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Nonlinear Programming

Lec-08
Quadratic Programming Problems-II

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So welcome to lecture series on nonlinear programming. In the last lecture we have seen about quadratic programming problems. We have seen that how can we write KKT conditions for a given QPP, and the KKT conditions is sufficient if Q in the objective function is positive semi definite matrix okay. Now let us discuss the same thing by an example okay. Suppose we have this problem.

-2x1x2 – 6x1-8x2 subject to we have only one constraint here 2x1-x2 less than equal to 13 and x1 x2 non negative okay. So we will try to solve this problem, there are different method to solve a QPP, one method is ULF base method, we will see what is the method is and try to solve this problem using ULF base method okay. So first it is a quadratic programming problem because involved objective function is quadratic and all concerns are linear so it is a quadratic programming problem.

Now what is Q here, Q is 1, 2, -1, -1 if you check for this Q and D1 is 1 okay one cross matrix determinant 1 and D2 is nothing but determinant of this 1, -1, -1, 2 which is 2-1 and again 1 greater than 0. So this Q is positive definite and hence the function is another function f is convex okay. The function of f is convex, because Q is positive definite and hence the function is convex function. So it will be a convex programming problem.

So the KKT conditions which we obtain becomes sufficient okay. So what will be the KKT condition for this problem, so to write KKT condition either we apply those conditions directly which we discuss in the last class or we can first define a Lagrange function like this L which is f, f is $x1^2+2x2^2$-2x1 x2-6x1-8x2+ multiplier correspond to this constraint is suppose U, for this suppose V1, for this constraint suppose V2 then U time the first constraint 2x1-x2-13-v1x1-v2x2.

So this we write as Lagrange function L then what is ∂ L/ ∂ x1 = 2x1-2x2-6+2u-v1=0 what is ∂ L/ ∂ x2 it is 4x2-2x1-8-u-v2=0. So these are the first KKT condition okay, you can obtain directly also by writing that condition okay. Then now by remembering that condition, other conditions are the feasibility conditions that is 2x1-x2+s=13 and x1 x2 non negative. Of course uv1 v2 was non negative.

And one more condition that is ui si=0 for all i, and vj sj = 0 for all j. So here it is uxs=0 only one constraint and one variable is involved and v1s1=v2s2=0. So these are the KKT conditions for this problem okay for this problem these are the KKT conditions. Now we can readily see that these KKT conditions, these conditions accept these conditions are linear.

So we can apply some method of linear programming problem to solve this particular problem, so before starting before solving this problem let us recap on linear programming problems okay let us see how we can solve a linear programming problem and then we will come back to this problem this and we will see how to solve this problem, okay.

So first we will come back to go to LPP's okay so this is a KKT condition we have already seen okay for this problem. Now what are linear programming problems?

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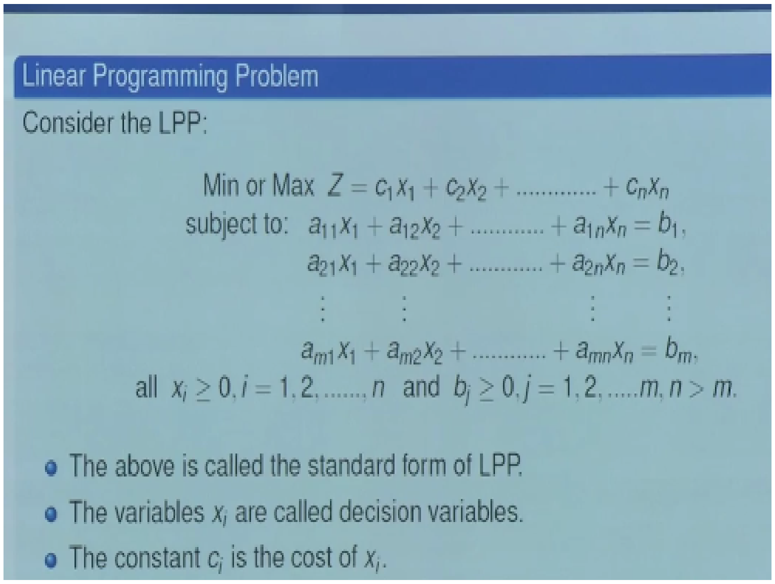

We already know that if objective function and all concerns are linear so such problems are called linear programming problems, here if all constrains are if equal type if we make it an equation if all constrain we make it an equation keeping right and side done negative then such problems are called standard form of LPP, okay. Here X_i is, are called decision variables and the, coefficients are called cost of x_i okay.

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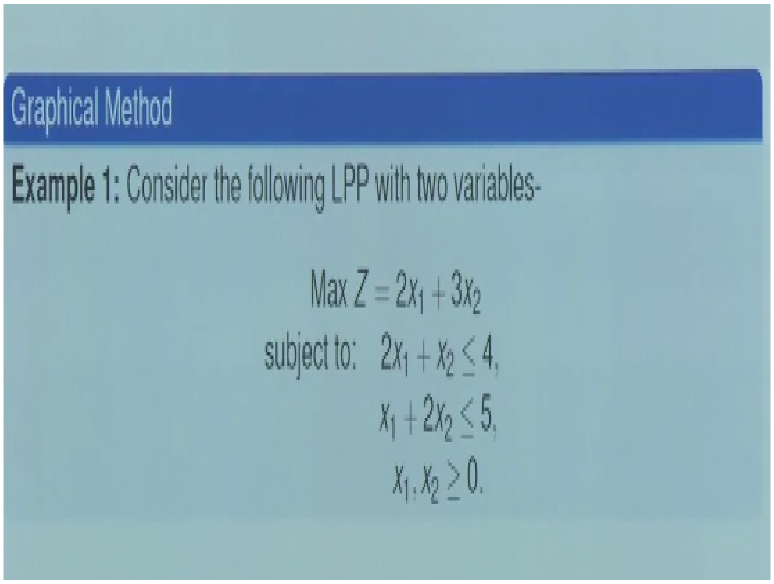

Suppose let us consider the simple problem of two variable problem 2x1 maximizing of 2x1 + 3x2 subject to this linear constrains, so this problem we can solve graphically also very readily we can solve this problem graphically let us see how. What is the first constrain, first constrain it is what is the problem okay so this is the graphic approaches solve LPP now come to this problem because same problem and we will try to solve same problem using simplest technique simplest method how can we solve this simple let us see this is an in equality which is less than equal to type.

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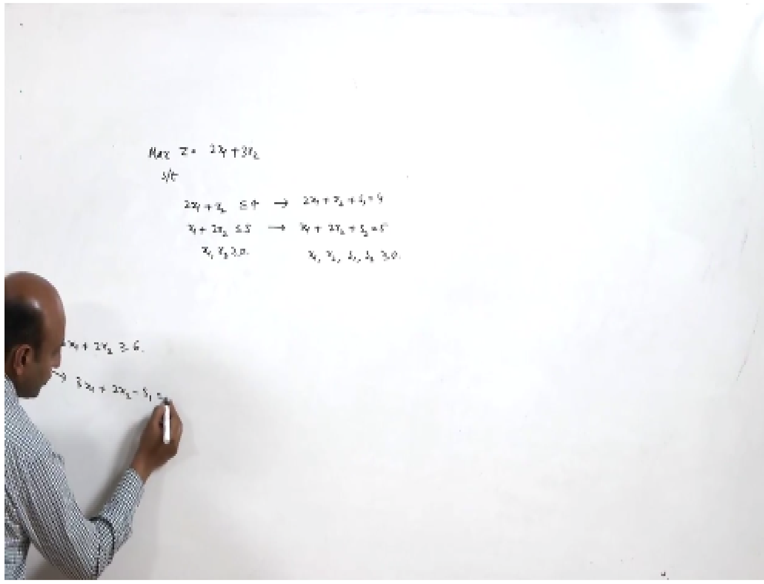

So to make it an equation we add a variable here in the left hand side because this quantity is less than equal to this that means we have to add some non negative value here to make it an equation okay so this you can write it less to x1 + x2 + s1 = 4 suppose that variable is s1 so we called at variable as slat variable s1 okay here also x1 + 2x2 + s2 = 5 and all variables are non negative okay, if we have > = type constraint suppose this constraint we are having 3x1 + 2x2 > = 6 so to make it an equation.

We subtract a variable which we call as superset variable okay, so this we can write it like this 3x1 + 2x2 – s1 = 6 and this s1 is called suppress variable okay to make it an equation because first a fall is simplest method we need to write the problem in the standard form for the standard form all in equalities must be question and right hand side muse be non negative.

And all variables must be non negative since the standard form of NPP standard form means all constraint must be of equation all variables non negative and right hand side must be non negative okay, now how can we start over process and how we can try to solve this problem let us see here quickly by see this problem, so we make a table here let us see how we can solve this problems how many by very first we are having four variables x1 x2 s1 s2 so four variables we are having x1 x2 s1 s2.

Whatever matrix I mean coefficient of x1 2 and 1 coefficient of x2 it is 1 and 2 s1 is 1 0 and s2 is 0 1 okay, now in the simplest method first start with that variable those variables which make identity in the first table now here we are having identity cross point to these two variables so we start with f1 and s2 we start with s1 and s2 and what is the right hand side, right hand side is 4 and 5 of solution, solution is 4 and 5 okay now writing this means s1 and s2 are variables whose values are 4 and 5.

And all other variables are 0 so these two variables are called basic variables and all other variables which added 0 level are called non basic variables okay, so here s1 is 4 and s2 is 5 and x1 and x2 the remaining variables are z 0 level so if you see if x1 and x2 are 0 then s1 is 4 x1 x2 and 0 and s2 is 5 so s1 is 4 and s2 is 5 okay what are the cost of s1 s2 in the objective function in the objective function s1 is not here the cost is 0 in the objection function s2 is also not here the constant 0.

In the objective function s2 is also not here the constant 0 cj is in the cost in the objective function cost is 2 cost is 3 cost is 0 cost is 0 okay, now how we will start over iterations however we start over process we first find relative profits what relative profits are relative profit means you see we have to maximize this objective function subject to these conditions okay and in each iteration is want to improve the solution relative profits means change in the value z per unit change in the value of variable suppose you increase the variable x bar x1 is in 0 here.

Suppose you change s1 from 0 to 1 so what will be the value, what will be the change in the value of z that is relative profit okay. Similarly, you observe that if you change s2 from 0 to 1 then what will be the corresponding change in the value of z okay, if it is positive if the changes positive that means if you enter that variable the value of z will increase definitely because if you are increasing this variable from 0 to 1 the value of z increase the value of relative profit is positive that means that increase.

So you have to enter that variable okay, so how to find relative profits so I am not going into much detail here I am simply finding the relative profits so relative profit is nothing but you simply multiply this cb into xj 0s2+0s1 that is 0 okay, and this minus this that is -2 this is negative of relative profit, relative profit is 2 okay two units. Again this into this plus this into this is 0, 0-3 is -3, 0s1 is 0, 0-0 is 0 and 0,0 okay.

So why this that we are not discussing here this is two units it is 2 I mean if you +2 so that are relative profit correspond to s1 and relative profit means if you change the value of s1 from 0 to 1 the net increase in the value of z is two units, you can verify also if you again increase s2 from

0 to 1 the net change in the value of z is 3 units, so this is giving higher value than this because if you increase s2 from 0 to 1 then increase the value of z is 3 units.

So of course if you enter s2 then the value of z will increase much higher than if you enter variable s1 so you enter s2, entering s2 means now s2 will become a basic variable and out of these two basic variable one will go to non basic variable, it is that okay. Now out of these variables which variable will leave the basis that will be decided by minimum ratio rule what is that, it is solution up on that entering column 4 up on 1 is 1 is 4 and 5 up on 2 is 2.5.

The value which is minimum will leave the basis and this rule is called minimum ratio rule and the intersectional these two arrows this point is called pivot element okay, why we are following minimum ratio because if we leave this and not leaving this then the feasibility condition will be disturbed feasibility condition means both should be non negative because all variables must be non negative.

We have to maintain that condition all variables must be non negative. If we are not following minimum ratio rule then feasibility condition will be disturbed okay, so now we will make the next table how to make the next table now you see, it is s1 and it is s2 okay, and you make 1 here by dividing with 2 that is 1/2 10,1/2,5/2 and make 0 here with the help of this that is make identity correspondent to the basic variables always.

W ith the help of pivot element you divide by 2 here that is if it is r1 r2 r3 you replace r3 / 2 and r2 you take r2 -1/2 are far r3 this – half of this is 3/2 this one is half of o this one is half of this is one this is – half and this +3 / 2 r1 may r1n, r1 you can take r1 + 3/2 times r3 so this plus 3/ 2 time this is -1/2 0 3/2 okay. Now what a relative profit corresponds to s1 is 1/2 what a relative profit correspond to s1 is -3/2 it is simply negative of relative profit it is negative of -3/2 that means if we enter it this variable the value of z will decrease.

Now it is 1/2 relative profit correspond to s1 that means if we enter this variable again the value of z will increase then what we have obtain in this table okay. So when this process will continue this process will continue till all this relative profit all this values are greater than equal to 0, greater than equal to 0 means relative profits are less than equal to 0. So you can verify your calculation from this table itself you see in the first table we are obtaining this relative profits and this thing it is -1/2 and it is one let us see it is one or not.

S1 is 0 toward this is 3 this is 3/2 tangos this 3/2 here we have mistaken, now again we will apply minimum ratio and here it is this -1/2 of this so this -1/2 of this 4- 5/2 is a3/2 okay, now this enters because this is relative profit is positive okay and what is the minimum ration this upon this is if you take here then 3/2 upon 3/2 it is 1 and this is 5/2 upon 1/2 is 5 so this is minimum so this leave and this is the pivot element.

So again you will come to a table you will make one here by multiplying the entire row by 2/3 make 0 here with the help of this, this -1/3 of this in the entire row and this + 1/3 of this make 0 here, so you will compare a table in this way so you will obtain this table and what is the optimal solution we have obtained that is x1 = 1 and x2 = 2. Which we have also obtained using graphical approach 1 and 2 okay, you can easily check.

And you can also verify that in each table you are go then increasing your value of z you see in this table value of z is what value of z is 0 because x1 x2 both are 0 x1 x2 are 0 means value of z is 0 in this table x1 is this x2 is this, so x1 is 0 if x1 is 0 and x2 is 5/2 that is 15/2 so here that is 15/2 and the last table in the last table if you see the last table x1 is 1 x2 is 2 so what will be z, z is 8.

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	c_j	2	3	0	0		
c_B	B.V.	x_1	x_2	s_1	s_2	Sol	Ratio
	Z	−2	−3 ↓	0	0	0	
0	s_1	2	1	1	0	4	4
0	← s_2	1	**2**	0	1	5	5/2
	Z	−1/2 ↓	1	0	3/2	15/2	
0	← s_1	**3/2**	0	1	−1/2	3/2	1
3	x_2	1/2	1	0	1/2	5/2	5
	Z	0	0	1/3	4/3	8	
2	x_1	1	0	2/3	−1/3	1	
3	x_2	0	1	−1/3	2/3	2	

Optimal solution: $x_1 = 1, x_2 = 2,$
$Z = 8.$

So in the last table z is 8, that means in each table the value of z is increases and the last table is in the optimal table because all this z capital Z are greater than equal to 0 and feasibility condition is maintain. So feasibility condition definitely maintain because of the minimum ratio rule okay and this all are greater than equal to 0 that means this is in the optimal table and x1 is 2 and x2 is 2 is an optimal solution okay.

So this is one of illustration of simplex algorithm because we have to apply this algorithm so that’s why I have explained it here now the same problem of LPP in the three variable problem can be sorted in same way you can easily see now come to ULF base we are solving this problem basically and for this problem.

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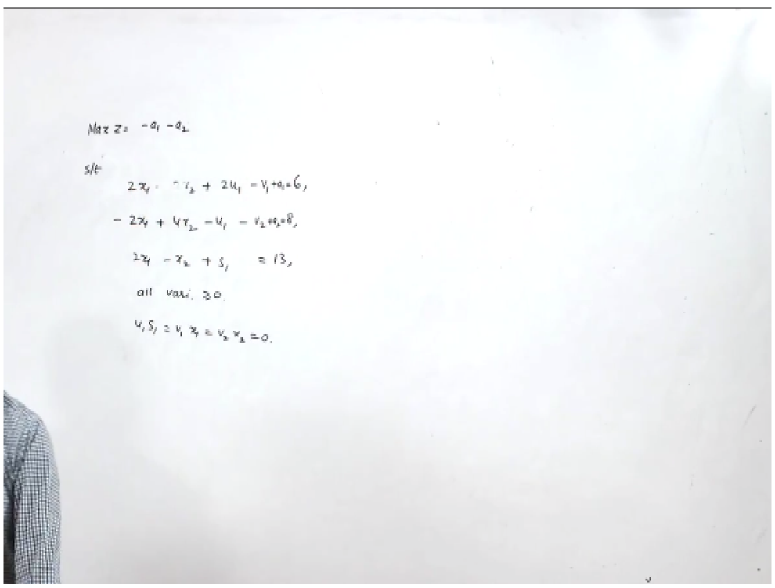

We have already right KTT conditions what are KTT conditions, KTT conditions are $2x_1$- $2x_2$+2_u or u_1-v_1 =6, the second constraint is -$2x_1$ where it is +$4x_2$-u_1-v_2=8 then it is $2x_1$-x_2+x_1=13 all variables are negative and u_1 and s_1 = v_1 x_1=v_2 x_2=0 so these are the KTT conditions which you obtained for this QPP okay.

Now how to solve this problem let us see so first except this condition all constraints are linear okay and this is u_1 s_1 v_1 x_1 v_2 x_2 equal to 0 this condition make this problem non linear okay now to apply simplex algorithm okay we need we first need a objective function number 1 and your first need variables such that you have point to which we have identity in the first step okay.

You see here it is –v_1 so coefficient of v_1 in the trial problem is – this one 0, 0 coefficient v_2 is 0, -1, 0 it is in the standard form okay it is in the standard form okay but we are not having identity correspondent to very verse but s_1 we are having s1 is 0, 0, 1 yes for s_1 we are having identity one column of identity matrix okay.

Now first how we will introduce identity and then how can be formulate objective functions so what we do we add variables what we are calling as artificial variables correspondent to first and second constraint because here we don't have identity correspondent to these constraints so we deliberately introduce to more variables here and those two variables are called artificial variables okay.

Now if you see a_1 a_2 and s_1 we are having identity corresponding to the these constraints if you see a_1 the coefficient a1 is 1, 0 a_2 0,1,0 S_1 0,0,1 so now we are having identity correspondent to these three constraints these three equations okay but this is not our problem we are change the problem by adding artificial variables by introducing two more variables in the problem okay.

Now to be constraint over objective function as and we are letting a_1 a_2 also non negative okay all variables are non negative now we want that theses two variables must leave the basis because must leave, must leave means both becomes zero if both becomes zero then only we get back to our original problem.

Our original problem is we doubt a_1 a_2 okay that we will obtain without a_1 a_2 only a_1 a_2 both are zero we construct our objective function like that only we construct that objective function like –a_1, -a_2 why this because you see a_1 a_2 both are non negative so it will be maximum only when both are zero.

It will maximum only when both are zero and as soon as both are zero we get back to our original problem what we need basically we need our point, our point x_1 x_2 we satisfy all these constraints all these conditions that will globally minimize our problem that will be the optimal solution of that problem of our problem okay and solving this problem is 25 to 30

So this is one of illustration of simplex algorithm because we have to apply this algorithm so that's why I have explained it here now the same problem of LPP in the three variable problem can be sorted in same way you can easily see now come to ULF base we are solving this problem basically and for this problem.

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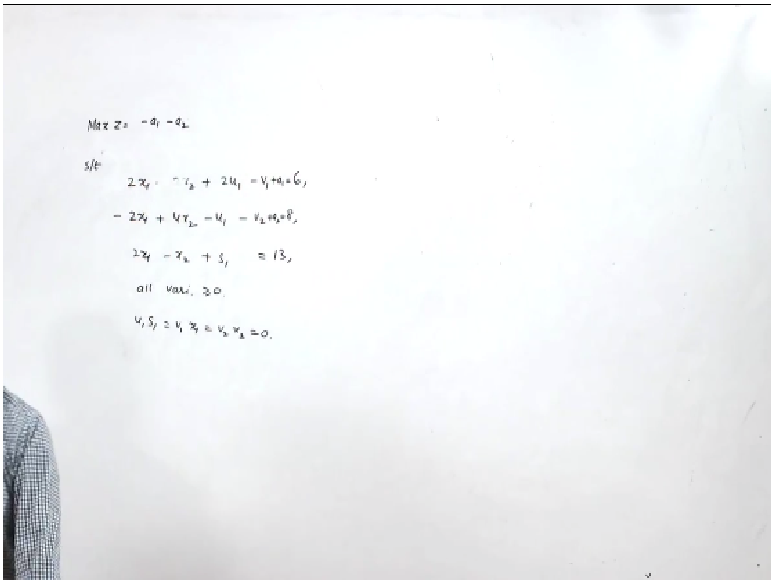

We have already right KTT conditions what are KTT conditions, KTT conditions are $2x_1$- $2x_2$+2_u or u_1-v_1 =6, the second constraint is -$2x_1$ where it is +$4x_2$-u_1-v_2=8 then it is $2x_1$-x_2+x_1=13 all variables are negative and u_1 and s_1 = v_1 x_1=v_2 x_2=0 so these are the KTT conditions which you obtained for this QPP okay.

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Our original problem is we doubt a_1 a_2 okay that we will obtain without a_1 a_2 only a_1 a_2 both are zero we construct our objective function like that only we construct that objective function like – a_1, $-a_2$ why this because you see a_1 a_2 both are non negative so it will be maximum only when both are zero.

It will maximum only when both are zero and as soon as both are zero we get back to our original problem what we need basically we need our point, our point x_1 x_2 we satisfy all these constraints all these conditions that will globally minimize our problem that will be the optimal solution of that problem of our problem okay and solving this problem is not a easy task.

I mean it is not that easy because we are having this condition also okay, so to apply simplex algorithm we add new variable which we are calling artificial variable and adding an objective function like this because as soon as this will be maximum only when both are a_1 both are 0 we get to our original problem.

If a_1 2 are not 0 a_1 or a2 not a 0 that means our problem is infeasible, our problem as no solution because a1 and a2 are not becoming 0 that means we are not getting back to our original problem that means our problem is infeasible our as no solution okay, so now we have objective function is linear these constrains and we apply the same simplex algorithm to solve this problem. Now how we can solve let us see here by bio tables.

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B.V.	x_1	x_2	u_1	v_1	v_2	a_1	a_2	s_1	Sol	Ratio
Z	0	−2↓	−1	1	1	0	0	0		
a_1	2	−2	2	−1	0	1	0	0	6	–
← a_2	−2	4	−1	0	−1	0	1	0	8	2
s_1	2	−1	0	0	0	0	0	1	13	–
Z	−1↓	0	−3/2	1	1/2	0	1/2	0		
← a_1	1	0	3/2	−1	−1/2	1	1/2	0	10	10
x_2	−1/2	1	−1/4	0	−1/4	0	1/4	0	2	–
s_1	3/2	0	−1/4	0	−1/4	0	1/4	1	15	10
Z	0	0	0	0	0	1	1	0		
x_1	1	0	3/2	−1	−1/2	1	1/2	0	10	
x_2	0	1	1/2	−1/2	−1/2	1/2	1/2	0	7	
s_1	0	0	−5/2	3/2	1/2	−3/2	−1/2	1	0	

Now you can easily construct this table you see a1, a2 and s1 this form the identity in the first table, so they can act as basic variable in the first table. All other variables are at 0 level, so the value of a1 is 6, value of a2 is 8, value of s1 is 13 in the first table okay, again you will find Z which are negative of profits okay, you know usual manner as we have already discussed. Now this relative profit is 2units and relative profit is 1 unit, all other relative profits are negative. You see this is negative of relative profit, you simply multiply with -1 and see what the relative profits are.

So you will either enter x2 or you will enter u, If you enter x2 now what additional thing is here besides simplex algorithm is there,, you see we have extra condition also u1 x1 = 0 v1 x1 =0 v2 x2 = 0, that means the both the variables u1 and s1 or v1x1 or v2x2 be not there simultaneously okay, what does that mean you see? Here it is most negative so it is entering, the compliment of x2 is what? V2 is not given in the bases in the 1st table in the 1st table v2 is not there okay, v2 is not there in the bases that means v2 is at 0 level.

V2 = 0 so if we enter this variable v2 x x2 = 0 remain satisfied because v2 = 0, so we can enter this variable, enter this leave by minimum ratio rule this only this will be counted because this is positive for minimum ratio we take ratio from only positive column entries, so this 4, this 8 up on 4 is 2 so this will leave the basis. You consider the next table, you make 1 here by dividing with 4 you make 0 here this bio element. Now you consider next table, now this is mopst negative -3/2 .

Now if you enter this u1, what is the compliment of u1, s1 u1 s1 must be 0, compliment means u1 s1 0, u1 s1, if you enter this u1 s1 is already there in the basis, so u1 x s1 may not be 0 we have to maintain the condition u1 s1 must be 0 it will possible only when 1of them is in the basis, then only because other value will be 0, then only we will be having u1 s1= 0 okay. So we cannot enter this variable because s1 is already on the basis now the next is x1 so enter this, if we enter x1 what is the compliment of x1?

V1 in the basis no, so yes you can enter x1 because v1 is not in the basis means v1= 0 the x1 = 0 condition is satisfied, so we can enter x1, we can enter x1 in the minimum ratio 10 up on 1s it is negative sop leave it 15 up on 3 it is 10, you can now leave any one because the minimum ratio rule is the value is same. So you can leave anyone it is better to leave artificial variable okay, so you leave artificial variable construct the next table and the next table is optimal one, so this is the optimal solution x1 is 10 x2 is 7 is the optimal solution of this problem.

You can easily check what are the values x1 10 x2 7 and s1 0 and all other variables are obviously 0 u1 v1 v2 all are 0. Now ig=f you take our original problem because artificial variable leave the basis now these definitely will satisfy all the variables 20 – 14 is 6 and these are all 0 = 6 it is -20 +28 is 8 all are 0 it is 20 – 7 it is 13, 13c + 0 is 13 all variables are non negative this condition is satisfied, so this point satisfy all the constrains and hence solved our problem ore QPP, so we have to apply simplex algorithm with additional condition that if we enter our variable, it is compliment must not be in the variable basis, that is the only condition we have to take care of.

If the condition holds enter the variable leave by minimum ratio rule complete table and apply till you obtain the optimal solution okay, so that is all for ULF base method and you can solve the problem by using some software which is freely available thank you very much.

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