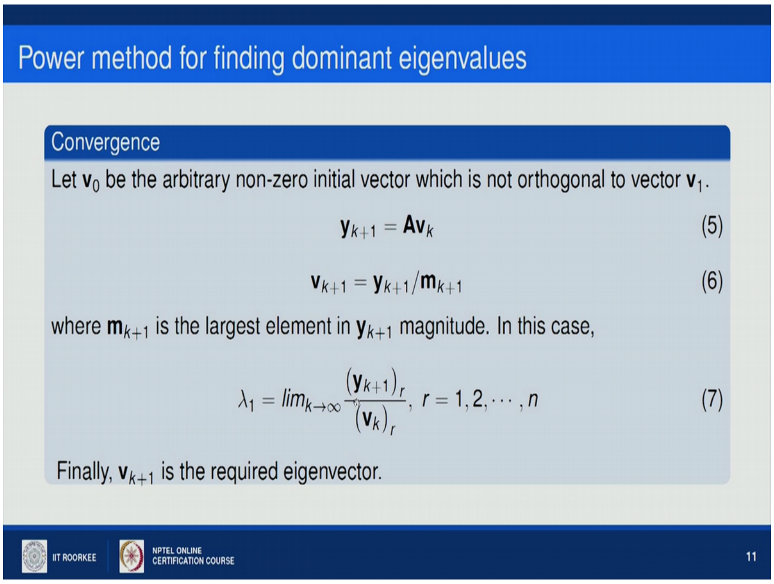

If we talk about eigenvector, so we start with V0 as the initial vector and the condition is that this V0 should not be orthogonal to vector V1. So and it should be a non-zero vector obviously, because we are finally converge in this particular vector is conversing to the eigenvector. So yk plus 1 equals to A of Vk then I will find out Vk plus 1 as yk plus 1 upon mk plus 1 and as I told you where mk plus 1 is the largest element in yk plus 1 in magnitude. So in this case, lambda1 will be limit k tending to infinity, yk plus 1 r upon Vk r. finally, Vk plus 1 will be the required eigenvector corresponding to lambda1.

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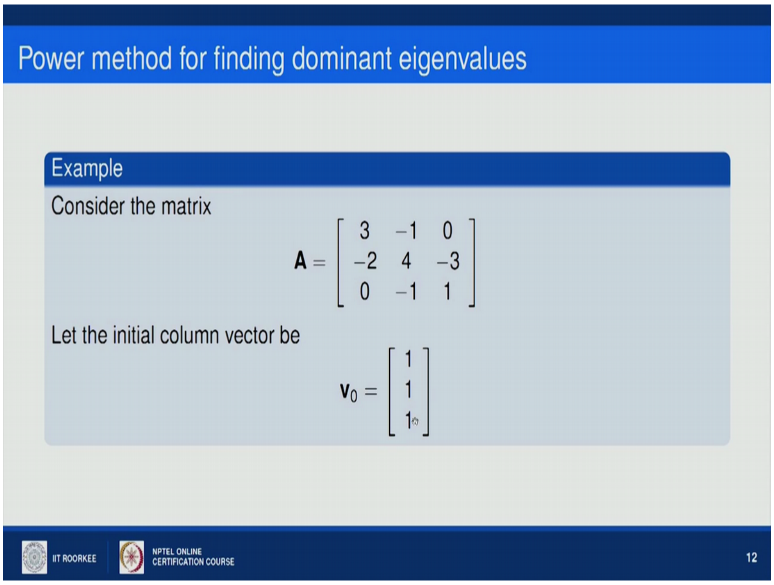

Let us take an example of this method, just consider this 3 by 3 matrix and let the initial column vector be 1, 1, 1.

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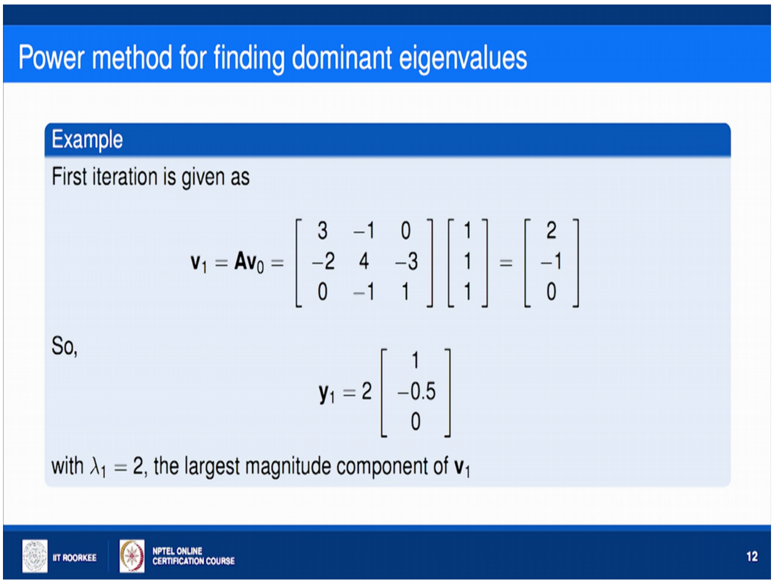

So first of all I will find out V1, uhh V1 will be A into V0, A is this 3 by 3 matrix, V0 is 1, 1,1 this column vector after multiplying I am getting another column vector, which is V1, 2,-1 and 0. Now what I will do first of all, I will find out from V1 to y1 and y1 will be, I will see which is the biggest component in this vector in terms of a float value and here it is 1. So I will divide this V1 by 2. So y1 will become 1 by 2, 1, 0.5, 0. So here I can say that, in tis iteration my eigenvalue is 2 and the eigenvector is 1, -0.5, 0.

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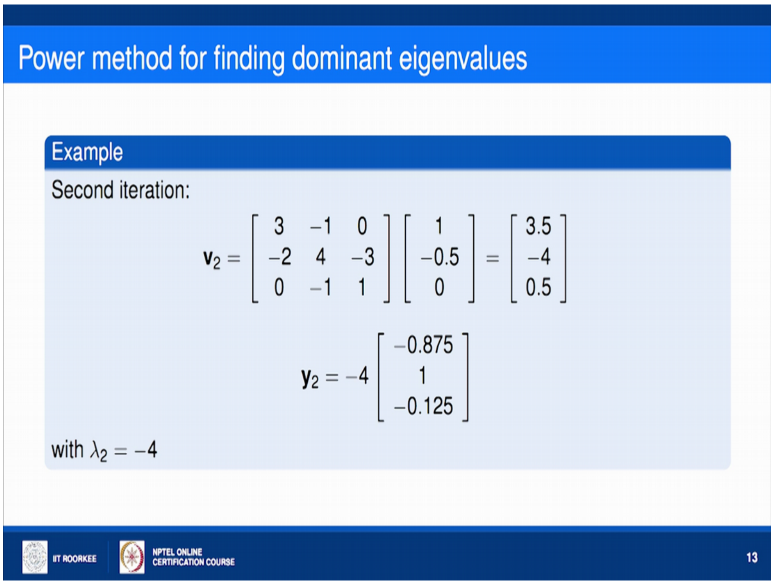

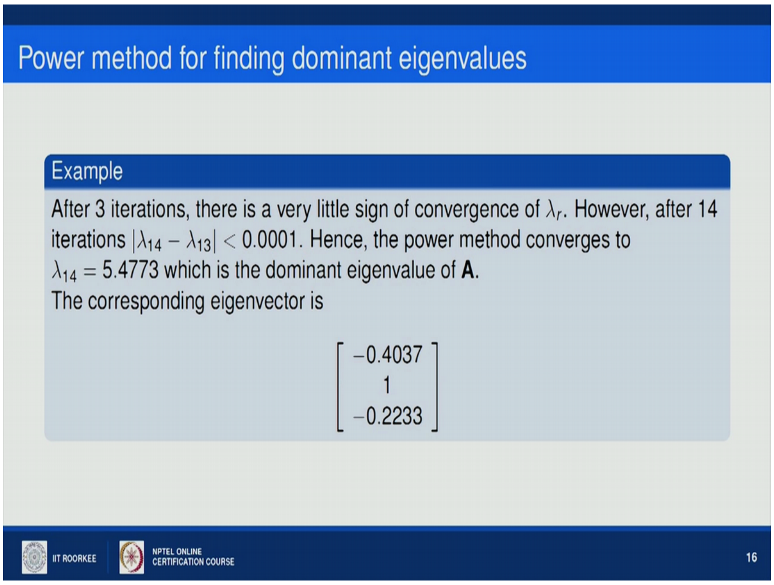

Then I will calculate V2. V2 will be A into y1 and A into y1 when I will calculate it will become 3.5 minus 4.5, again I will divide this vector by 4 so that this term will become 1, -0.875, 1 and -0.125. So here in this iteration eigenvalue the approximation of eigenvalue is minus 4. So just look in first iteration it was 2, in the second iteration, it is coming -4 and similarly, we are getting a deviation is eigenvector. In the third iteration, eigenvalue becomes 6.125 that is approximation and y3 becomes -0.918, 1 and -0.1837.

So we are not getting any sign of convergence so far in 3 iterations however if we go up to 14 iterations what we found that the approximate of eigenvalue which I am getting in 14th iteration minus which I am getting in 13th iteration that absolute difference between these two is less than 10 race to power -4 and hence, the power method converge to eigenvalue 5.4773, which is the dominant eigenvalue of the given matrix and the corresponding eigenvector becomes -0.4037, 1 and -0.2233.

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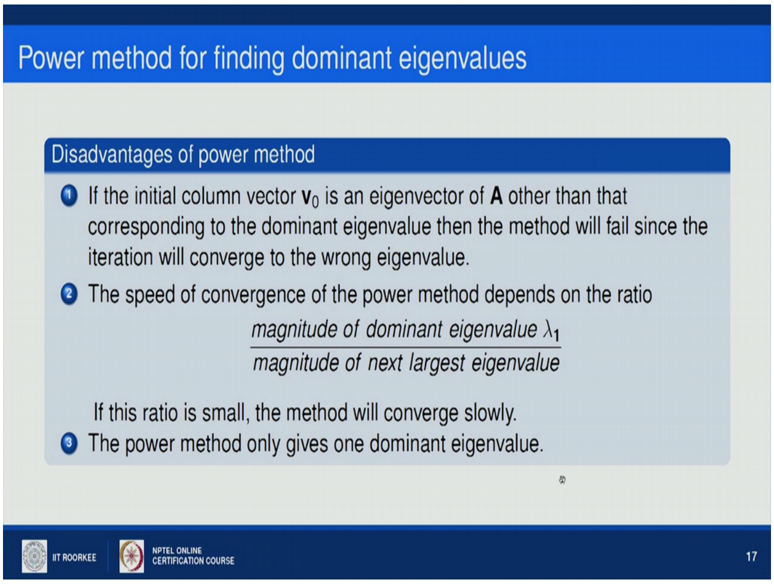

There are some disadvantages or limitations of this method. The first one is if the initial column vector V0 is an eigenvector of A other than the dominant eigenvector, then the method will fail since the iteration will converge to wrong eigenvalue, moreover the speed of convergence depends on the ratio magnitude of dominant eigenvalue lambda1 upon magnitude of the next largest eigenvalue. If the ratio is small, the method will converge slowly. The power method only gives one dominant eigenvalue at a time; okay I will tell you how can we find out other eigenvalues using this method.

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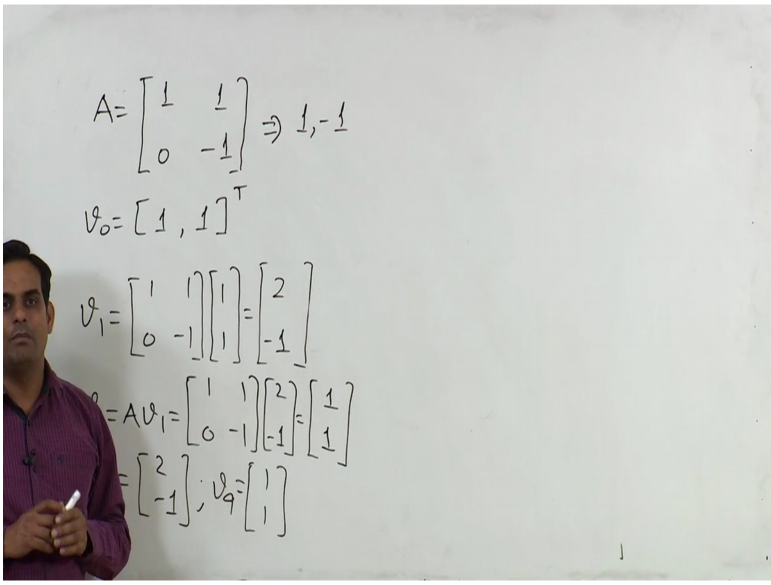

So as I told you, there are some limitations when I told you the assumption for applying this method, I told you that there should be a dominant eigenvalue, if this is not the case what will happen, whether the method will converge or not. Let us see it with an example. So if I take a 2 by 2 matrix, let us say 1, 1, 0,-1. Let us find out the eigenvalue of this method eigenvalue of this matrix using power method. So let me start with a initial vector V0 which is 1,1. So V1 will become 1, 1, 0.-1 that is my matrix A into V0 1, 1 so 1 plus 1, 2 and -1.

Now I will calculate 2, V2 will become A of V1. So it is 1, 1, 0, -1 into 2 minus 1. So 2 minus 1 will become 1 and it became 1, 1, again then if I will calculate V3, V3 will come 2 minus 1, V4 will come 1, 1 and so on. So my method will stuck here in these two vectors in either I will get for the odd iterations of V, I will get 2, -1 for the even iterations like 2, 4,6 I will get 1,1 and I will it will never converge. Why it is happening. This is clear from here, if you see the eigenvalue of this matrix, it is an upper triangular matrix and here eigenvalues are 1 and -1 and the region of this oscillation is very simple that the matrix is not having the dominant eigenvalue that is why the condition that the matrix should have dominant eigenvalue is quite important for applying the power method up to now we have seen that using the power method, we can calculate up only dominant eigenvalues and corresponding eigenvector, suppose I want to calculate other eigenvalues also.

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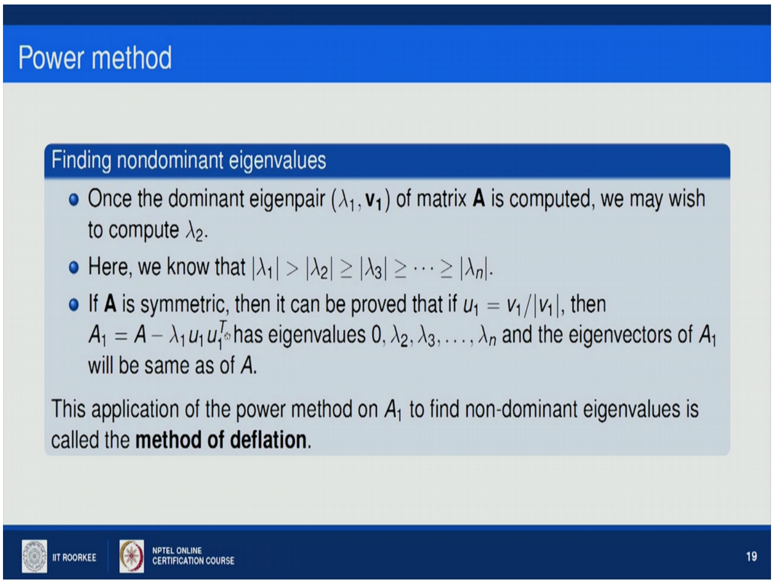

So we can modify this power method in such a way that we shift the dominant eigenvalue to 0 in a new matrix such that the second 2 dominant eigenvalue become the dominant, for example, if you are having eigenvalues lambda1, lambda2, lambda3, if lambda1 is dominant what we will do we will shift tis lambda1 to zero in some other matrix such that the lambda2 becomes the dominant and then in this new matrix we will apply the power method. So this method is called method of deflation and it is based on deflation theorem. So how it works?

So once you calculate the dominant Eigen pair that is lambda1 V1 of a matrix as A you will calculate the or you want to calculate lambda2. Here, so I will take an example of symmetric matrix, but it can be generalized for any other matrix also. So if A is a symmetric matrix then it can be prove that if U1 is V1 upon mod of V1 then A1 is A minus lambda1 U1 U1 transpose has eigenvalues 0, lambda2, lambda3, lambda n and the eigenvector of A1 will be the same as of A. so this is one of the result of deflation theorem.

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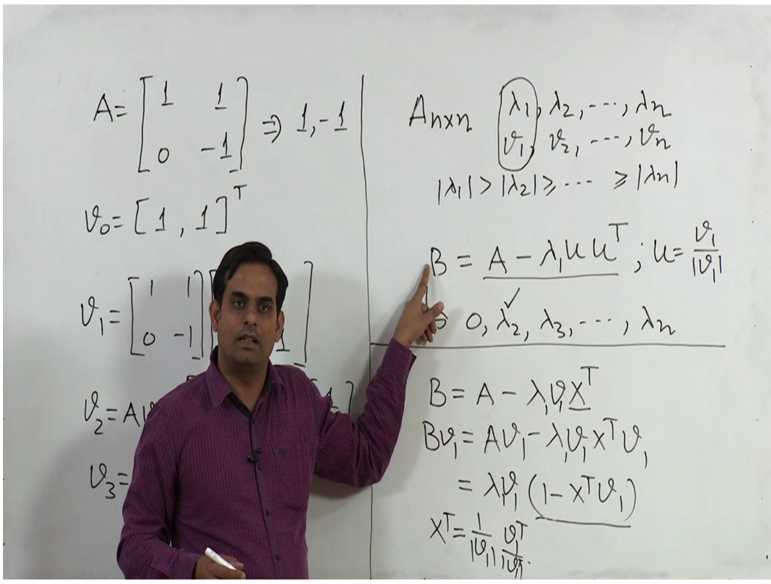

So here we are saying that if A is n cross n matrix having eigenvalue lambda1 lambda2, lambda n and corresponding eigenvectors are V1, V2, Vn. Now and also we are assuming that lambda1 is the dominant eigenvalue and the corresponding eigenvector to this V1 that is a dominant eigenvector is V1. Now I am saying, if A is a symmetric matrix, I can calculate a new matrix A1 or let us say it B, which is A minus lambda into U into U transpose, where U is the unit vector in the direction of V1 then, this matrix B will be having the eigenvalues 0, lambda2, lambda3 up to lambda n and the eigenvectors will remain same like V2, V3, Vn for this new matrix B. so what we can do, suppose using the power method on this matrix A, we calculate the dominant eigenvalue and corresponding eigenvector that is lambda1 and V1 and here, this is my lambda1 is the dominant eigenvalue in this result.

So I calculate these two what I will do I will apply this transformation, I will get a new matrix B and again I will apply the power method on B so that I can calculate the next eigenvalue to the dominant that is lambda2 and corresponding eigenvector that is the dominant eigenvalue of B will be the next two dominant eigenvalue of A and how we are getting this result? Suppose A is a symmetric matrix, so I want a new matrix B, which is something A minus lambda V into X transpose. So I am taking a vector X, if I multiply this vector X in this second term of the right hand side of this equation, I get a new matrix B, which is having the one of the eigenvalue as 0 if lambda is the eigenvalue of A.

Now how? If V1 is the eigenvector corresponding lambda of A, so I will be having B of V1 equals to A of V1 minus lambda V1 X transpose V1 and as I told you lambda1 and V1 are the dominant pair of matrix A. So AV1 can be written as lambda1 V1 and then it will become lambda1 V1 minus lambda1 V1 X transpose V1. So it will become 1 minus X transpose V1. Now how to choose this vector X such that one of the eigenvalue of B should be 0 and the corresponding eigenvectors would be V1 so here if this term become 0, then what I can have? If this term is 0 and I choose X in such a way then BV1 will become lambda1 V1 into 0 that is BV1 equals to 0 means one of eigenvalue corresponding for which this is the eigenvector should be 0. So for a symmetric matrix I can take this X as this one so that my this becomes U and lambda1 UU transpose transpose this becomes the deflation transformation.

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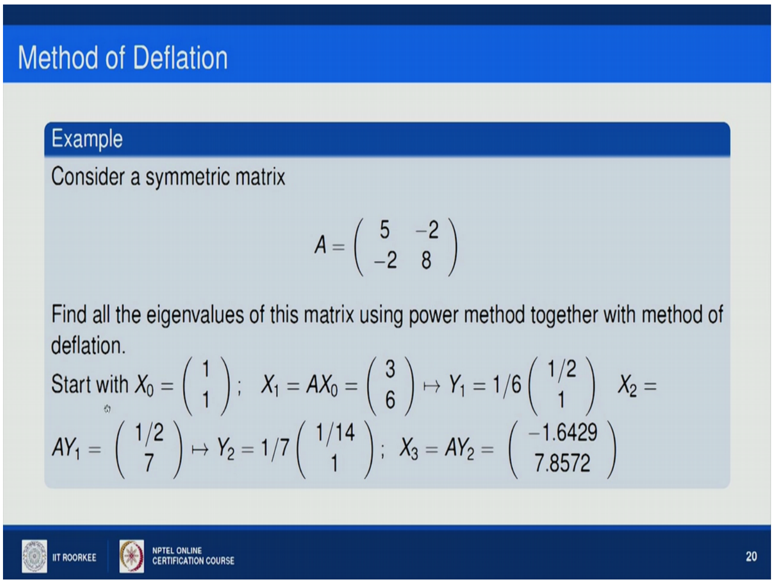

Let us check this with one of the example again I am taking a 2 by 2 matrix and now I question is find all the eigenvalues of this matrix using power method together with method of deflation. So it is a 2 by 2 matrix, so there will be only 2 eigenvalues, uhh one of the eigenvalue that is the dominant eigenvalue we can calculate using the power method and the other one, we will use the method of deflation in power method.

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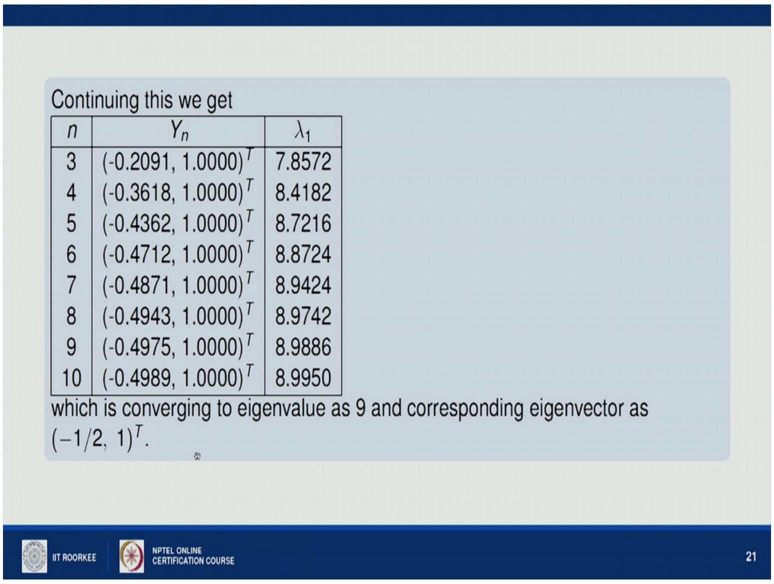

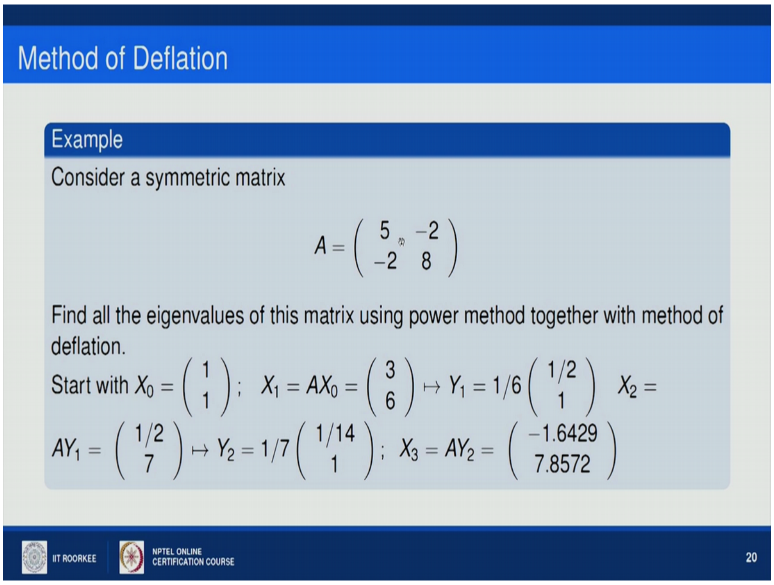

So let us start with 1, 1. So using the power method and after going up to 10th iteration what I am getting that eigenvalues eigenvalue that is the dominant eigenvalue is converging to 9and the corresponding eigenvector is converging to minus half 1. So hence one of the eigen value of this matric, the bigger one in terms of absolute value is 9 and the corresponding eigenvector is minus 0.5, 1.

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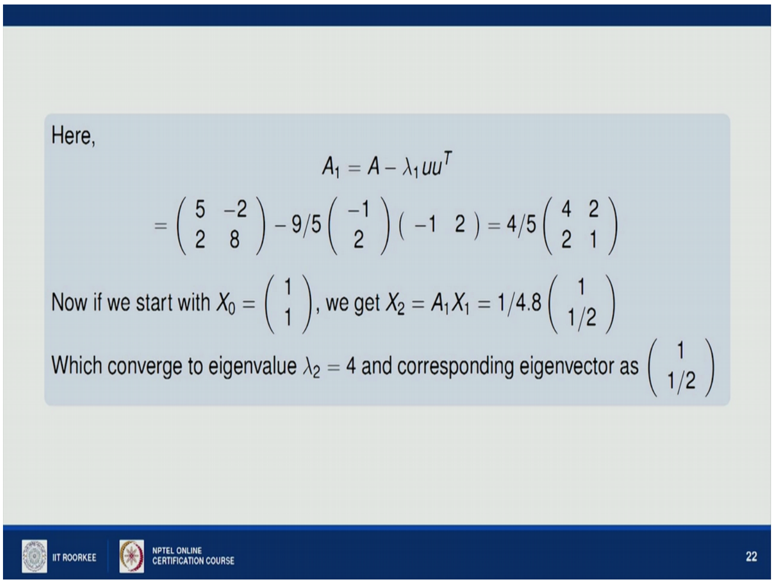

Now I apply this transformation deflation, so A will become A minus uhh B will become or A1, I have written A1 will become A minus lambda1 UU transpose. So after applying this my A1 is coming 4 upon 5 into (4, 2, 2, 1). Now again I will apply and from the method deflation theorem, one of the eigenvalue of this matrix will be 0 and the corresponding eigenvector will be V1 that is minus 0.5 and 1, which is corresponding to 9 or A. So by applying the power method again on this new matrix A1 starting with 1,1 we get X2 as this 1,1 upon 4.8 into 1 upon 1 by 2 and after going this way, we will see that the method is converging to 4 as the eigenvalue dominant eigenvalue of A1 and corresponding eigenvector as 1 and 0.5.

So hence this lambda2 equals to 4 is the dominant eigenvalue of A1, but it is the other eigenvalue that is the second eigenvalue of A and the corresponding eigenvector is 1 with and 0.5 as the two components. So hence using method of deflation with power method, we can calculate all the eigenvalues of a given matrix. So in this lecture we learn how to use power method for finding the dominant eigenvalue and corresponding eigenvector of a given matrix later on we have seen method of deflation if we apply together with power method, we can calculate other eigenvalues also, those are not dominant of the given matrix, thank you very much.