Numerical Methods
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Lecture 11
Introduction to Eigenvalues and Eigenvectors

Hello everyone, so today we are going to start module 3 of this course and in the first lecture of this module, I will give an introduction to eigenvalues and eigenvectors. Why I am giving this introduction to you, because in this unit, we will focus on, finding the eigenvalue and eigenvectors using the numerical methods for a given matrix. Basically in real life problems quite frequently, we need to find out eigen values or eigen vectors of a matrix for getting some idea about the system and hence, it is very important, to know about these two things that is eigen values and eigen vectors for a given matrix and how to apply a numerical methods for finding eigenvalues and eigenvectors of a given matrix.

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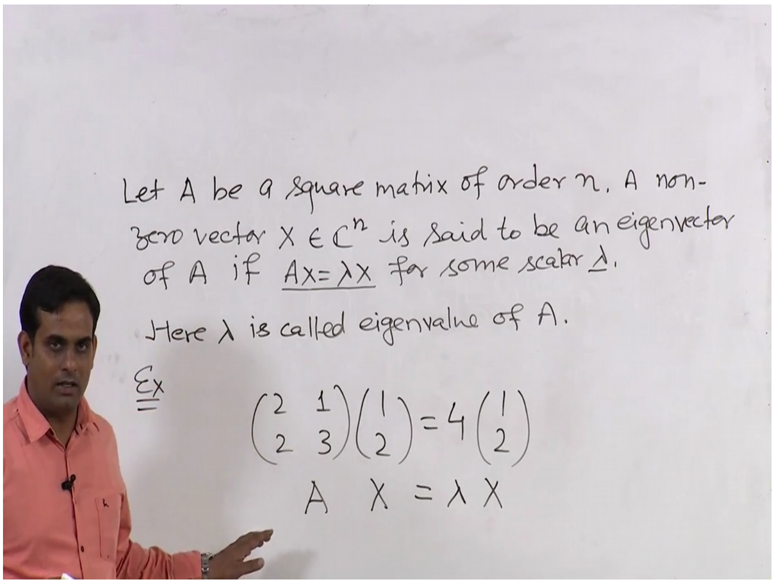

Let me define the eigenvalue and eigenvectors. So let A be a square matrix of order n. So it means, this matrix say is the element of the vector space Cm by n, then a non-zero vector X belongs to Rn or more generally, you can write it belongs to Cn is said to be an eigenvector of A. If AX equal to lambda X for some scalar lambda, notice this that I will said at X is an eigenvector of A, first of an it should be a non-zero vector, seconding it should be satisfy this particular condition that is AX equals to lambda X.

Now let us talk about this scalar lambda. Here, lambda is called eigenvalue of A and hence, we can say that X is an eigenvector corresponding to the eigenvalue lambda, for example, take a 2 by 2 matrix (2 1 2 3), so if I take a vector let us say (1 2). So this I can write it 4 times (1 2). So this is my matrix A, this is the eigenvector X. This is equals to lambda times X. So it means this particular vector (1 2) is an eigenvector of this matrix A and 4 is an eigenvalue of A and this eigenvector is an eigenvector corresponding to eigenvalue 4.

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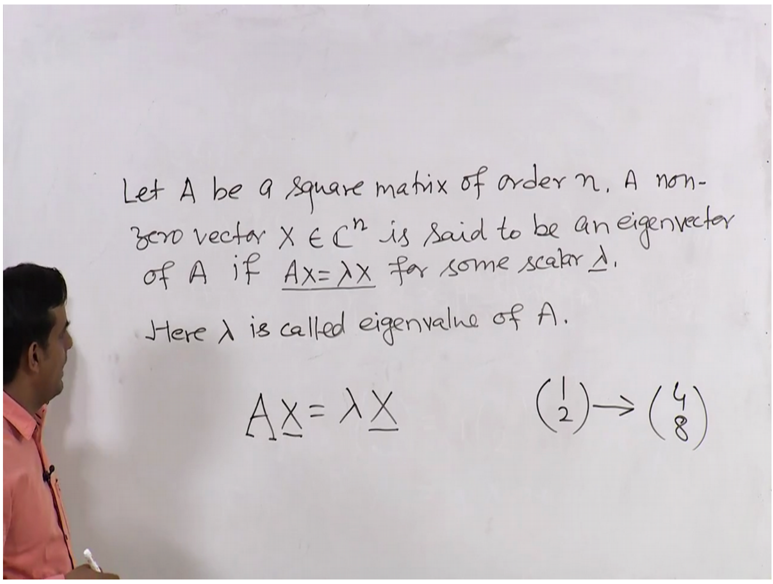

Now what this equation is telling, what we are having X is a vector and A is a matrix. So every matrix is a transformation, basically linear transformation. So what we are doing to be here applying a linear transformation on a vector X and we are getting a scalar change in the vector X scalar time change. So either the vector will expand or vector will (())(5:50). This will depend on the value of lambda, example for example, in the earlier one, I was having initially vector (1 2) and after applying the transformation, it is becoming 4 times (1 2) that is it is going to (4 8). So there was a magnification in the vector of 4 times. So hence this is the another interpretation of eigenvalue and eigenvector of a transformation or matrix A.

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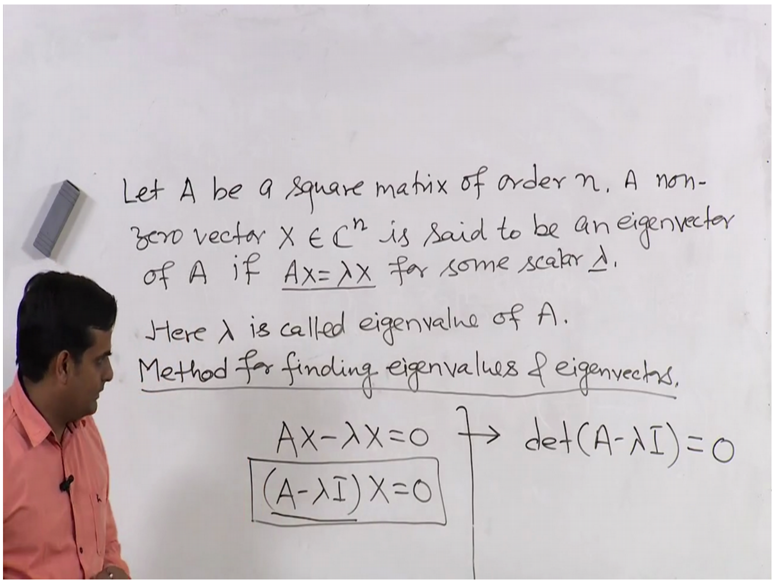

Now how to find eigenvalues and eigenvectors of a given matrix? So methods for finding eigenvalues and eigenvectors. Now look at this definition of eigenvectors. Here, I am saying that X is a non-zero vector and it is said to be an eigenvector of A, if A times X equals to lambda X. So A into X equals to lambda X can be written as AX minus lambda X equal to 0 or this I can write this A minus lambda I equals to 0, where I is the identity matrix of the same order as A.

Now from here you can see, we are having here system of homogeneous equation with where we are having n equations and n unknown; I am saying that X is a non-zero vector. So if X is a non-zero vector means, this system is having non-zero solutions and if I talk that this system is having non-zero solution, it means that is the null space of this particular transformation A minus lambda I, is having dimension more than 0. Now if this is having the non-zero solution, it should be having go and place then 1 less than n. So rank of this A minus lambda I should be less than n, it means determinant of A minus lambda I should be 0, because if rank is less than n.

Now if I get the determinant of A minus lambda I, it will be a polynomial of n degree in lambda and the zeros of that particular polynomial are the eigenvalue of matrix A. So using this concept we can find the eigenvalues of a matrix means what you need to do? You have to write the matrix A minus lambda I, you have to find out the polynomial, which is coming from the determinant of A minus lambda I and solving this equation, now linear equation that is a polynomial in of degree and in lambda equals to 0 you will find the eigenvalues of A.

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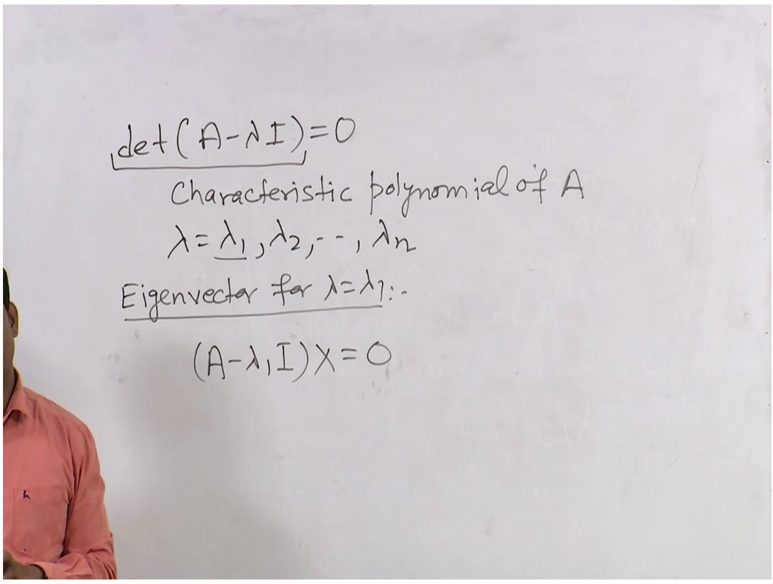

Now as I told you by solving, this equation you can get the values of lambda for which determinant of A minus lambda I equals to 0 and these values of lambda is called the eigenvalues of A. This particular polynomial is called the characteristic polynomial of A and hence, eigenvalues are also called the characteristic values of the given matrix. Now once you find out the eigenvalues, let us say eigenvalues are coming like this lambda1, lambda2, lambda n, then what we need to do? We need to find out the eigenvectors corresponding to each eigenvalue. So eigenvector corresponding to eigenvalue lambda equals to lambda1 can be calculated just by solving the homogeneous system of equations A minus lambda1 IX equals to 0.

Since, we have choosed such a lambda1 for which, the system is having non-zero solution and hence we will get the get a non-zero vector X as a solution of this system and then non-zero vector X will be the eigenvector corresponding lambda equals to lambda1. Then similarly we can find the eigenvalues, eigenvectors corresponding to other eigenvalues. So this is the classic way of finding eigenvalues and eigenvectors of a given matrix.

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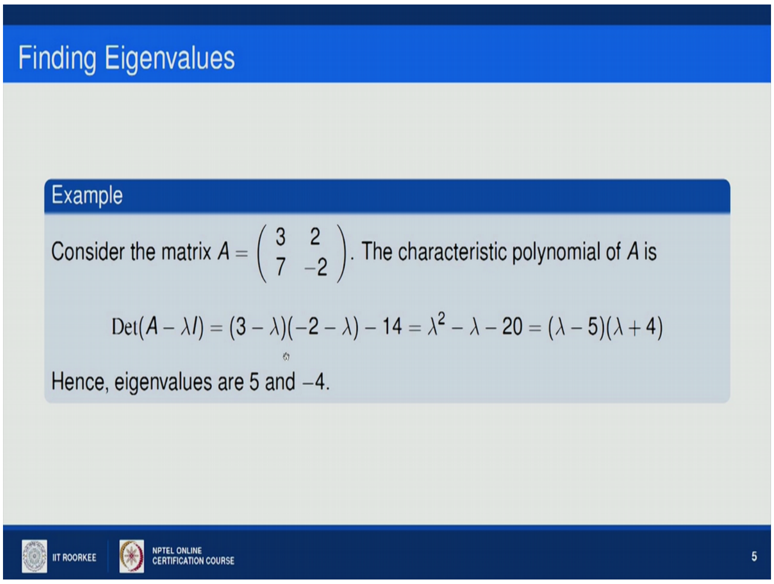

So let us take an example of it. Let us consider tis 2 by 2 matrix. So first row is (3 2 7 -2), we need to find out the eigenvalues and eigenvectors of this matrix. So the characteristic polynomial of A is determinant of A minus lambda I, which becomes 3 minus lambda minus 2 minus lambda 2 and 7 into minus 14, lambda square minus 20 and zero of this polynomial is 5 and minus 4. So hence the eigenvalue of this matrix is 5 and minus 4.

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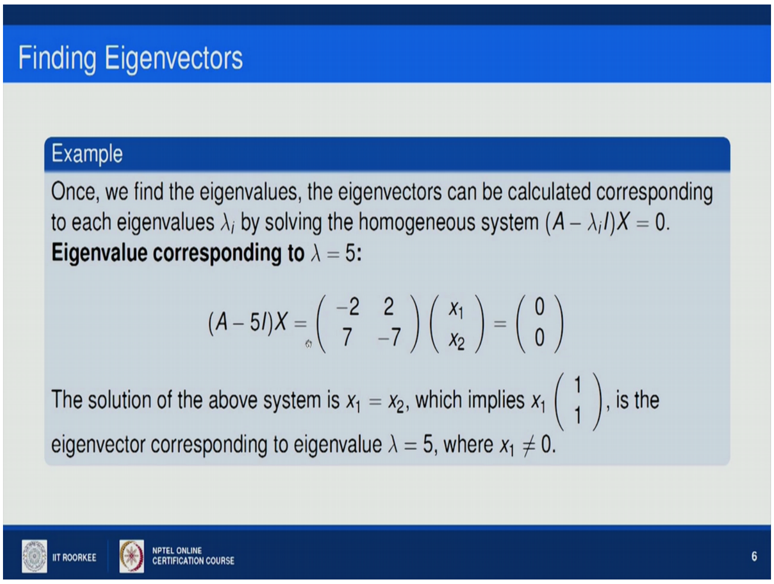

Once we find the eigenvalues we need to calculate eigenvectors. So eigenvector corresponding to lambda equals to 5 is given by A minus 5I into X equals to zero that is, we got 2 equations, one is minus 2X1 plus 2X2 equals to 0, another one 7X1 minus 7X2 equals to 0, basically both are the same equation. They linearly dependent and the solution of this equation is X1 equals to X2. So we choose X2 as 1, so X1 will be 1. Hence (1 1) is an eigenvector corresponding eigenvalue lambda equals to I.

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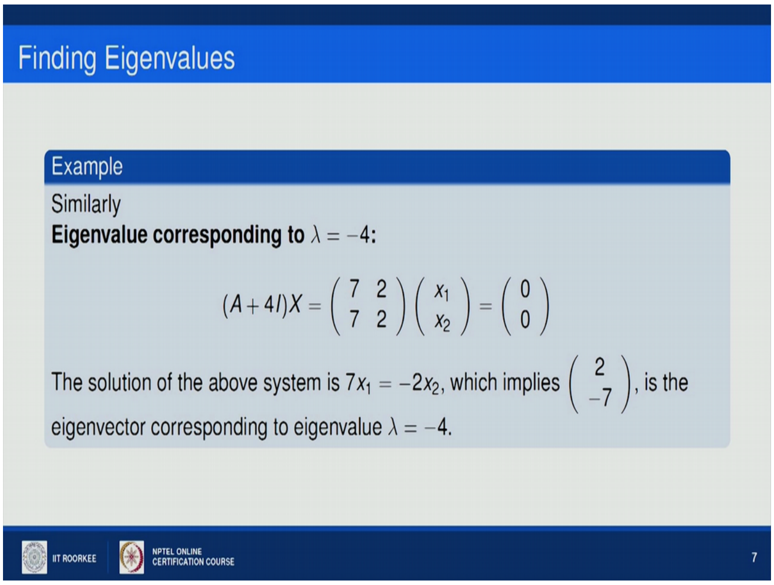

Similarly, we can calculate the eigenvector corresponding to lambda equals to minus 4 and it is coming out 2 and minus 7, you can verify that both of these eigenvectors with respect to eigenvalue 5 and minus 4 satisfy the relation X equals to lambda X.

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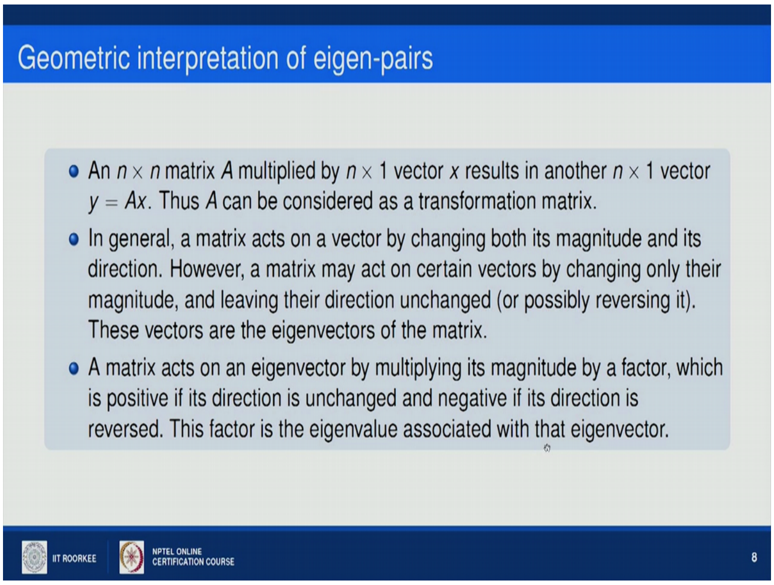

Now geometrically, an n by n matrix A we are multiplying it by n by n vector X. So resulting into another n by 1 vector Y equals to AX. Thus, A can be considered as a transformation matrix, which is transforming a vector into another vector of the same dimension. In general, a matrix X on a vector by changing both its magnitude and its direction; however a matrix may (())(14:54) certain vectors by changing only their magnitude and leaving their direction unchanged or possibly reversing the direction. These vectors are the eigenvectors of the matrix, okay.

So if we are having a matrix, it is a we are applying this matrix on a set of vectors for some of the vectors, it will change magnitude as well as direction; however for certain vectors what it will do, if you are having n by n matrix, there will be a vectors n or less than n for which just what it will do, linearly independent vectors. It will only change the magnitude and those vectors will be the eigenvectors of the matrix. So in this way we can differentiate or we can take out the eigenvectors of a matrix from the set of vectors.

So a matrix sets on an eigenvector by multiplying its magnitude by a factor, which is positive if its direction is unchanged and negative if its direction is reversed. This factor is eigenvalue associated with that eigenvector. So you can see, X equals to lambda, we are A is acting on X and we are getting lambda X. what is lambda? Lambda is just a scalar, so what will happen if lambda is some positive number? What will happen the direction of the vector will never change, because if it vector is X1 X2, it will become lambda times X1 lambda time X2.

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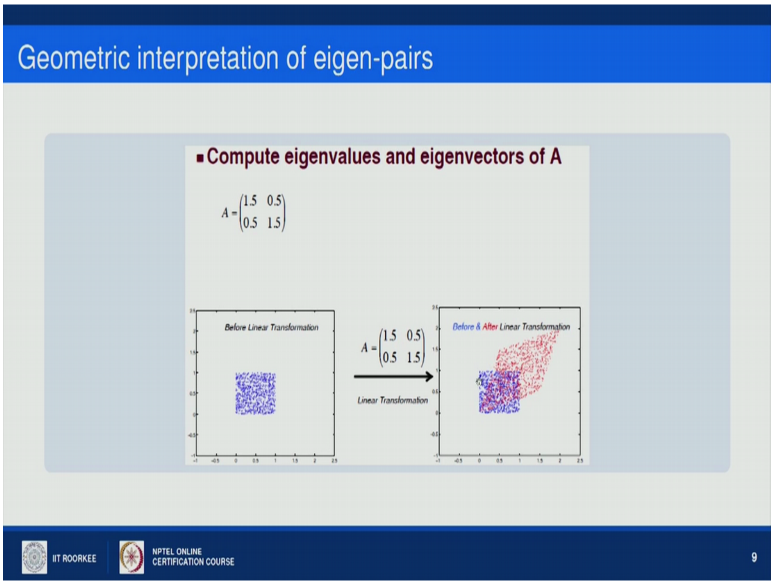

If lambda is negative number, it will become just the direction of the vector becomes in the reverse direction; however magnitude will certainly changed, just take this beautiful example here we are having this 2 by 2 matrix A, I am applying this matrix A on the set of points. So after applying this transformation or this matrix on this set of points, I am getting these red points. So blue points are before transformation and red cluster of points are after transformation. So what is happening, it is changing. This square say point to in this shape.

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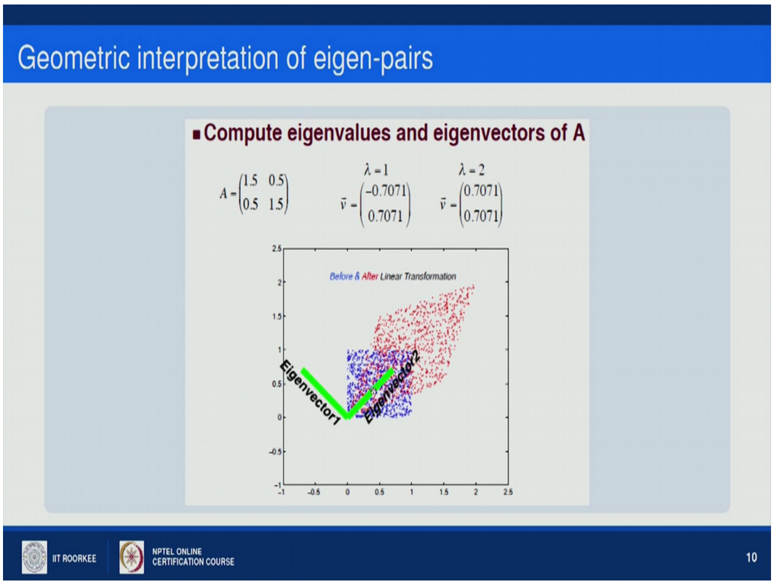

Now if I calculate the eigenvalues and eigenvectors of this matrix. Eigenvalues are coming 1 and 2 corresponding to 1 eigenvector is minus 0.71 and 0.7071 means it is something aligned in the direction of Y equals to X and here, it is coming same, so it is aligned in the direction of Y equals to minus X, sorry this is the line in direction of y equal to minus X, this is aligned in the direction of direction of Y equals to X.

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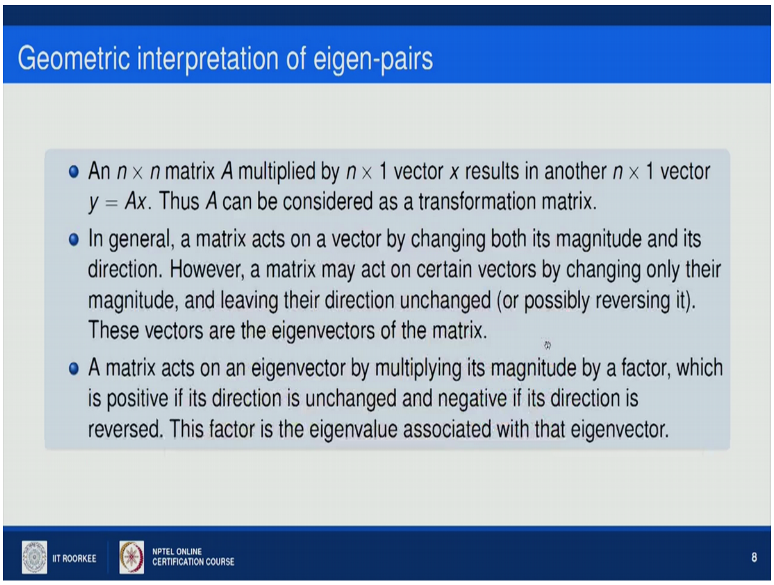

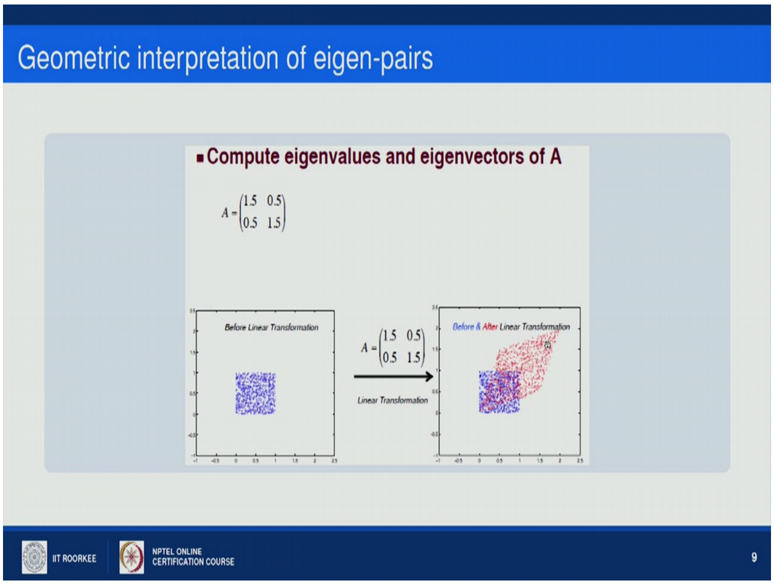

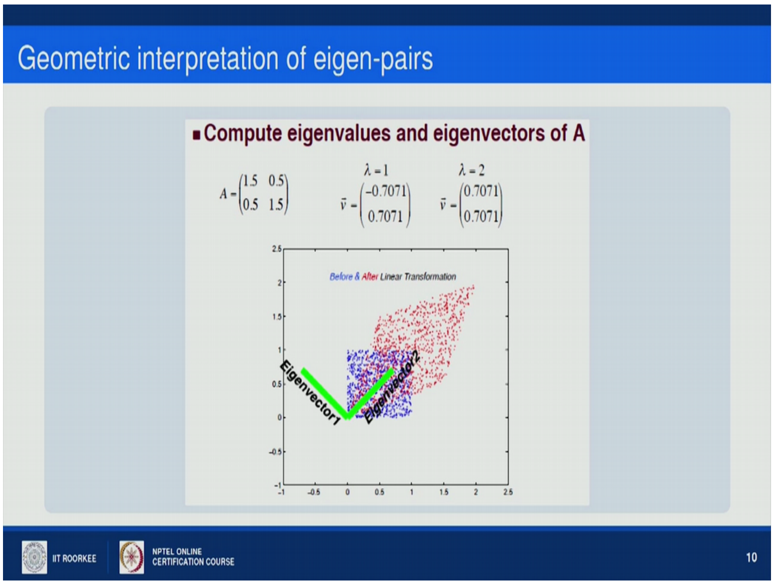

And hence, you can see what are having, this change is the maximum in the direction of Y equals to X and what is the scale of the change. This is scale of change is just double so what we I want to say that in the direction of eigenvector corresponding to largest eigenvalue we are having the maximum change and change is just multiple of that particular eigenvalue and no change in the direction of this Y equals to X, because here eigenvalue is 1. So 1 into that vector remains the same.

So here the role of eigenvector corresponding to the biggest eigenvalue becomes very important for analyzing patterns or in when you are talking about data analytics or pattern classification on these areas. So we will learn some numerical methods to finding to find out the maximum eigenvalue and corresponding eigenvector of a given matrix in the coming lectures.

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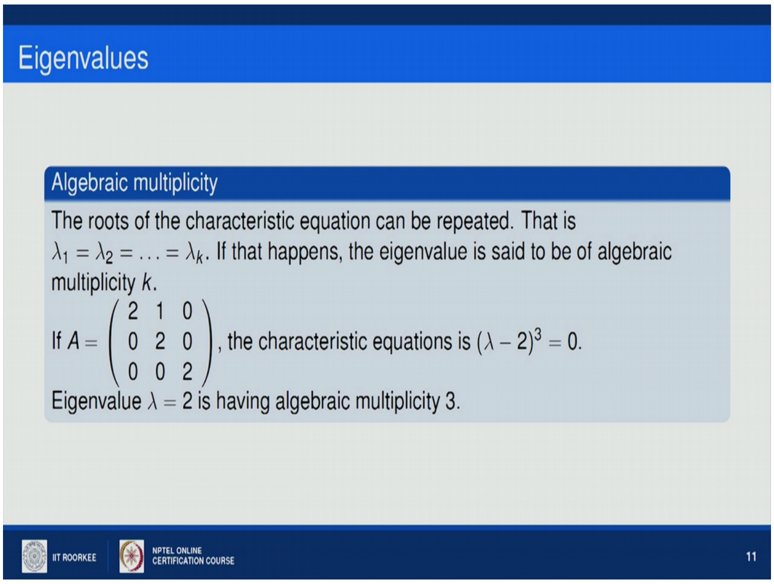

Now as I told you the eigenvalues are the roots of the characteristic polynomials and a polynomial can have repeated roots. So for example, I can have lambda1 equals to lambda2 equals to up to lambda k. So if that happens, the eigenvalue is said to be of algebraic multiplicity k. So what is algebraic multiplicity of an eigenvalue? The algebraic multiplicity is the number of times, it is repeated, for example, if I am having a matrix 3 by 3 order matrix A and eigenvalue is (2, 3, 5). So hence all the eigenvalues are having algebraic multiplicity 1. If I am having eigenvalue as (2, 2, 5) so here eigenvalue 2 is having the algebraic multiplicity 2 and 5 is having algebraic multiplicity 1. If I am having eigenvalue as (2, 2,2), so 2 is repeating 3 times, then algebraic multiplicity of eigenvalue lambda equals to 2 is 3.

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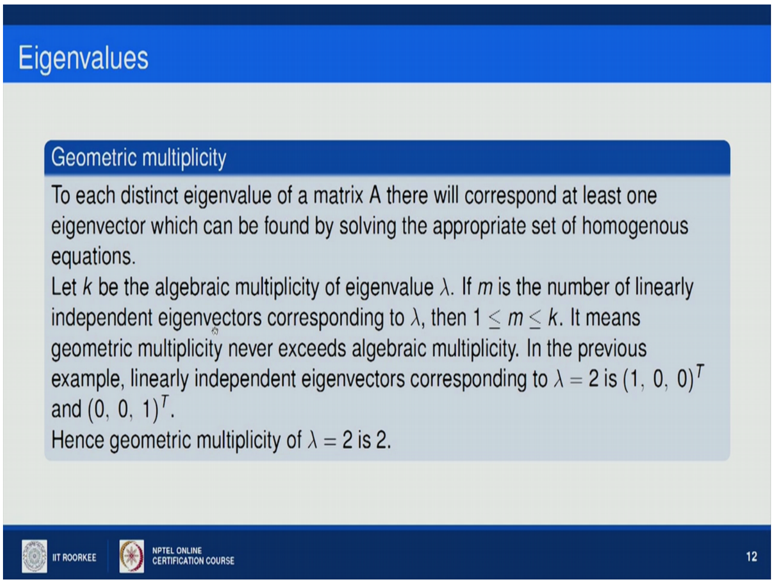

Now for each distinct eigenvalue of matrix A there will be correspond at least 1 eigenvector, which can be found by solving the appropriate set of homogeneous equations. Let k be the algebraic multiplicity of eigenvalue lambda. If m is the number of linearly independent eigenvectors, please note that linearly independent corresponding to eigenvalue lambda, then 1 will be always lie between 1 to k means the number of linearly independent eigenvectors corresponding to an eigenvalue will be always less than equals to the algebraic multiplicity of the eigenvalue and this particular number of linearly independent eigenvectors corresponding given eigenvalue is called the geometric multiplicity of the eigenvalue.

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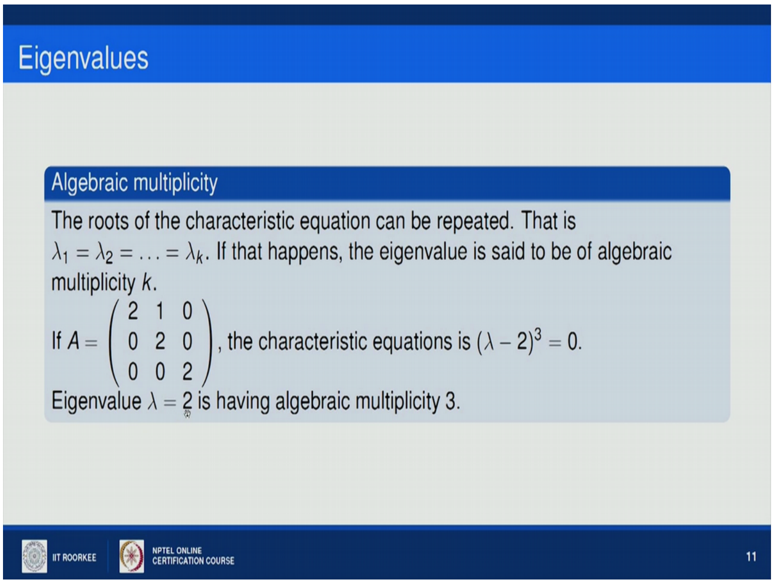

So I want to say that, geometric multiplicity never exceed algebraic multiplicity, like if you take this example, it is a 3 by 3 matrix. It is a upper triangular matrix. It is having characteristic equation lambda minus 2 cube equals to 0. So hence, it is having root 2 repeated 3 times algebraic multiplicity of eigenvalue lambda equals 2 is 3.

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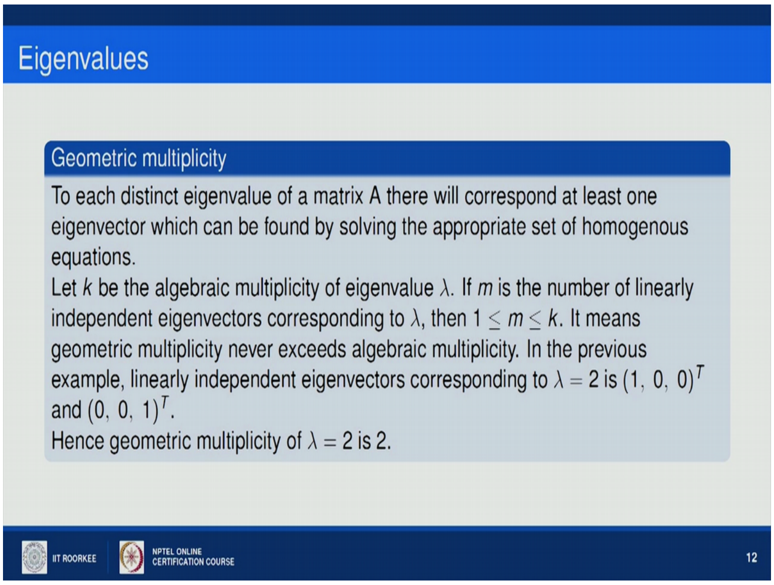

If I calculate the eigenvector corresponding to lambda equals to 2, then I am getting the eigenvector (1, 0, 0) and (0, 0, 1) so hence geometric multiplicity of eigenvalue 2 is only 2 whereas algebraic multiplicity is 3.

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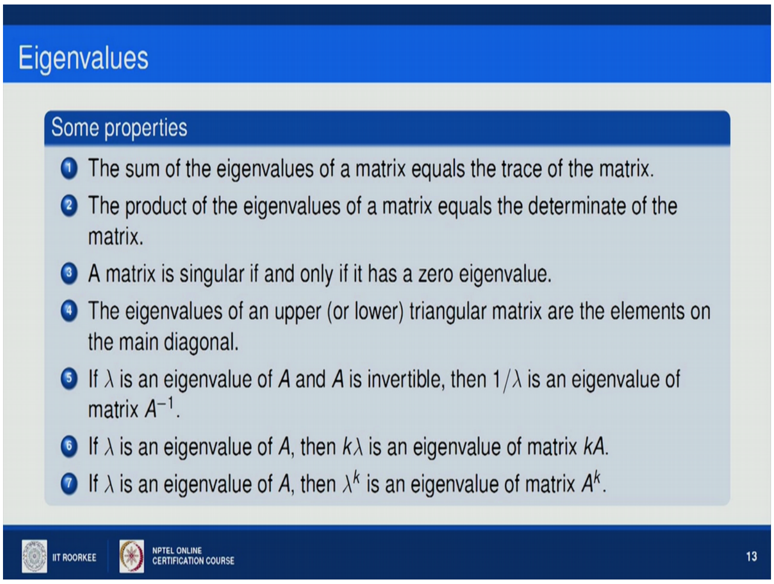

Let me tell you some properties of eigenvalues and eigenvectors. So the sum of eigenvalues of a matrix equals to the trace of the matrix. So how can we define that trace of a matrix. Trace is the sum of the diagonal elements of the matrix and hence, sum of eigenvalues equals to sum of diagonal elements of the matrix and that you can see from the characteristic equation very clearly. The product of eigenvalues of a matrix equals to the determinant of the matrix. Hence, if an matrix is having a zero eigenvalue, one of the eigenvalues as zero means the matrix is a singular matrix, because determinant is the product of eigenvalues, 0 is coming there. So product will be 0. Hence, determinant will be 0.

The eigenvalues of an upper or lower triangular matrix are the elements of the main diagonal. Similarly, the eigenvalues of a diagonal matrix are the elements of the diagonal. If lambda is an eigenvalue of A and A is an invertible matrix, then eigenvalue of A inverse will be 1 upon lambda and what will be having the same corresponding to same eigenvectors.

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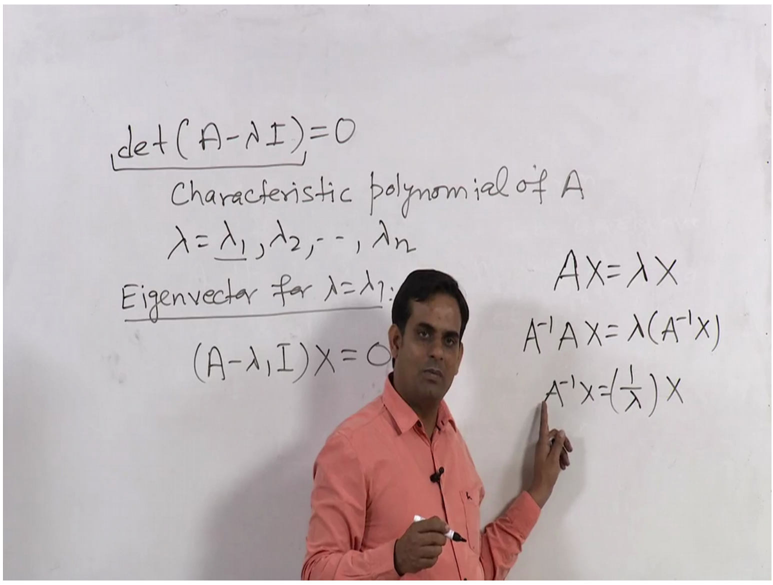

If lambda is an eigenvalue of A, then k lambda is an eigenvalue of matrix kA and this can be in the proof is given very easily, suppose you are having a matrix AX equal to lambda X. now what you do multiply both side by A inverse, keep A inverse exists (())(24:37). So A inverse AX will become lambda times A inverse X and from here, I can write A inverse X equals to 1 upon lambda into X. So if X is an eigenvector of A corresponding to eigenvalue lambda, then X will also be an eigenvector of A inverse corresponding to eigenvalue 1 upon lambda. So if lambda is an eigenvalue of A, then lambda race to power k will be the eigenvalue of A race to power k, for example, if a 3 by 3 matrix is having eigenvalue as (5, 10, 20), then I square of this matrix will be eigenvalue is a 25 that is square of 5, 100 and 400 moreover, it is very important result for the topic which is I am going to introduce you in the next lecture that is the similarity transformation.

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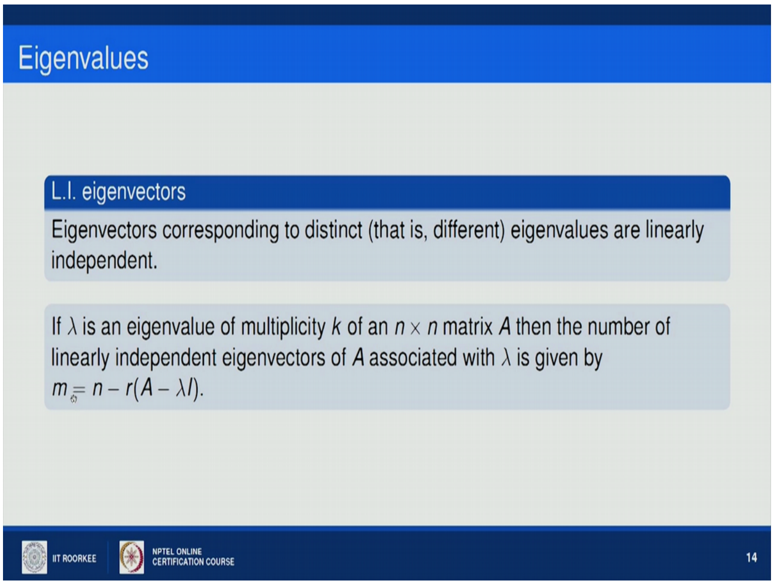

So this result tells us that eigenvectors corresponding to distinct eigenvalues are linearly independent moreover, if we do not have the distinct eigenvalues for a given matrix and sum of the eigenvalue lambda is having algebraic multiplicity k, then the number of linearly independent eigenvectors of A associated with this eigenvalue lambda is given by m and where m is thus n minus rank A minus lambda I.

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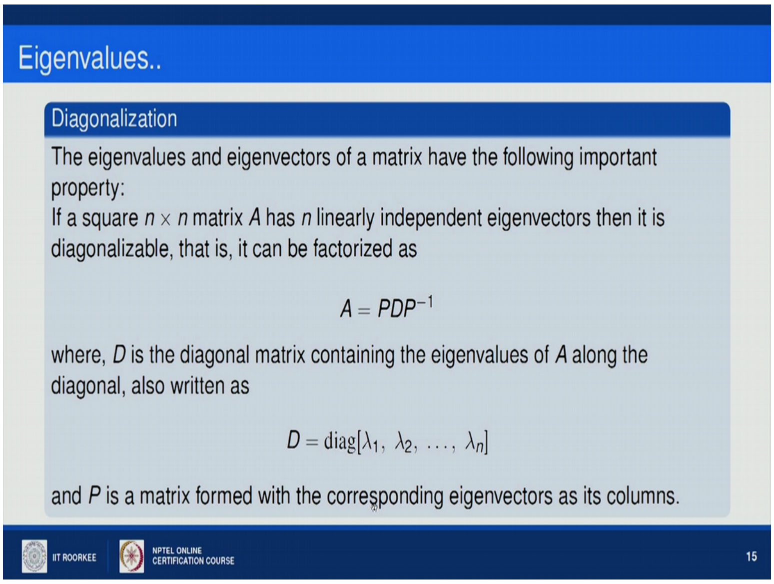

Now what is the diagonalization of a matrix? So the eigenvalue and eigenvectors of a matrix having a very important property that is if a square n by n matrix A has n linearly independent eigenvectors then it is diagonalizable that is, it can be decomposed as A equals to PDP inverse, where, D is the diagonal matrix containing the eigenvalues of A along the diagonal, so D can be written as diagonal lambda1, lambda2, lambda n and P is the matrix, which is formed with the corresponding eigenvectors writing the in the columns.

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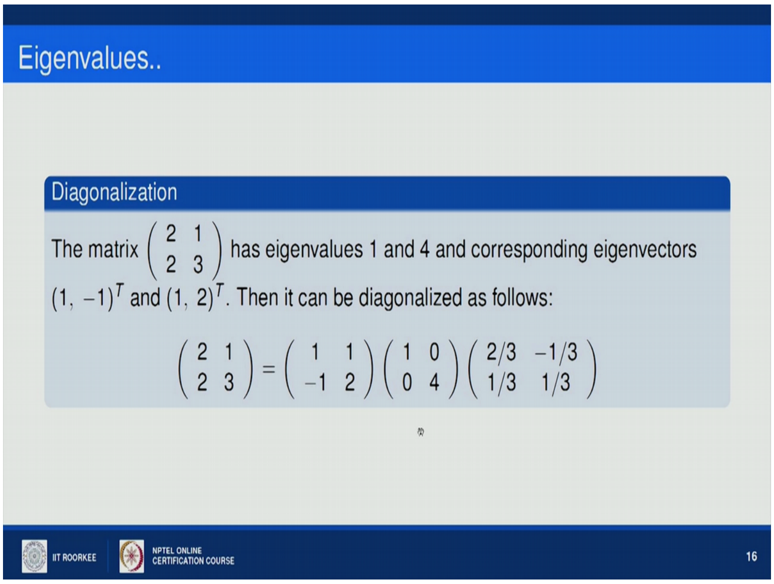

For example, if you take a matrix this (2 1 2 3), it is having eigenvalue 1 and 4 and corresponding eigenvectors (1, -1) and (1, 2) respectively. Then it can be diagonalized as (2 1 2 3) equals to (1 1 -1 2). So it is my matrix P and you can note here, I have written this first eigenvector as the first column of P, second eigenvector as the second column of P into D. So D is a diagonal matrix having the eigenvalue as the diagonal entries, only thing you have to note down that first eigenvalue is 1 in the first row. So the eigenvector corresponding to 1 should be the first column.

Similarly, 4 is in the second row, so the eigenvector corresponding to 4 should be the in the second column and then P inverse. So if you multiply these three maxtices the product of these 3 comes out to be this matrix and hence, this particular property is very important when you are talking about eigenvalues, because the eigenvalue of this P A and the eigenvalues of D will be same and since, D is a diagonalized matrix diagonal matrix, so eigenvalues will be the elements in the main diagonal.

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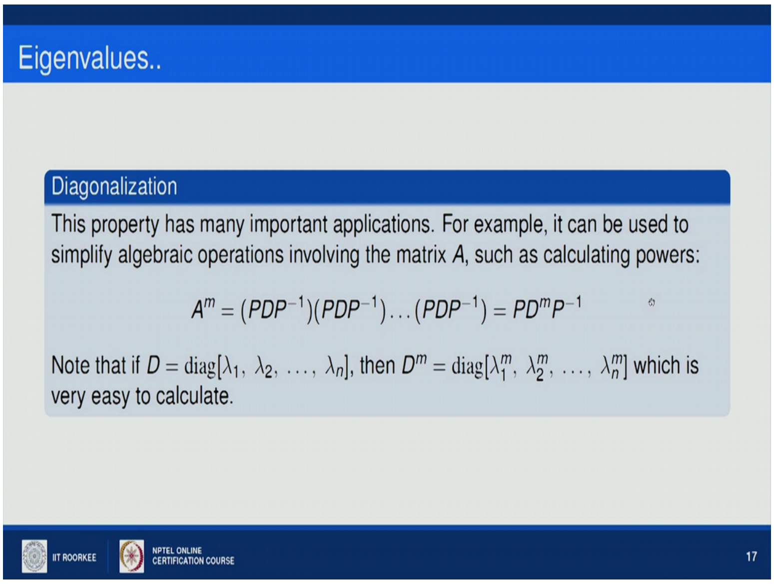

Moreover, we can use this property in many other ways like suppose; you need to find out A race to power m. So this means PDP inverse, PDP inverse, PDP inverse m times. It is coming out to be P into D race to power m into P inverse where, D is a diagonal matrix. So D race to power m can be calculated very easily just by taking the power of lambda1, lambda2 m power m and easily calculated and hence, we can calculate A race to power m in a very say manner; however all the matrices not have this property that all the matrixes cannot be writurn in the form PDP inverse.

So in the next class what we will do we will take some example, where some of the matrix I can write as PDP inverse, but some of them I cannot write, I will tell you the condition which is necessary for writing A equals to PDP inverse, I will tell you if I cannot write A equals to PDP inverse then, is there any transformation other transformation, which is which able to which makes this a factorized into product of different matrices. So now I will stop myself and I will start from the next lecture that is similarity transformation, thanking you.