**Numerical Methods**
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**Lecture 13**
**Jacobi's Method for Computing Eigenvalues**

Hello everyone, so welcome to the 3rd lecture of this module. So in this lecture we will learn a method for computing eigenvalues and here, the method will be a numerical, methods because so far you learn how to calculate eigenvalues from as a root of characteristic polynomial. So here what we will do? We will apply some sort of similarity transformations on the given matrix such that after a sequence of similarity transformations, the matrix convert into a diagonal matrix and from the diagonal matrix we can see the eigenvalues directly as the diagonal elements.

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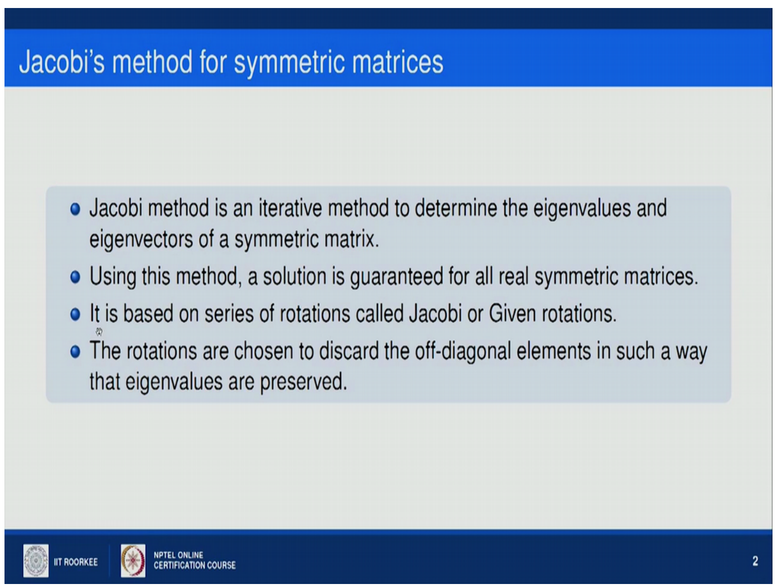


Furthermore, the sequence will also contain the information about the eigenvectors of the matrix. So this method is called Jacobi method and this method gives a guarantee for finding the eigenvalues of real symmetric matrices as well as eigenvectors for the real symmetric matrix. So, as I told you, it is based on the sequence of similarity transformations and those transformations will be based on the rotation matrices or given rotations, we will apply the rotation matrices in terms of similarity transformations to the given matrix in such a way that all the off diagonal elements become zero after a series of transformations. Here, off diagonal elements become zero means, there should not be any change in the eigenvalue of the matrix and that is why I am telling you the similarity transformations, because the two similar matrices will be having the same spectrum.

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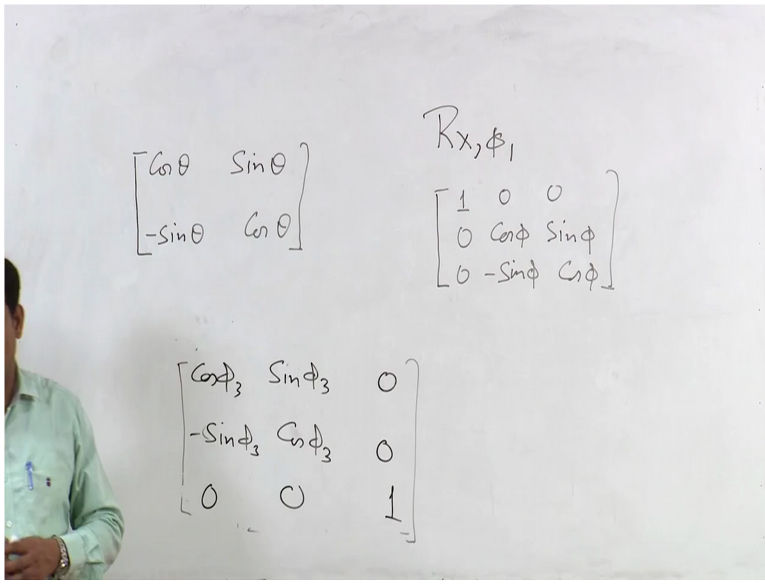


So first of all, here we should know or we should have the idea about a rotation matrix. So what we mean by a rotation matrix? So a 2 by 2 rotation matrix in a plane is given by this particular matrix. So it is an orthogonal matrix you can check, the having determinant as one A transpose will be equal to A inverse and so on. This is about the rotation in a plane by an angle theta. If I talk about the rotation in the space then, first of all we should decide where I want to make the rotation whether the rotation about X axis or Y axis or Z axis, the second thing by which angle? So if I want to a rotation about X axis by an angle phi1 then the rotation matrix will be (1, 0, 0), (0,cos phi, sin phi), (0,minus sin phi, cos phi), if I want to make the rotation about Y axis by an angle phi2, so it will become, so the rotation matrix will be this one and if I want to make it about Z axis by an angle, so I want to make it about Z axis by an angle 3 phi3, then the rotation matrix will become the third column will become (0,0,1) 3$^{rd}$ row will become (0,0,1) and here, rotation will be cos phi, sin phi3, cos phi3 minus sin phi3 and cos phi3.

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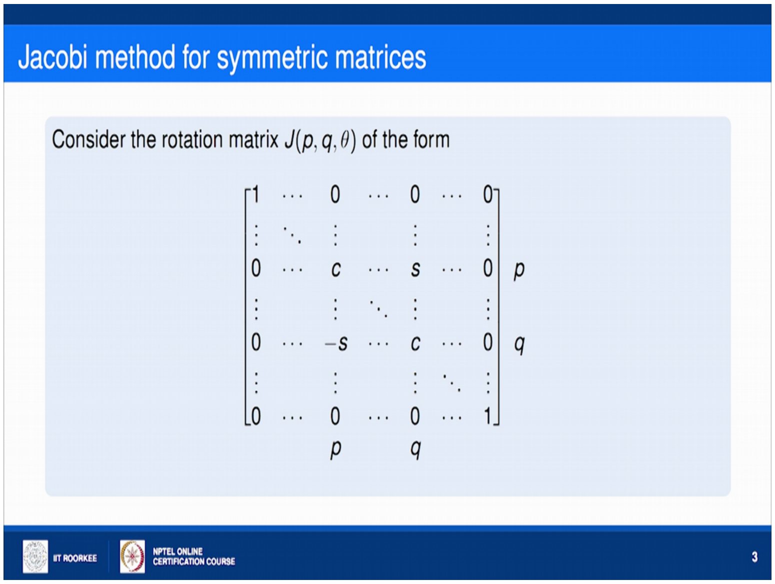


So these are some examples of rotation matrix in 2D and 3D space 2D plane and 3D space, but suppose a n by n matrix is given to us, then a n by n rotation matrix let us define denote it by J(p, q, theta). So here we want to put the cos and sin terms in pth rows and qth row pth column and qth column and this will be of the form this one. So except these two rows rest of row will be like (1,0,0,0 0,1,0,0 like that in this two rows we will be having the terms of cos theta and sin theta like here in pth row I mean 0,0,0,0 then at the diagonal of pth row and pth column it will be cos theta at the intersection of pth row with qth column, it will be sin theta. Similarly at the intersection of qth row pth column will be minus sin theta and the intersection of qth row with qth column will be having term cos theta.

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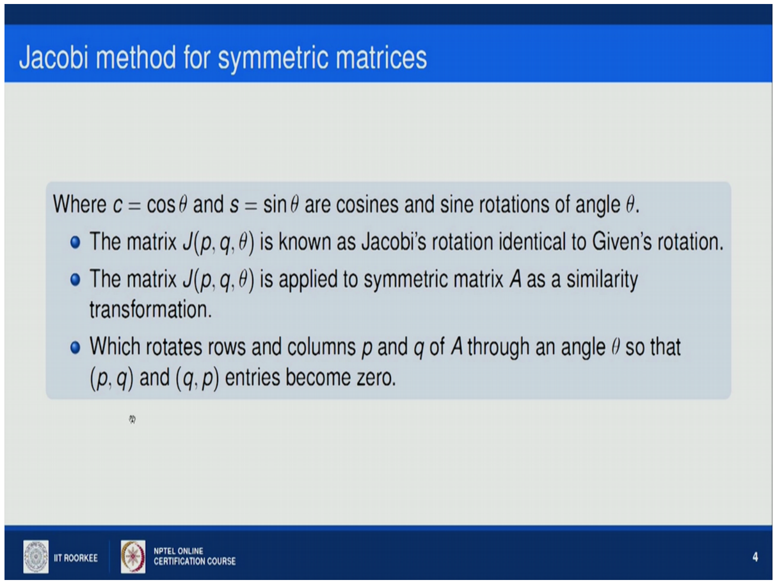


So let us denote this cos theta by C and sin theta by S, then the matrix J(p, q, theta) is known as Jacobi rotation matrix or given rotation matrix. The matrix j(p, q, theta) is applied to symmetric matrix A as a similarity transformation and once, we applied to A, this will rotate row and columns p and q of A through an angle theta so that (p, q) and (q, p) entries become zero. So these are two of diagonal entries (p, q) and (q, p) and these two entries will become 0 and I told you this method is applicable only for symmetric matrices. So hence (p, q) will be (q, p).

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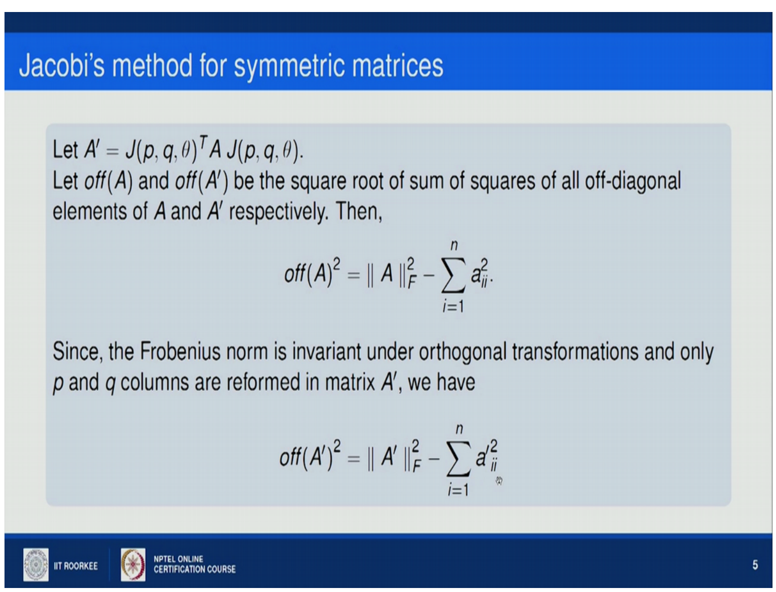

So let us denote this the similarity transformation like this, we are having pre-multiplication of J transpose and a post-multiplication of J with matrix A and I am getting my next matrix A dash. So A dash is J transpose A into J, where J is a rotation matrix with the p and q and by an angle theta. So let off (A) and off (A dash) be the square root of sum of squares of all off-diagonal elements of A and A dash respectively. Then off (A) square will be frobenius norm of (A) muns square of the frobenius norm minus i equals to 1 to n aii square. So what we are doing we are taking square of all the elements and we are subtracting diagonal elements. Since, the frobenius norm is invariant under orthogonal transformations and only p and q columns are reformed matrix A dash. So we can have the sum of squares of off diagonal elements of matrix A dash equals to square of the frobenius norm of A dash minus square of the diagonal elements of A dash.

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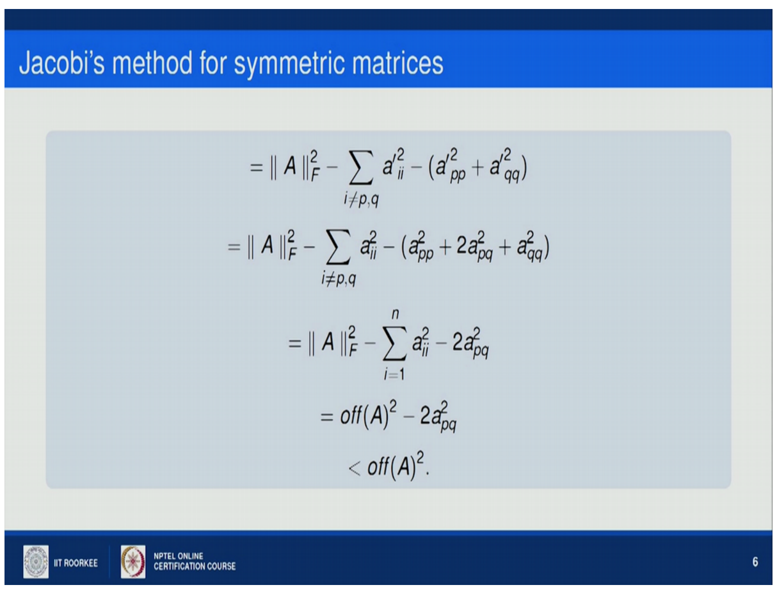


This equals to a square of the frobenius norm of A minus i not equals to p,q a dash ii square, because there will be change only in p and q elements pth row (p, q) and (q, p) elements minus a dash pp square plus a dash qq square, this will become this particular term and finally, it comes out that the square sum of the squares of off diagonal elements of A dash will be less than sum of a square of off diagonal elements of A, it means the square elements are the elements from off diagonal are eliminating going towards zero and this is a basic motivation for the Jacobi method.

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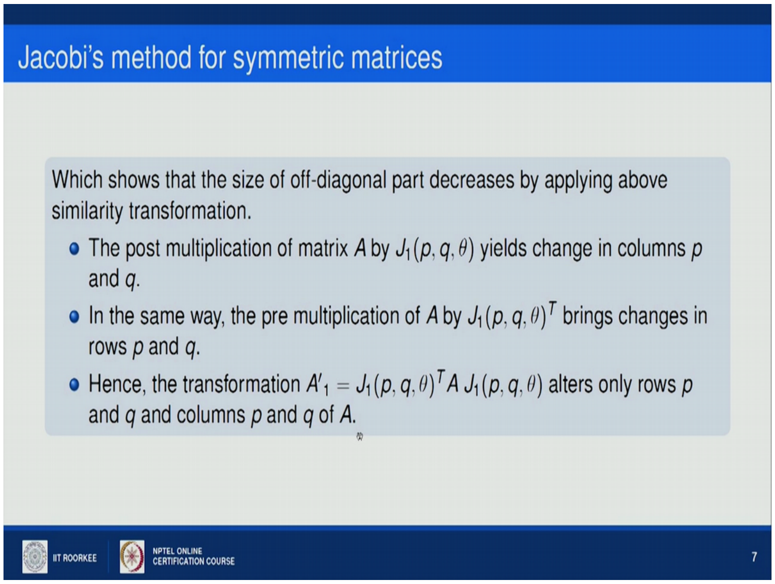


So here it shows that the size of off diagonal per decrease is by applying Jacobi transformation. The post-multiplication of A by J1 will change in columns p and q in the same way the pre-multiplication of J1 transpose bring changes in rows p and q. Hence, the transformation A dash equals to J1 transpose A j1 alters only rows p and q and columns p and q of A and there is no change in rest of the rows and columns.

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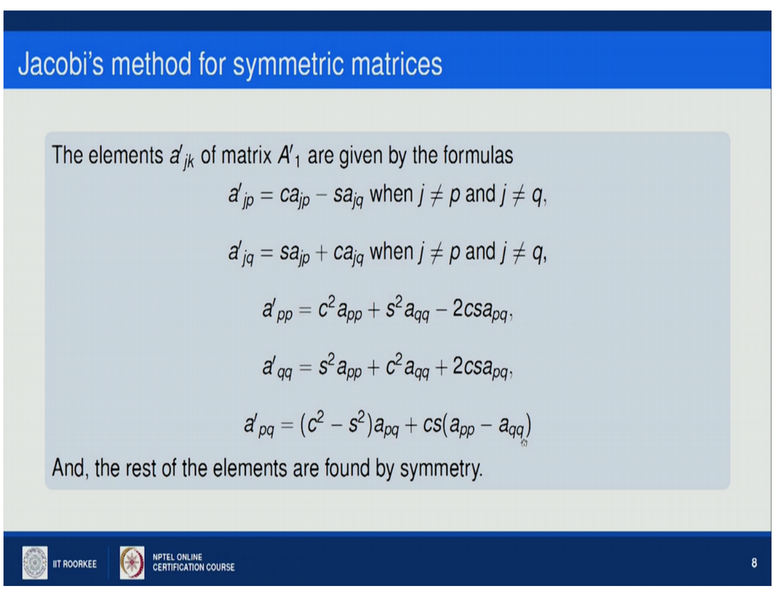


If we see this transformation in n by n set up, so what will be the relation between matrix A and A dash? So the elements said a jk of the matrix A dash are given by the formula. So when j not equals to p and j not equals to q, a dash jp will be C times ajp, please not that here, C is cos theta minus Sajq. Similarly a dash jq is given by Sajp plus Cajq when j not equals p and j not equals to q. The diagonal entry in the pth row is given by a dash pp equals to C square app plus S square aqq minus 2 CSapq. Similarly the diagonal entry in the qth row is given by S square app plus C square aqq plus twice of c into S into apq. The elements on the intersection p th row and qth column or qth row and pth column is given by that is a dash pq c square minus S square apq plus CSapp minus aqq and please, here note that the last and rest of the elements we can find by the symmetry, but here please not that this element. This is the off diagonal element in the pth row and qth column or qth row and pth column and we want to make them zero.

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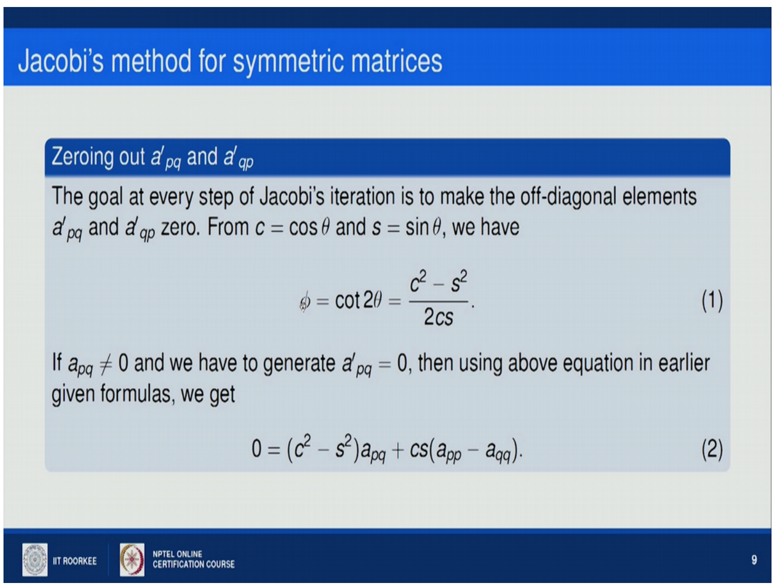


So if I make them zero, I can write zero equals to C square minus S square apq plus CS app minus aqq like this. Given by equation 2, moreover the goal of every step of Jacobi equation is to make the off diagonal elements a dash pq and a dash qp zero. So from this we can have phi equals to cot 2 theta C square minus S square. So it will be cos 2theta minus twice of cos theta into sin theta that is sin 2theta. So this is phi.

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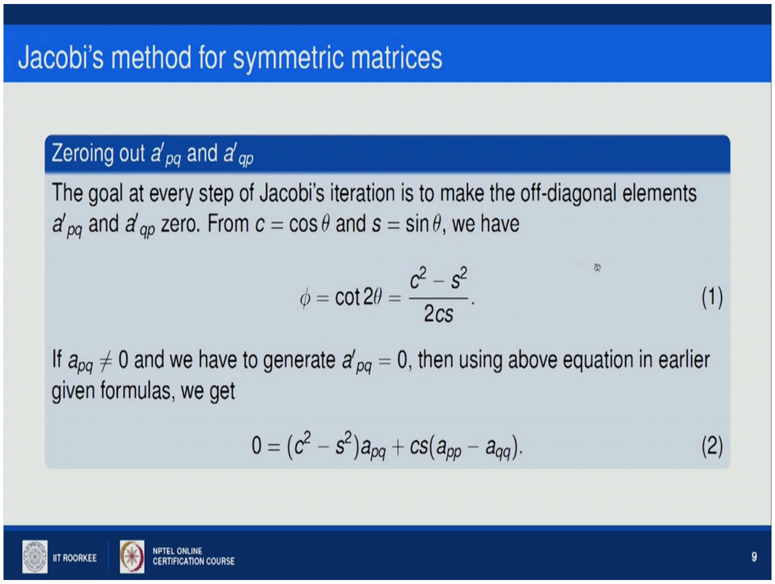


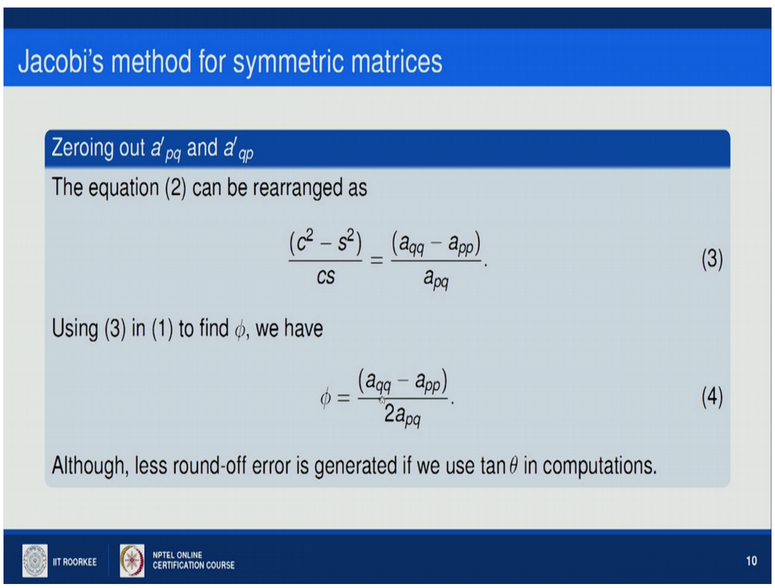

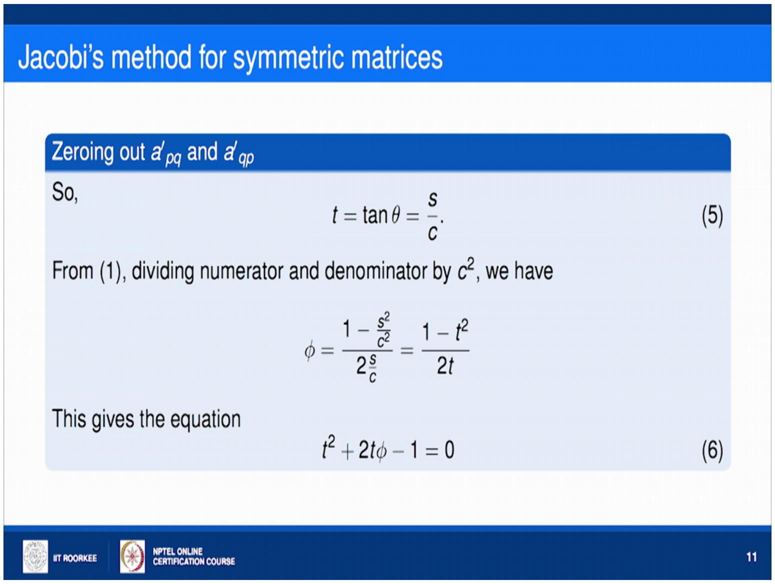


Now from equation 1 and 2, I can write C square minus S square upon CS equals to aqq minus app upon apq. So from here, I can write phi equals to instead of this I can write this term. So aqq minus app upon twice of apq. So this is equals to cot of 2 theta, so tan of 2theta will become this value 2apq upon aqq minus app and theta will become 1 by 2 into 1sintan inverse twice apq upon in the denominator we will be having aqq minus application; however a less round of error generated if we use tan theta in computations like let us assume t equals to tan theta which is sin theta upon cos theta. So from equation 1 that is my equation 1. So I divide the numerator and denominator by C square, so it will become 1 minus S square upon C square upon twice of S upon twice of S upon C. So 1 minus t square upon 2t, because S upon C is t and tis gives a quadratic equation t square plus 2t phi minus 1 equals to zero.

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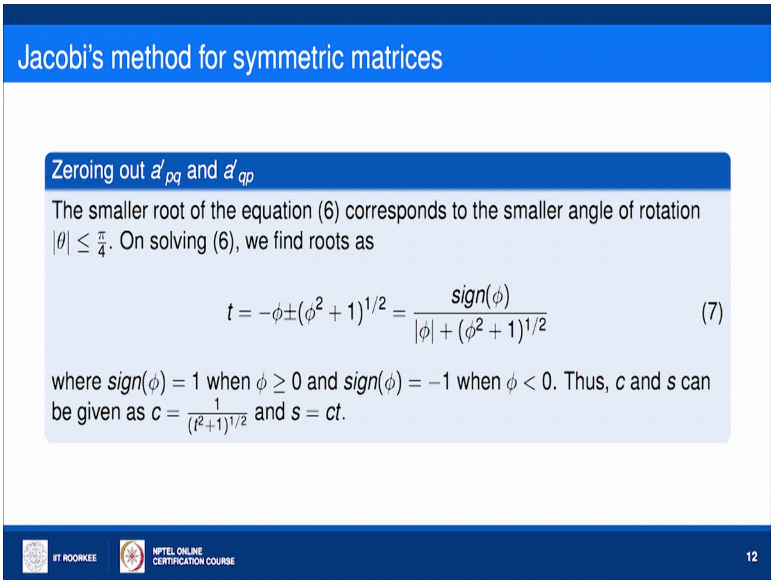


The roots of this quadratic equation is given by minus phi plus minus square root of phi square plus 1 and that will be sign of phi upon absolute value of phi plus square root of phi square plus . Here, sign of phi is 1, when phi is non-negative and it is minus 1, when phi is a negative number. Thus, once we get t we can calculate C and S by this formula C will be 1sign upon square root of t square plus 1 and S will be C times t.

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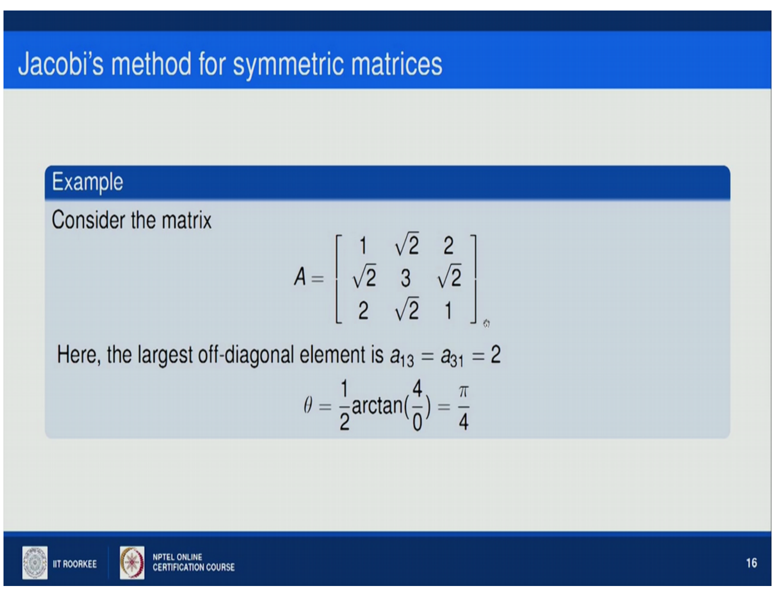

Let us take an example of this method. Consider the matrix this. So this is a 3 by 3 matrix having elements (1, square root 2, 2),( root 2, 3, root 2), (2, root 2to, 1) and let us solve this matrix or let us apply.

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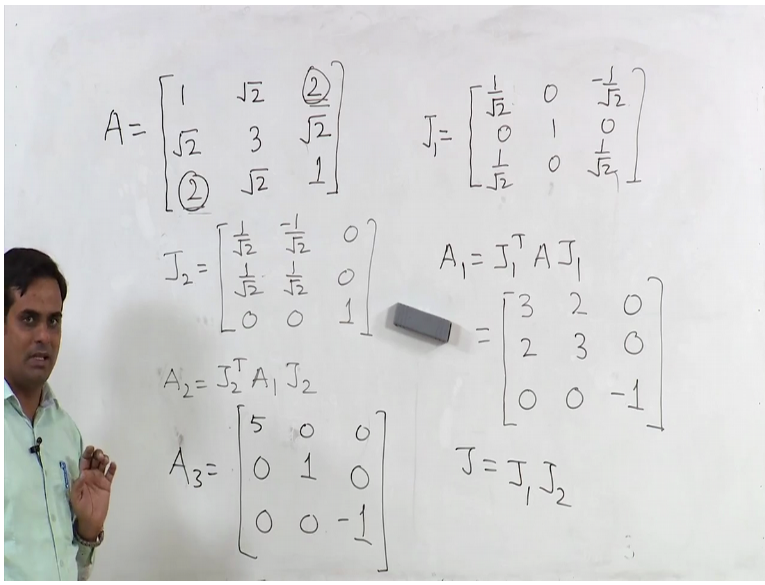


So let us take a 3 by 3 matrix and find out the eigenvalue of this matrix as well as eigenvector using Jacobi method. So matrix is (1, square root 2, 2), (square root 2, 3, square root 2), (2, root 2, and finally 1). So it is a real symmetric matrix. Now first of all, in Jacobi method we will look for the off diagonal element having the maximum absolute value, because we will perform a similarity transformation to make it 0. So if I look at the off diagonal elements the biggest off diagonal elements is this one that is a31 equals to a13 equals to 2. It means my p is 1, q is 3. So it means app is 1, aqq is 1 and apq equals to aqp equals to 2.

So here if I calculate theta, theta will be 1 by 2 tan inverse twice of apq upon aqq minus app and it is coming out pi by 4, because here it will be 0 in the denominator. So tan inverse infinity and so it will be pi by 2 into 1 by 2, it will become pi by 4. This is one of the way of calculating theta, we can calculate it using the t that quadratic equation, first by calculating phi then t and then from t we will directly calculate cos theta and sin theta as I told you. Basically that is more accurate compare to this one.

Now so define matrix J. So J will be cos pi by 4 that is 1 by root 2, 0, that is cos pi by 4 minus sin pi by 4, sin pi by 4 and cos pi by 4. This is J1. Calculate A1that is J1 transpose A into J1 and it comes out 3, 2, 0,2,3,0 and finally 0, 0 ,-1. So please compare A and A1, just see these two elements, I have made these two elements 0, okay just by applying this Jacobi rotation.

Now to make this particular matrix as a diagonal matrix what I need to do, I need to perform make these two elements are also zero. So for making these two elements 0, again I will do that so my p is 1 and q is 2 and a12 equals to a21 equals to 2 that is apq equals to aqp equals to 2, app equals to aqq equals to 3 these two diagonal elements again if I calculate theta, theta again will be 1 by 2 into tan inverse twice of apq that is 4 upon 0. So it is coming again pi by 4 and my next matrix will become J2. So J2 will become (1 by root 2, -1 by root 2, 0), (1 by root 2, 1 by root 2, 0),(0,0,1). If apply again this particular similarity transformation on A, the matrix obtain in this one, I got the matrix A3, A3 will be (5,0,0), ( 0,1,0) and (0,0,-1). So please look at A3, A3 is a diagonal matrix and hence, the eigenvalue of A3 is 5, 1, -1 and so these are the eigenvalues of A, because I obtained A3 just by applying the two similarity transformations on A. Now the eigenvector of this matrix A is given by the product of J1 into J2 and I will tell you while I am writing. So eigenvectors will be columns of this matrix that is J1 into J2. So I got the eigenvalue, I got the eigenvectors and hence, I will be able to solve this particular problem for finding eigenvalues and eigenvectors using Jacobi method.

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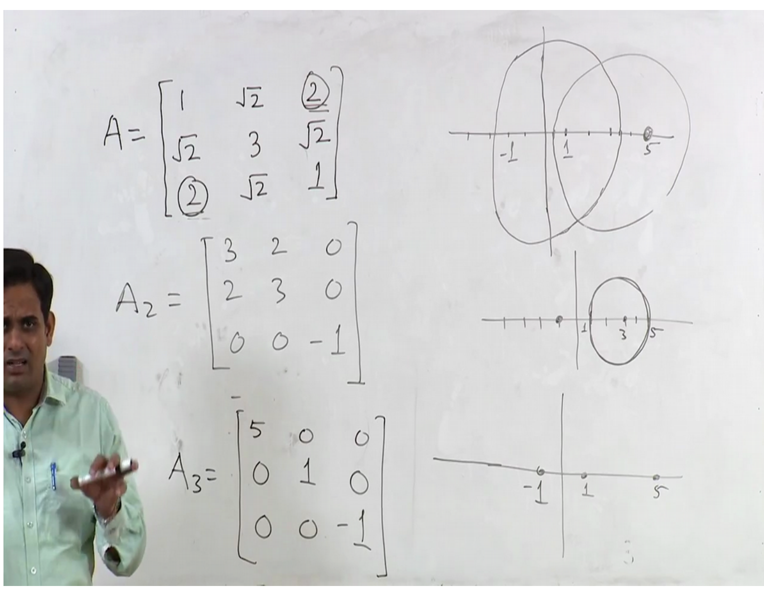


Now actually what is happening let me tell you few things about this method. Let me write a 2 here, which I obtained just by applying J1 transpose A J1 on A. So it was something like (3, 2, 0)

(2, 3, 0) (0,0, -1). So from here I can here and from here I can here. Now if you remember previous lecture there we talk about gershgorin theorem. So let us see how eigenvalues are changing or gershgorin circles are changing in different iterations of Jacobi method.

So if I talk about this and this first disc is having center at 1 and radius is 2 plus root 2, so something 3.4, so 2, 3, 4,-1,-2,-3,-4. So this will be the disc from the first row, from the second row, center at 3 and radius is 2 times root. So it will be something radius at 3, so 1, 2, 3 and center at 2 times root 2, so something 2.8. So it will be like this and the third row is again center at 1, radius is 2 plus root 2, that will be just your first disc and as I told you eigenvalues are 5, 1, -1. So one of the eigenvalue will be here at 5, one will be here 1 another one at -1.

So gershgorin theorem holds for this matrix; however here what I am having one of the eigenvalue is the common area of two disc. The Gershgorin discs are not disjoint here if I say in this matrix so let us say 1,2,3,4, -1,-2,-3,-4. So first is lambda minus 3, so center at 3 and radius is 2. So this one, so 1, 3,4,5, second is giving center at 3, radius 2, which is similar to first one and the third one is having at center at minus 1 and radius is 0. So in the second matrix which I got in the first iteration of Jacobi method what is having the disc as overlapping each other and one is disjoint.

Now see the third matrix that is A3. Here, simple I am having one of the eigenvalue is here at 5, another one at 1 and the third one at -1 and these are the gershgorin disc for the given problem. So basically what I am doing by applying the Jacobi rotations to the given matrix, I am reducing the size of gershgorin disc in such a way they become disjoint or they (())(27:09) to a single point that is your diagonal matrix we are getting here.

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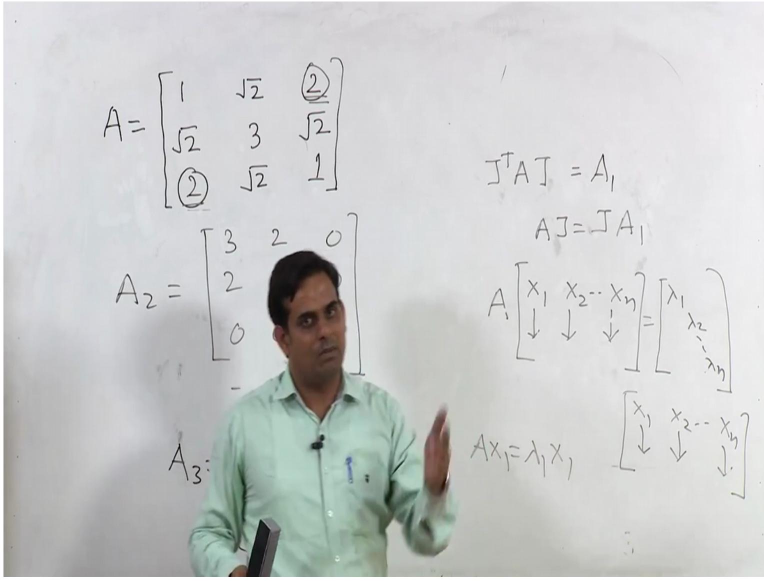


Another thing I want to tell you here why I am saying that the product of J1, J2 like that will give me the eigenvectors. What is happening let us say A1 is the eigen matrix which is having diagonal entries as the eigenvalue and this I am getting just by applying a Jacobi rotation on J. Now as you know that J is an orthogonal matrix, it is a rotation matrix. So J transpose will be equals to j inverse. So what happens if I multiply both sides, so it will become, I can write like this or this is identity. So A into let us say x1, x2…xn, are the eigenvectors of A and as I told you they will be the columns of matrix J. So I have written like x1 is the first column of J, x2 is the second column of J; xn is the third column of J.

Now look here what are these A1 is a diagonal matrix having the eigenvalues of A. So it will be something lambda1, lambda2, lambda n. this matrix into again J, so J is x1, x2, xn. So when I multiply this with first column what I will get Ax1 equals to lambda1 x1, which is the first eigenvalue and corresponding eigenvector for A from the Ax2 will be lambda2 x2, Axn will become lambda n xn. So hence this matrix J is having eigenvectors of A as its column, which I claimed earlier that you can see from here.

So in this lecture we have learn a method for calculating eigenvalues and eigenvectors just by using the similarity transformations or a sequence of a series of transformations and those transformations are formed just using the rotation matrices, in such a way that the off diagonal elements become zero. In the next lecture we will learn another method for finding the largest eigenvalue and its corresponding eigenvector for a given matrix and that particular method is called power method. So thank you very much for this lecture, by see you in the next lecture.