Numerical Methods

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Lecture No 10

System of Nonlinear Equations

Hello everyone so welcome to the last lecture of the 2nd module of this course. In the past lectures in this module we have learned that how to solve a non-linear equation in particular for finding the roots of non-linear equations we learned several methods including bisection method then Regular Falsi and Secant method. Later on we have have gone to the Newton Raphson method and finally fixed point iterations methods. In today's lecture we will learn about how to solve a system of non-linear equations. In particular we will focus on 2 methods that is Newton Raphson and fixed point iterations for doing this job.

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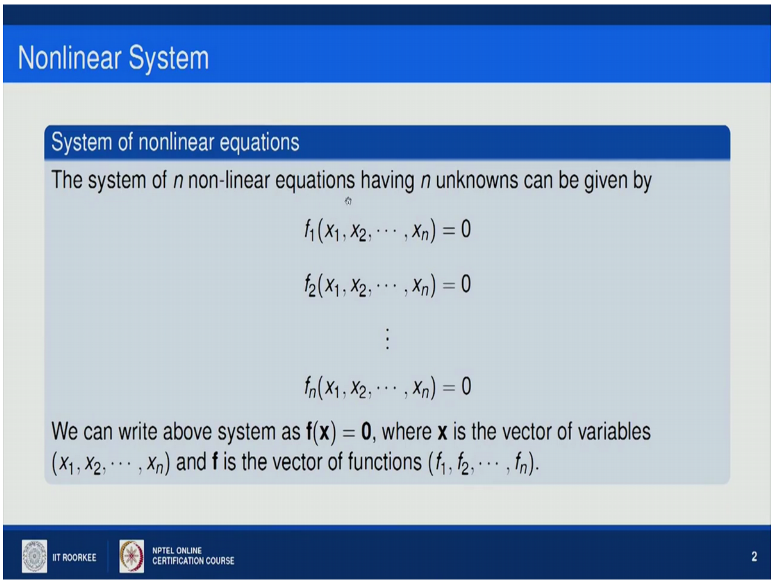

So 1st of all let me tell you what is a system of non-linear equations, so consider a system of n nonlinear equations having n unknowns and written as f 1 of x 1, x 2, x n equals to 0. f 2 again a non-linear equation in x 1, x 2, x n that is 0 and so on and finally nth equation is f n in x 1, x 2, x n equals to 0, so here you can say we are having n equations each one is denoted by f 1, f 2 up to f n and then each one is having n number of unknown variables x 1, x 2 up to x n. So solving this non-linear system means we need to find out the values of x 1, x 2, x n we satisfy all those equations.

Basically analytically it is very hard to solve and hence here we will I will introduce some numerical methods for solving these equations. So if I want to write this system into a vector form, so just consider a vector f which is a vector of functions f 1, f 2 up to f n and a vector x which is a vector of unknown variables x 1, x 2, x n. Then this system of non-linear equation can be written as f x equals to 0.

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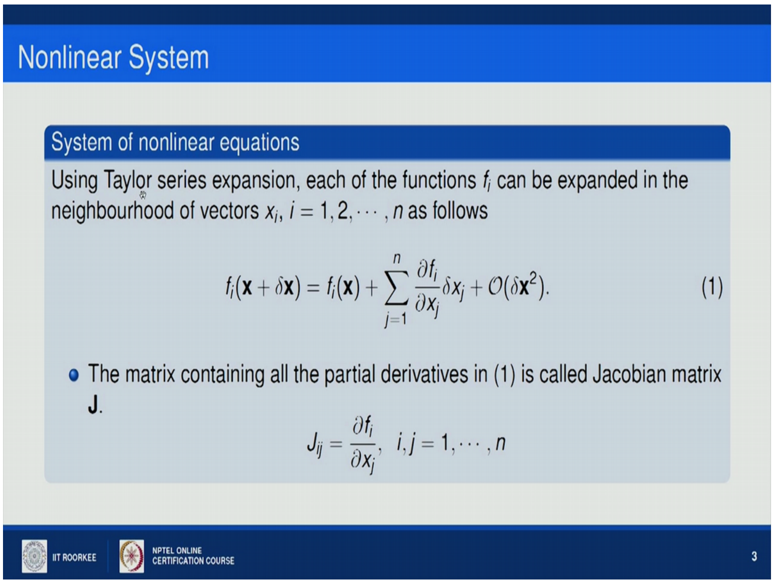

Now using Taylor series expansion each of the function f i where i is from 1 to n can be expanded in the neighborhood of the vectors x i that is x 1, x 2 x n. So using the Taylor series expansion we can write f i x plus del x equals to f i x plus j equals to 1 to n del f i over del x j into delta x j plus higher order term. Here you can notice that particular summation term comes out as a n by n matrix. So this particular matrix containing the partial derivative different f with respect to unknown variables x 1, x 2, x n is called the Jacobian matrix. So the i j^{th} entry that is an entry in i^{th} row and j^{th} column of this Jacobian matrix J can be given as del f i over del x j that is partial derivative of i^{th} function f with respect to j^{th} variable.

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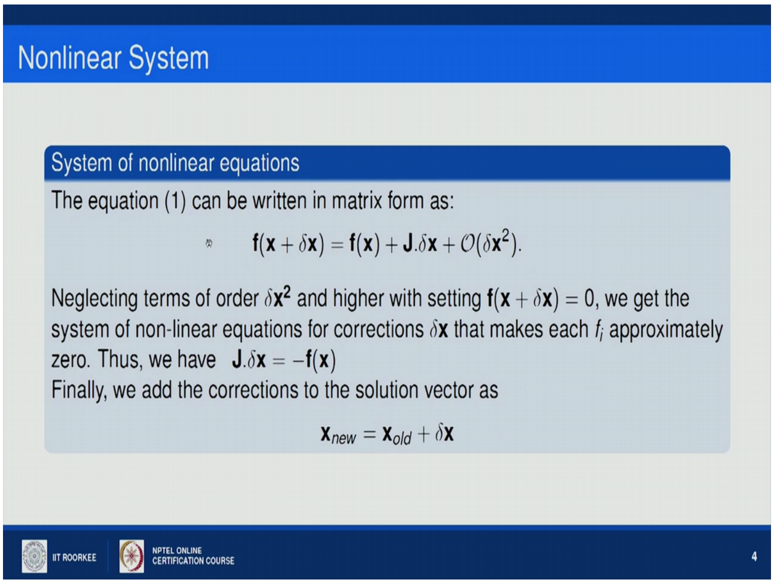

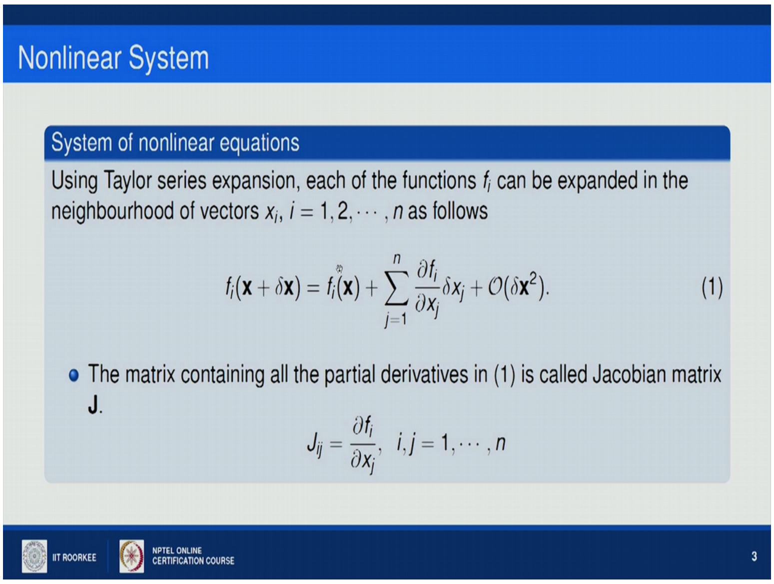

We can write the equation 1 in matrix form as f of x plus delta x equals to f of x plus J times delta x plus second order and higher order terms. So here you can notice here we were having a vector means f 1, f i x 1 plus f i x 2 plus f i x 3, so we have written it as f of x and then Jacobian matrix this is coming from the term summation J equals to 1 to n del f i over del f j, so for i^{th} equation will be like del f i over del x 1 plus into delta x 1 plus del f i over del x 2 into delta x 2 and so on up to n^{th}, so we have written it in matrix form in this way. Now as we are assuming that is delta x is a small quantity, so we can neglect the 2^{nd} order and higher order term and moreover the left-hand side f x plus del x we can set it to 0.

So we get the system of non-linear equation for corrections del x that makes each f i approximately 0 thus we have since this left-hand side is 0, so we can write J into delta x equals to minus of f x. So here you can notice we are having this system here Jacobian matrix is known to us for a given x naught or for a given vector x 1, x 2, x n and so we know the f of x and hence it is coming out as a system of linear equations. After solving this system of linear equation we add the correction to the older solution that is x new is equal to x old plus delta x and in this way we update our iterations. So let us modify this particular thing into combination of Newton Raphson method for solving the system of non-linear equations.

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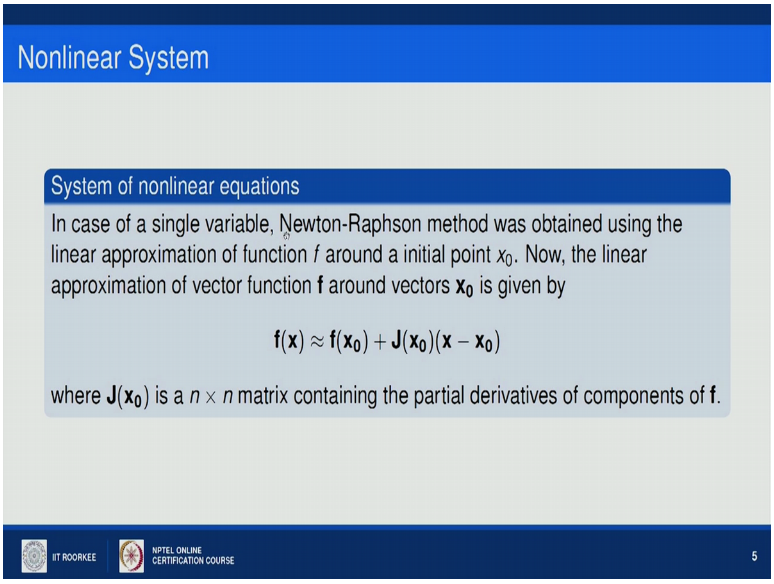

So in case of a single variable Newton Raphson method was obtain using the linear approximation of function f around an initial point x naught. So in case of a system of non-linear equations can write the linear approximation of vector function f around the unknown variables vector x naught in this way that is f x equals to f x naught plus J x naught into x minus x naught where J x naught is n by n matrix and as you know it is the Jacobian matrix which is forming using the partial derivatives of f.

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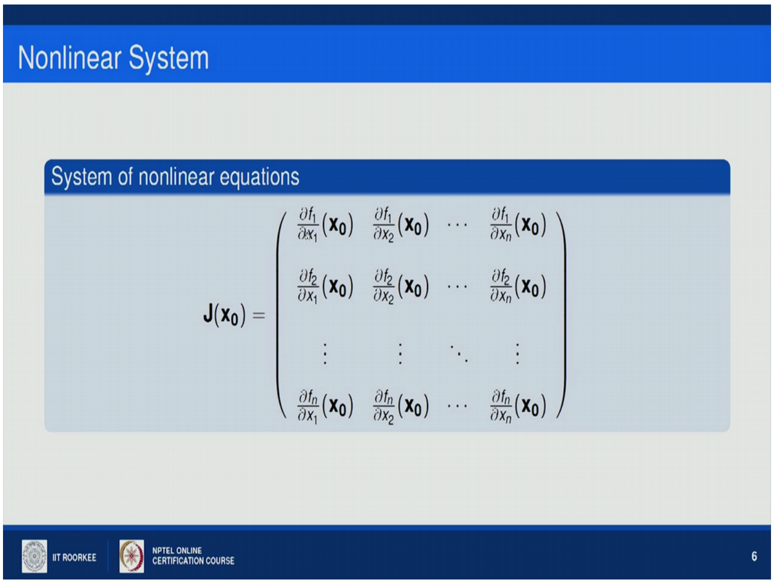

So like this one j f 1 over j x 1, j f 1 over j x 2 up to j f 1 over j x n then we are having in the second row we are having the partial derivatives of the function f 2 with respect to x 1, x 2 up to x n and so on in the last we are having the partial derivative of f n with respect to x 1, x 2 up to x n.

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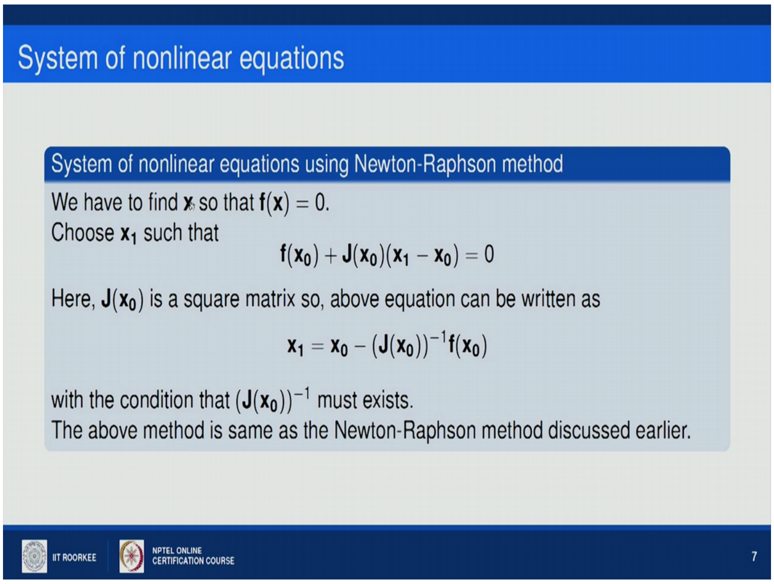

Now we have to find x so that the system f x becomes 0. So choose x 1 such that f x 0 plus J x 0 x 1 minus x 0 equals to 0. Please note that here all these f of x 0, x 1 minus x 0 all these are vectors and J is a matrix n by n matrix and so on. So here J x naught is a square matrix so we can write the system in this way where x 1 equals to x naught minus inverse of J into f of x naught. However we can write in this way only when the inverse of J exists and this method is called Newton Raphson method for the system of non-linear equations.

So in algorithmic way we can use this method provided inverse of J exist but suppose you are having hundred equations with hundred unknowns it means the size of J will become hundred times hundred and hence finding the inverse of hundred times hundred matrix is not an easy job in terms of computation complexity.

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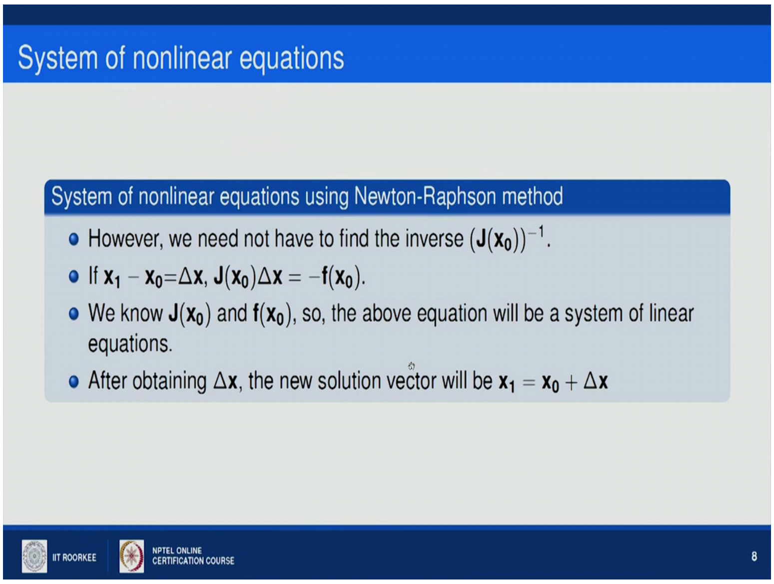

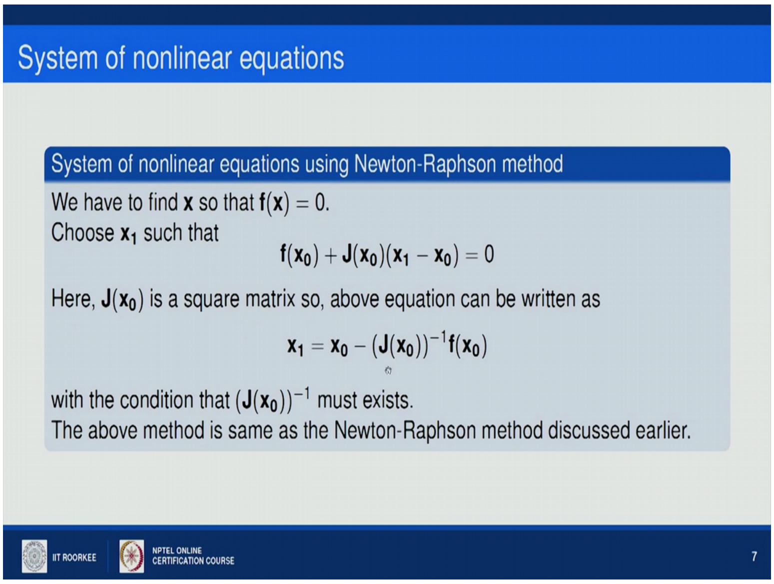

So we can use an alternate instead of finding the inverse of J, what we will do that we will write x 1 minus x 0 as the delta x, so delta x is the vector of differences in the 2 iterations let us say in 1st and 2nd iteration and then J x naught will become we can write the this scheme as J x naught into delta x equals to minus f x naught. We know J for a given x naught as well as f for a given x naught, so the above system will become a system of l linear equation with n unknowns and hence once we can got this system we can solve it using any scheme which I have told you in the module 1 like Jacoby or gauss Seidel those kind of approaches. Once you find x 1 using any one of the iterative scheme for the linear equation you can obtain delta x and from the delta x the new solution can be obtain using x 1 equals to x naught plus delta x.

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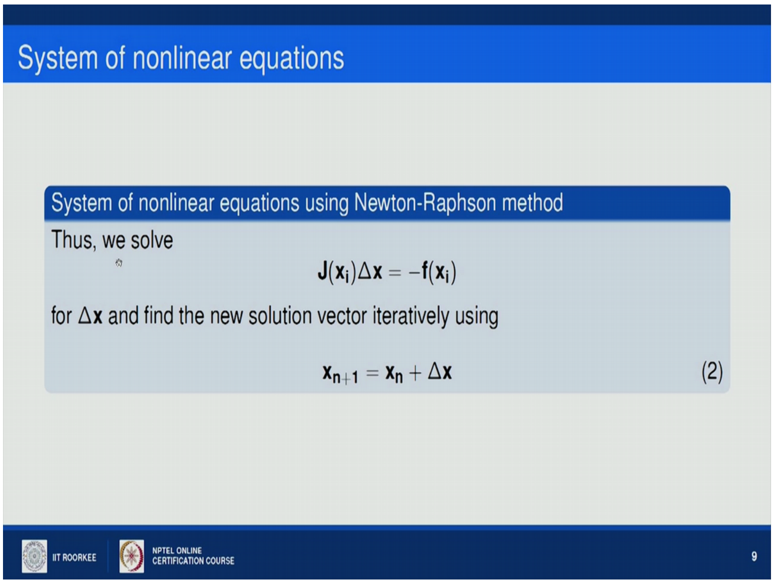

So in brief we can say if we are having the i[th] vector x i that is in the i[th] iteration we will solve this particular equation J of x I into delta x equals to minus f of x i and by solving this we will find delta x and vector x in next iteration will become x n plus delta x. So let us take an example of this system.

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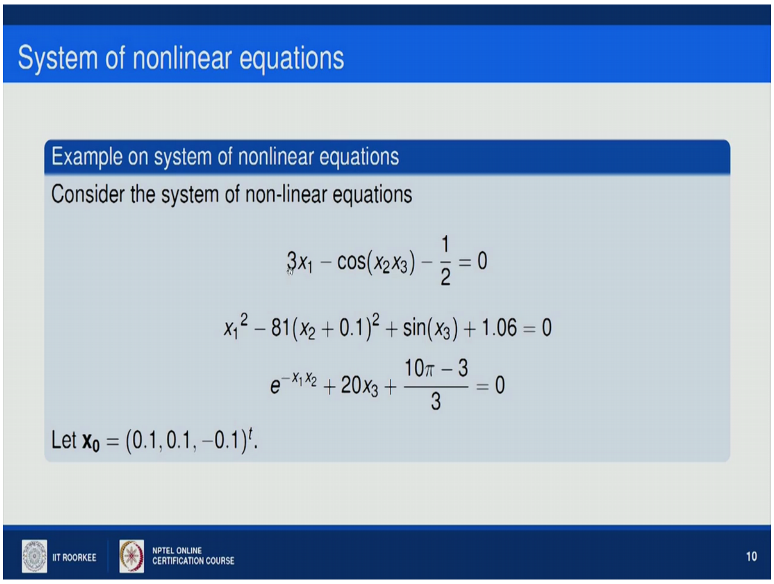

So here we are having 3 equations in 3 unknowns, so the 1st equation is 3 times x 1 minus cost of x 2 x 3 minus half equals to 0. The 2nd equation is given as x 1 square minus 81 times x 2 plus 0.1 whole square plus sin x 3 plus 1.06 equal to 0 and the 3rd equation is e raise to power minus x 1 x 2 plus 20 x 3 plus 10 pi minus 3 upon 3 equals to 0. So here you can note down at each of the equation is a transcendental equation like in 1st equation we are having cos sign term similarly in the 2nd equation we are having a sin term and the 3rd equation we are having an exponential term. Let us solve this system of transcendental equations with an initial guess 0.1, 0.1 and minus 0.1.

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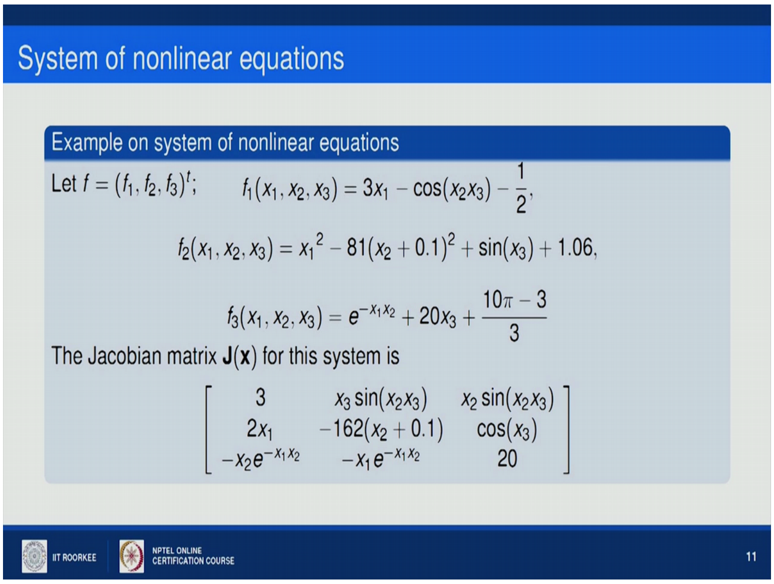

It means in the as an initial solution we will take x 1 as 0.1, x 2 as 0.1 and x 3 as minus 0.1. So now let us assume the 1st equation is f 1 that is this equation is a function of f 1 x 1, x 2 and x 3 similarly 2nd equation let us f 2 and 3rd equation is f 3, so we can write in this way and hence first of all we need to find out the Jacobian of this system.

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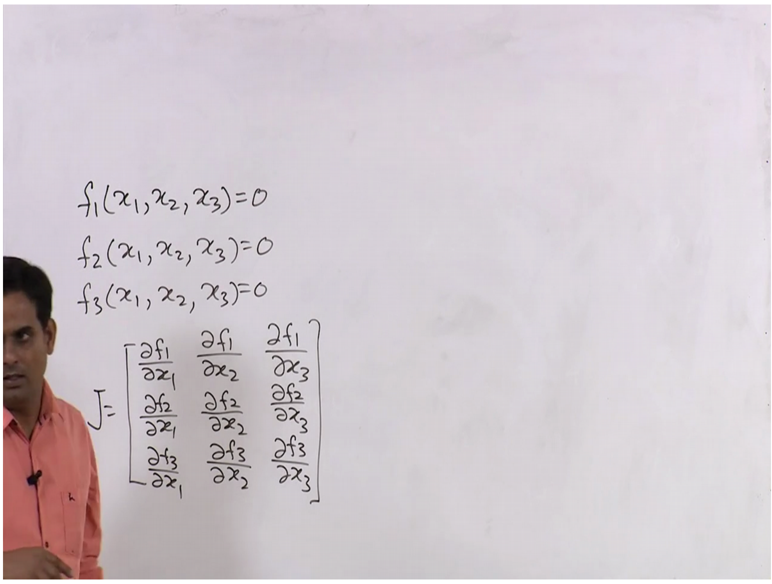

So for finding the Jacobian of the system we are having a system like f 1 x 1, x 2, x 3 equals to 0, f 2 x 1, x 2, x 3 equals to 0 and then f 3 x 1, x 2, x 3 equals to 0. Now the Jacobian matrix will be a 3 by 3 matrix which will be having del f 1 over del x 1 then this element will be del f 1 over del x 2, del f 1 upon del x 3. In the same way in the 2nd row we will be having instead of f 1 function f 2, so del f 2 over del x 1, del f 2 over del x 2 and then del f 2 over del x 3. In the final row you will be having function f 3 that is a partial derivative slope f 3 with respect to x 1, x 2 and x 3.

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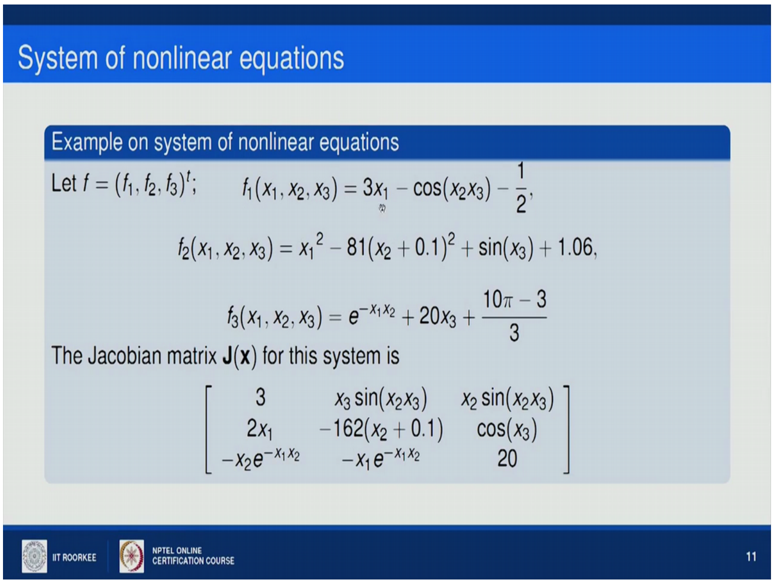

So hence Jacobian matrix will be a 3 by 3 matrix and here del del f 1 over del x 1 will become 3, del f 1 over del x 2 will become minus minus will become plus sign x 2 x 3 into x

3, so x 3 sin x 2 x 3. Similarly del f 1 over del x 3 will become x 2 times sin x 2 x 3 this will be the partial derivatives slope f 2 with respect to x 1 with respect to x 2 and with respect to x 3 and the final row is having partial derivative of f 3 with respect to x 1, x 2 and then x 3.

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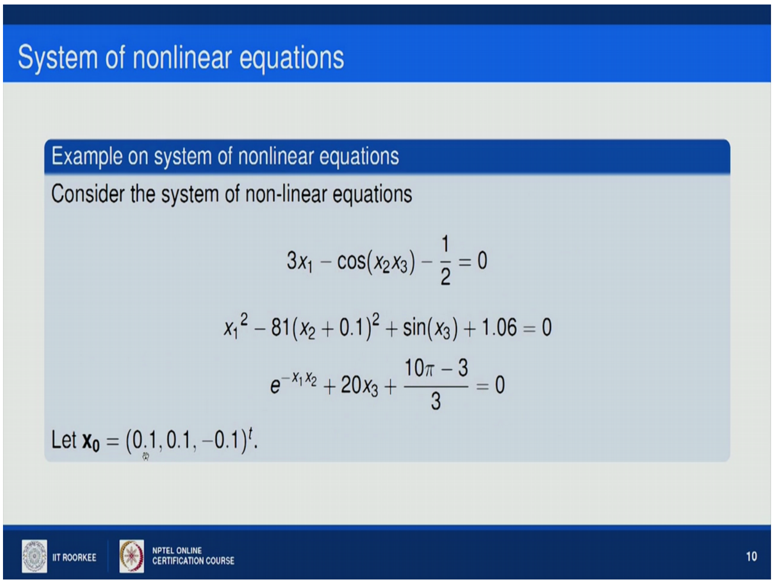

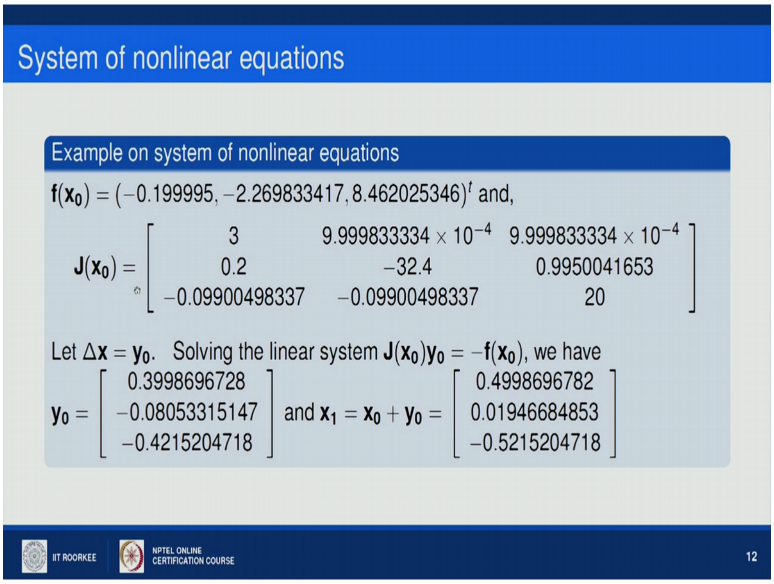

Now what I will do I will put these values in the Jacobian matrix that is x 1 0.1, x 2 0.1 and x 3 minus 0.1, so the Jacobian matrix or initial Jacobian matrix I will say become like this moreover for the initial solution 0.1, 0.1 and minus 0.1 the value of f 1, f 2, f 3 is given by these 3 numbers, so I am having J of x naught, f of x naught. So what I will do, I will solve the system J of x naught into delta x equals to minus f of x naught and after solving this I will get the solution for delta x like this that is delta x 1 will become 0.39986 delta x 2 will become minus 0.080533 and delta x 3 will become this number, so this is the change which we need to made in initial solution to getting the update in 1st equation. So adding this in the initial solution, I will get this solution that is 0.479, 0.019 and minus 0.521 as my 1st means solution of x in 1st iteration that is I will denote it as x 1.

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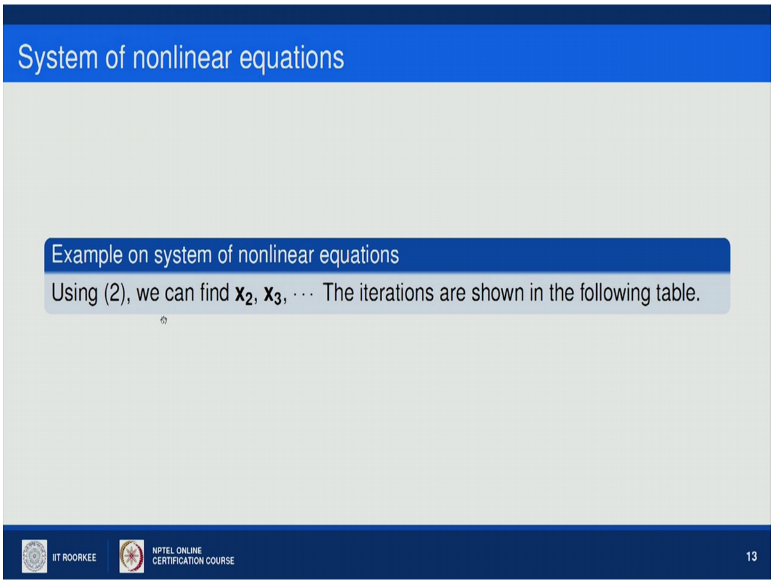

So continuing this now I will use this x 1 and I will find x 2, so it means I need to solve system J of x 1 into del x equals to minus f of x 1 and hence from there I will get x 2 that is x 2 will be x 1 plus this delta x which I get using solving this system.

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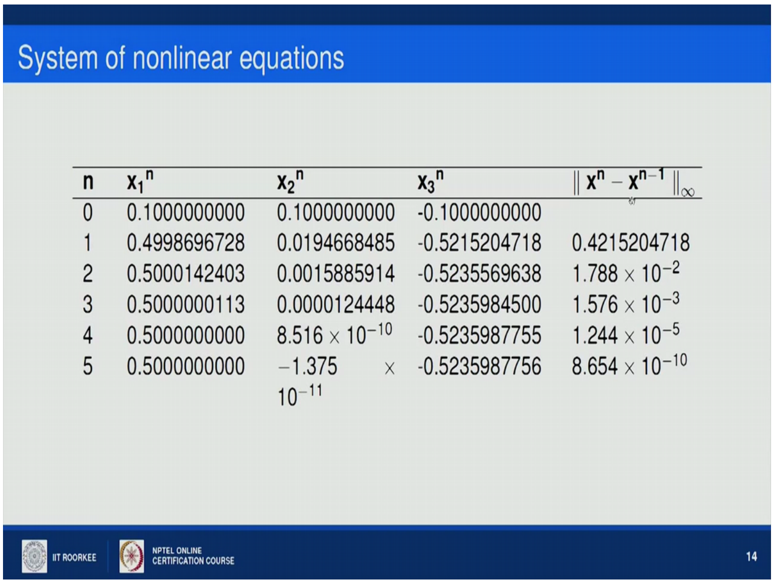

n	x_1^n	x_2^n	x_3^n	$\lVert x^n - x^{n-1} \rVert_\infty$
0	0.1000000000	0.1000000000	-0.1000000000	
1	0.4998696728	0.0194668485	-0.5215204718	0.4215204718
2	0.5000142403	0.0015885914	-0.5235569638	1.788×10^{-2}
3	0.5000000113	0.0000124448	-0.5235984500	1.576×10^{-3}
4	0.5000000000	8.516×10^{-10}	-0.5235987755	1.244×10^{-5}
5	0.5000000000	-1.375×10^{-11}	-0.5235987756	8.654×10^{-10}

So hence this stable so the iterations in for various for various n like earlier initial solution was 0.1, 0.1, minus 0.1 then this is the solution getting in 1st equation. The last column of the table shows the norm of the difference of the two vectors x of 10 in successive iteration means in current iteration and in the previous iterations. So for the 2nd iteration I will get x 1 as 0.5 then this is x 2 and this is x 3 and here this difference will be of order 10 raise to power minus 2. In the 3rd equation these are the values and difference is of 10 raise to power minus 3.

On the 4th iteration difference means these are the values and differences of 10 raise to power minus 5. So here you can notice the difference is increasing it means accuracy we are moving towards the exact solution. So in the 5th iteration these are the numbers value for x 1, x 2 and x 3 and the differences is of order 10 raise to power minus 10, hence this is the vector x which is very close to the exact solution and in this way using the successive iterations we can get the numerical solution of x for a system of non-linear equation using the Newton Raphson method.

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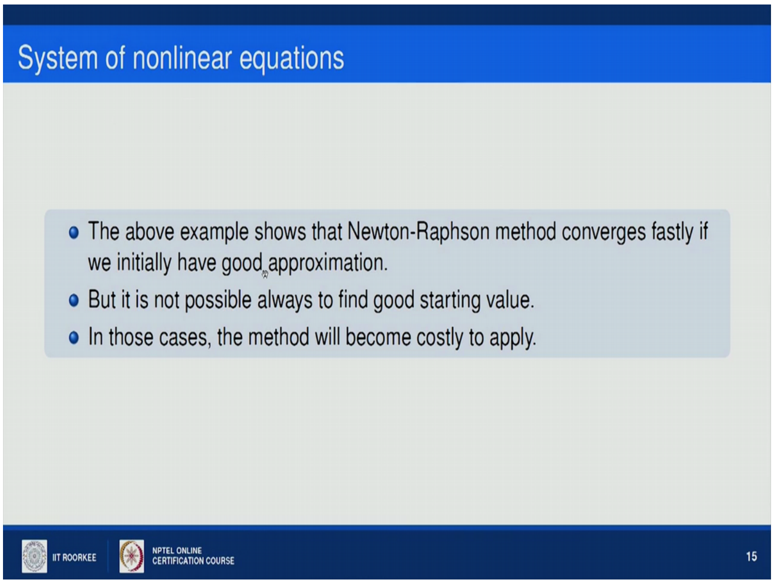

So the above example shows that Newton Raphson method converges fastly if we initially have a good approximation means our initial solution is close to the exact solution but it is not possible always to find a good initial solution for a given problem, so in those cases the method will become costly to apply. So for this we will take the another method that is the fixed point iteration method like the earlier which we have done for a single non-linear equation, we will extend it for a system of non-linear equations.

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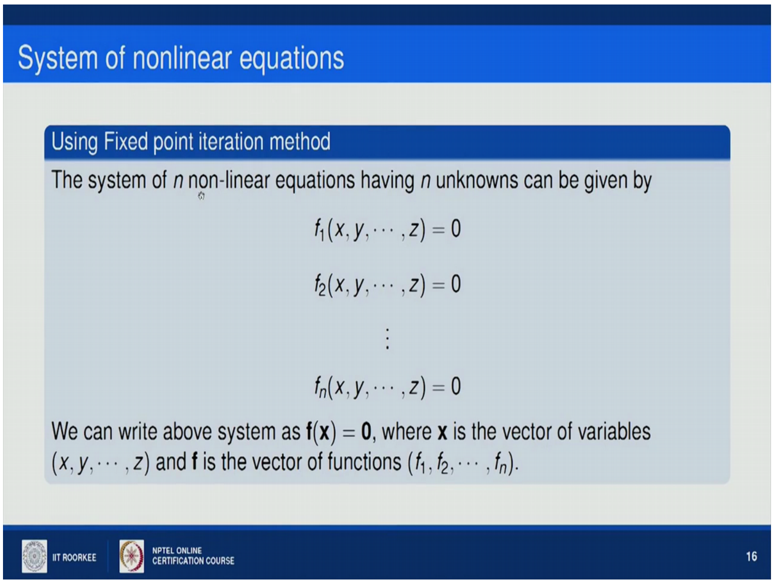

And here again consider a system of n nonlinear equations with n unknowns is given by this equations here just I have changed the variables, so here n variables are x, y up to z and like that functions are f 1, f 2, f n, so we can write this system again in a vector form like this f of x equals to 0, where f is the vector of functions f 1, f 2, f n and x is the vector of variables x, y up to z that is n variables.

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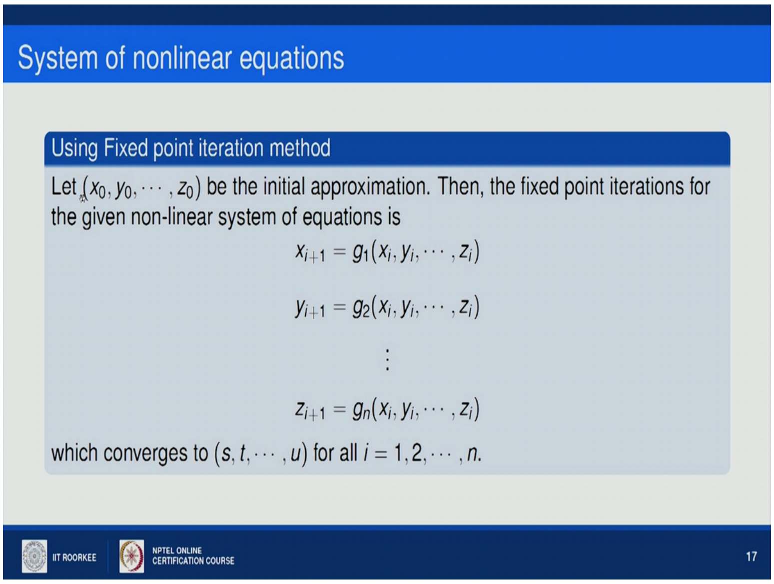

Now if I take the initial approximation as x naught, y naught, z naught, then the fixed point iteration for the given non-linear system of equation is given in this way, so here you can notice that we are having g 1, g 2, g n. So like you are having the 1st non-linear equation, so you have 2 write the 1st non-linear equation that is f 1 of x equals to 0 in such a way that x equals to g 1 of x. Similarly for 2nd equation and rest of the equations, so in the case of a single non-linear equation I told you what is the criteria for writing such a g, so that our method converge. So we need to write in this way which converge to the numerical solution s, t, u for i equals to 1, 2, n.

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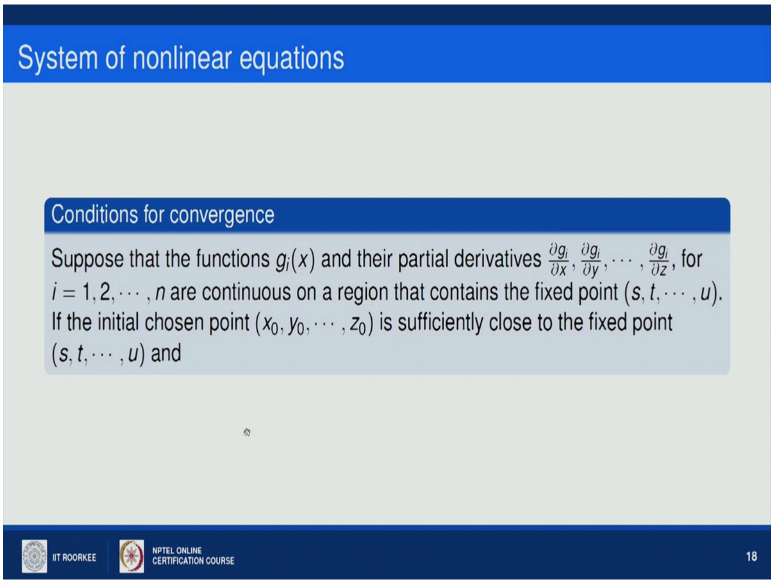

Now what is the condition for convergence in this case, so that is given by this particular result that is suppose that the functions g 1, g 2, g n x and their partial derivatives with respect to various unknown variables x, y up to z for i equals to 1, 2, n are continuous on a region that contains the fixed point s, t up to u. So here what we are saying that the region where you are having your initial solution partial derivatives your g. Your g should be continuous, partial derivative should be continuous on the same region and your fixed point also lies in the same domain.

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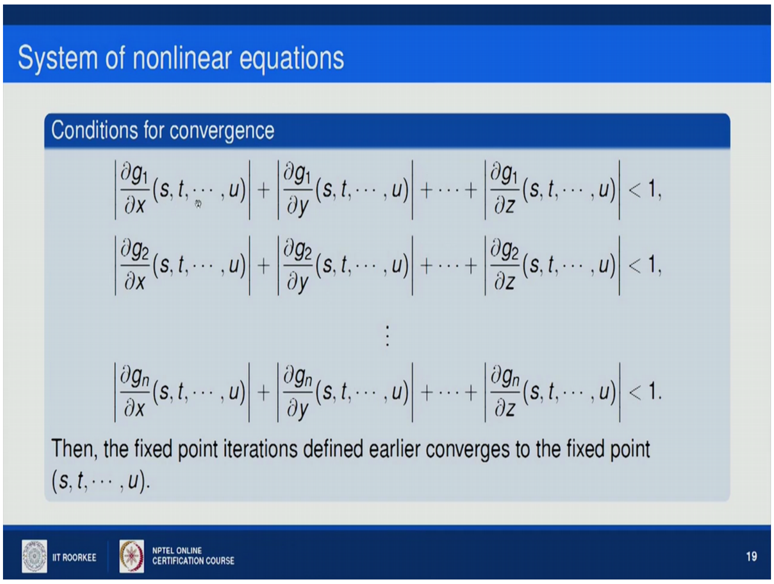

So if the initial chosen point is x naught, y naught, z naught is sufficiently close to the fixed point s, t, u and the partial derivative of g 1 with respect to 1st unknown variables at the fixed point plus partial derivative of the g 1 with respect to 2nd unknown variable and so on partial derivative of g 1 with respect to nth unknown variable and absolute sum of all these terms should be less than 1. Similarly for g 2 similarly for nth, g n then the fixed point iteration defined earlier converges to the fixed point s, t, u.

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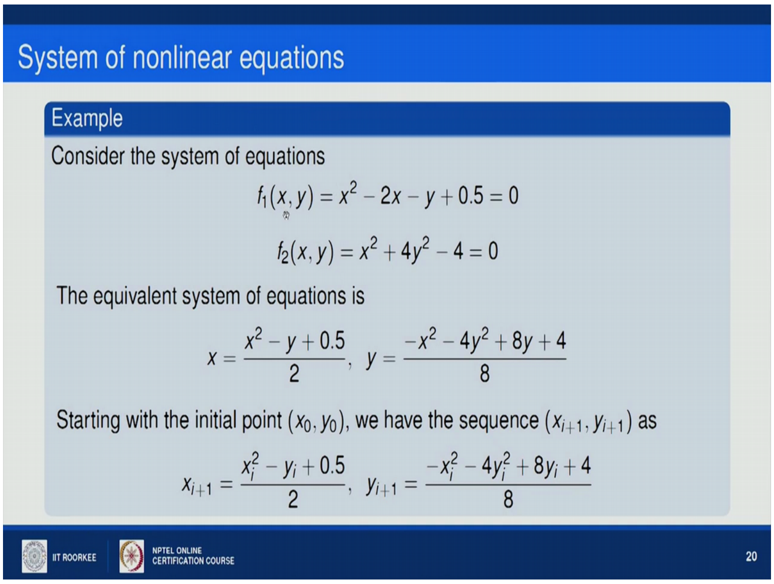

So for this let us take an example again, so here we are taking an example of 2 equations in 2 unknowns, so 1st equation is x square minus 2 x minus y plus 0.5 equals to 0, 2nd equation is x square plus 4 y square minus 4 equals to 0, so now we are writing the 1st equation in the form as x equals to g 1 x, so here we can choose we can write this equation like x equals to x square minus y plus 0.5 up on 2, so what we have done we have taken this minus 2 x term into right-hand side and then x will become half of x square minus y plus 0.5.

Similarly we need to write the 2nd equation as the 2nd unknown variable equals to g 2 of function of all unknown variable. So here we are writing it as y equals to minus x square minus 4 y square plus 8 y plus 4 up on 8, so starting with the initial point x naught y naught, we have the sequence x i plus 1 y i plus 1 as as you did it in case of a single non-linear equation do it separately for this g 1 x as well as for g 2 x, so we get this iterative scheme. Now start with an initial value of x and y and then you can get the successive approximation of x as well as y which converse towards the fixed point, so let us check the convergence condition.

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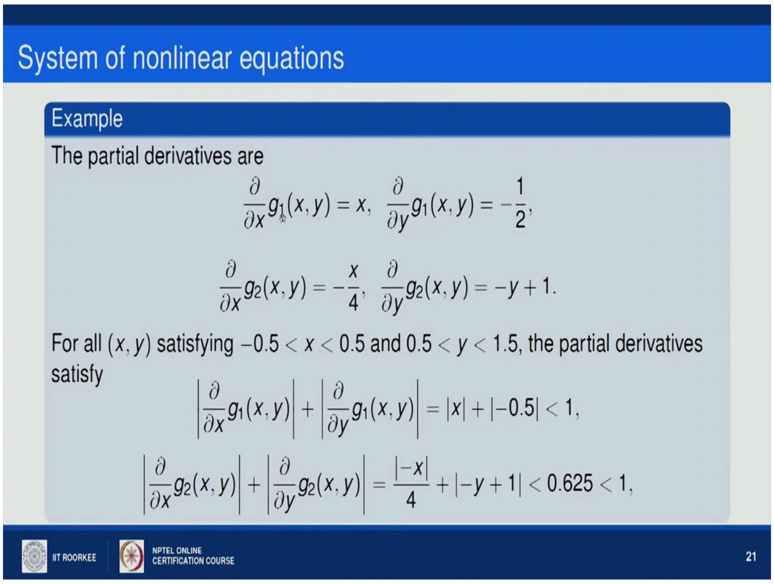

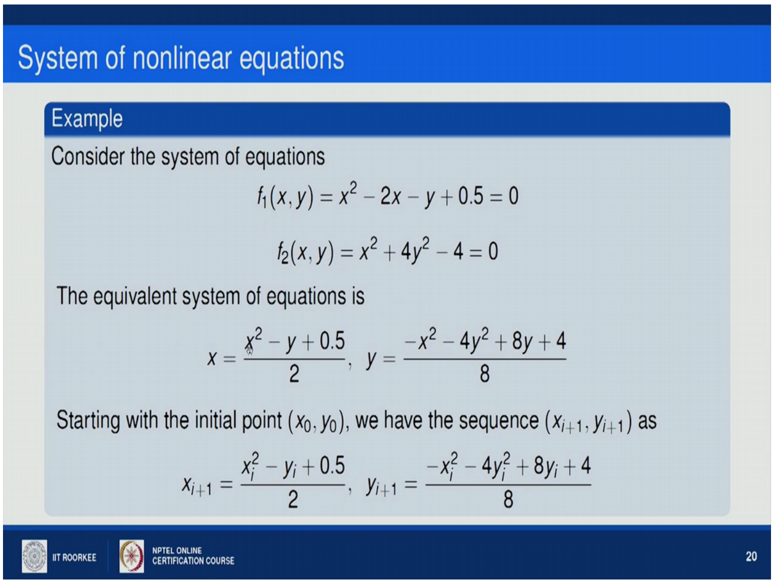

So here we are having these 2 functions, we have written it like this, so partial derivative of g 1 with respect to x is given by 2 x upon 2 that will be x here. Similarly partial derivative of g 1 with respect to y is given by, so this is my g 2. So this will be minus 2 x upon 8 sorry with respect to y, so it will become minus 8 y upon 8, so it is g 1 of y that is minus half g 2 upon x minus x by 4 and g 2 upon y will be minus y plus 1, so if you not take this rectangular domain that is x is between minus 0.5 to 0.5 and y is in 0.5 to 1.5 then you can note down that the partial derivatives satisfy the convergence conditions that is del g 1 over del x plus del g 1 over del y will be less than 1. Similarly del g 2 over del x plus del g 2 over del y will be less than 1 for all values of x and y belonging to this domain.

So here since if you choose any initial solution this rectangular domain then our system will converge to the fixed point of g 1 and g 2 and fixed point of this system will be minus 0.22 and 0.99 which is also in the given rectangular domain. Now the question is how to choose g various g in the system so that my method converge because as you know from the single equation, single non-linear equation when we did it using the fixed point iteration method we can have various choice of g. So as you know from the non-linear equation and the related fixed point iteration method that the choice of g is very important the convergence. The same case apply in case of system of non-linear equations, so here again choice of g is quite important.

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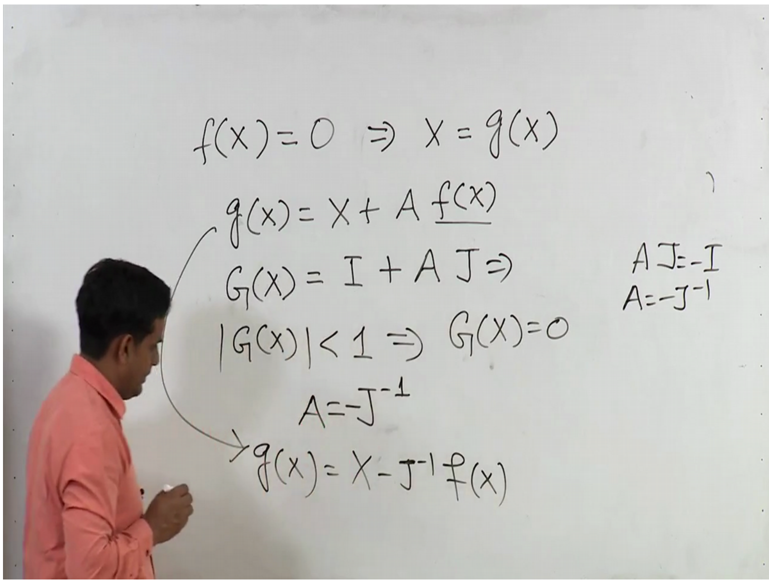

So how to choose the best g let me explain here, so let us take we are having a system of non-linear equations and which is represented in matrix form as f of x equals to 0, so here f is a vector containing the functions f 1, f 2 up to f n and x is again a vector containing the variables x 1, x 2 up to x n. So we are having n number of non-linear equations in n unknown. Now the fixed point iteration method for this can be written as x equals to g of x.

Now what should be this g the best choice of g, so let me write this gx as x that is the same equation plus a constant matrix A which is square matrix of order n into f of x. Since f of x equals to 0 so this term is 0 basically. Now if I differentiate it partially with respect to different x 1, x 2, x n then it here let us take I am getting matrix capital Gx, this will comes out as an identity matrix I plus A is a constant matrix, so it will be domain like this into the differentiation of this with respect to different x 1, x 2, x n. So this as you know will become Jacobian matrix. Now so my capital Gx equals to I plus A into Jacobian that is the matrix containing the partial derivatives of different at f with respect to different x. Now for the rapid convergence or for the convergence this Gx should be less than 1.

This gives me if I take Gx equals to 0 which is the minimum possible value of this G absolute value of G, I will get the rapid convergence or the best convergence. So if I put Gx equals to 0 here what am getting, I am getting A as the matrix J inverse because minus J inverse basically because this is 0, so I can write it 0 equals to I plus AJ, so I can write or AJ equals to minus I and then if I multiply by J inverse that is the post multiplication I will get A equals to minus J inverse. So A equals to minus J inverse so if I put here I will get gx as x plus or it is minus or minus J inverse into fx.

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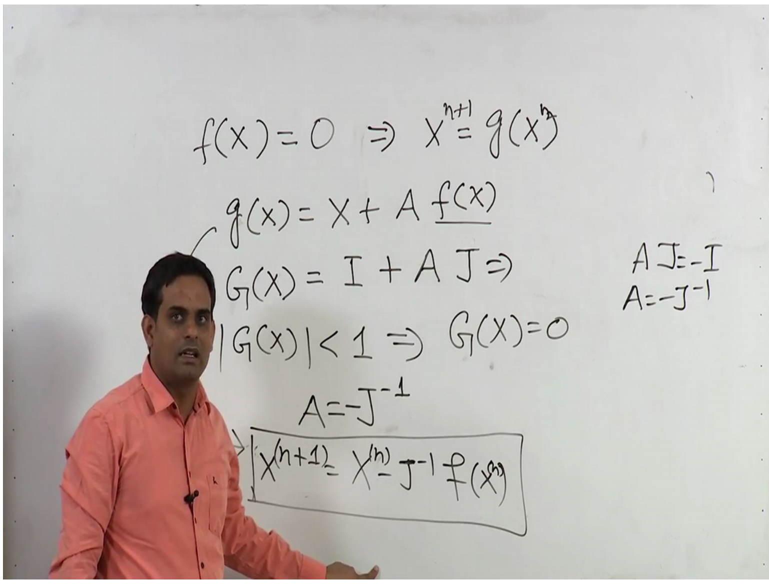

And what is this? This gx because my iterative scheme is x n plus 1 of g of x n, so if I write it x of n plus 1, x of n then what is this scheme? This is the Newton Raphson method for system of non-linear equations and hence the best choice of G gives me the fixed point iteration method same as the Newton Raphson method for the system of non-linear equations.

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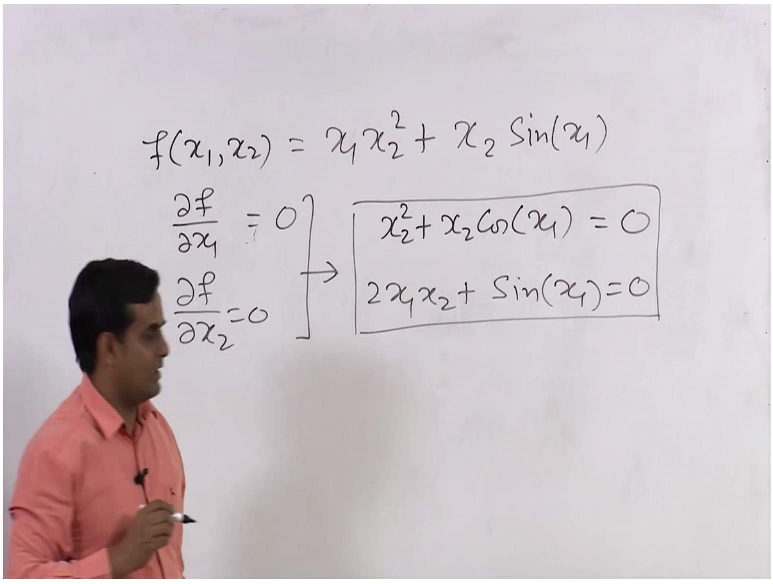

So when we do the find the extrema or maxima or minima of a system of more than one variable let us say this one, so let us take this function as x 2 square plus x 2 sin of x 1. So let us say I need to find out the maximum or minima or extrema of this function, so here the necessary condition is that del f over del x 1 as should be 0 and del f over del x 2 should be 0, so from this we find the points in the domain of x 1 and x 2 where we need to check the points for maxim or minima of the function f.

Basically we find the stationary points, so here if I do it so del f over del x 1 gives me x 2 square plus x 2 cos of x 1 equals to 0 and the 2nd equation gives me twice of x 1 x 2 plus sin of x 1 equals to 0. So please note that, what is the system? This is a system of non-linear equations which we have discussed just now. So here you will find the system quite frequently and if we consider a problem of minima of this system and we apply the sufficient condition we will find that the Jacobian matrix should be invertible for having the maximum or minima of this system on the points of solution of this system.

So hence this is an application of system of non-linear equations. Now in this lecture we learned how to solve a system of more than 1 non-linear equations using the Newton Raphson as well as fixed point methods. This is the last this was the last lecture of the module 2 and now we will come to 3rd module in the next lecture, so we will start our 3rd module where we will discuss about numerical methods for finding the eigenvalues and Eigen vectors of a matrix. Thank you very much.