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Nonlinear Programming

Lec-07
Quadratic Programming Problems-I

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Welcome to the lecture series on nonlinear programming. Now we will see in this lecture what quadratic programming problems are okay. We have already seen convex functions, their properties, KKT conditions, necessary and sufficient etc., okay. Now what QPP or quadratic programming problems are and how can we solve these problems we will see in this lecture, let us see. What is the quadratic programming problems first.

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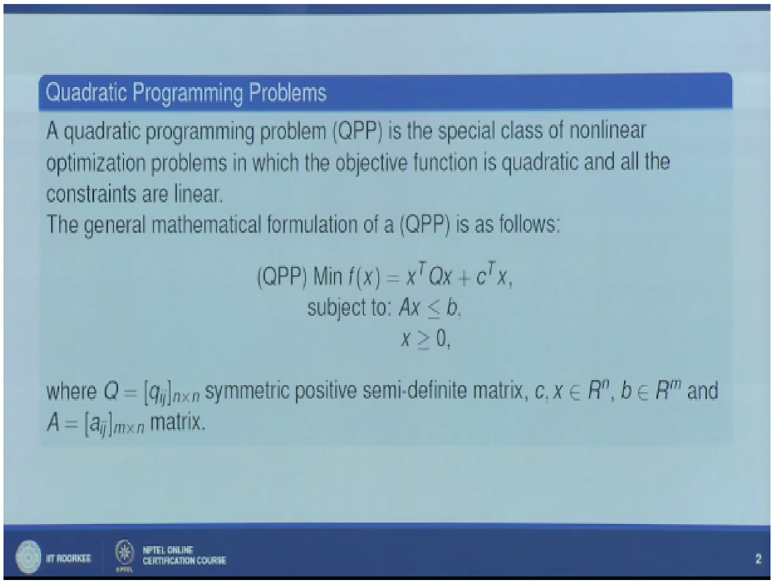

A quadratic programming problem QPP is a spatial class of nonlinear optimization problems in which the objective function is quadratic and all the constraints are linear. So if we have a nonlinear programming problem in which the objective function is quadratic, quadratic means of degree 2 and all the constraints are linear then such problems are called quadratic programming problems okay.

So the general format of QPP is minimizing the function f(x) which is equals to $x^T Qx+c^Tx$ subject to x less than equal to b and x non negative. Where Q, this Q is a matrix and this matrix is for nxn is a symmetric positive semi definite matrix. C and X belongs to R^n B belongs to R^m and A is an mxn matrix okay. So this is the general format of a quadratic programming problem. You see suppose you have this example it is minimizing.

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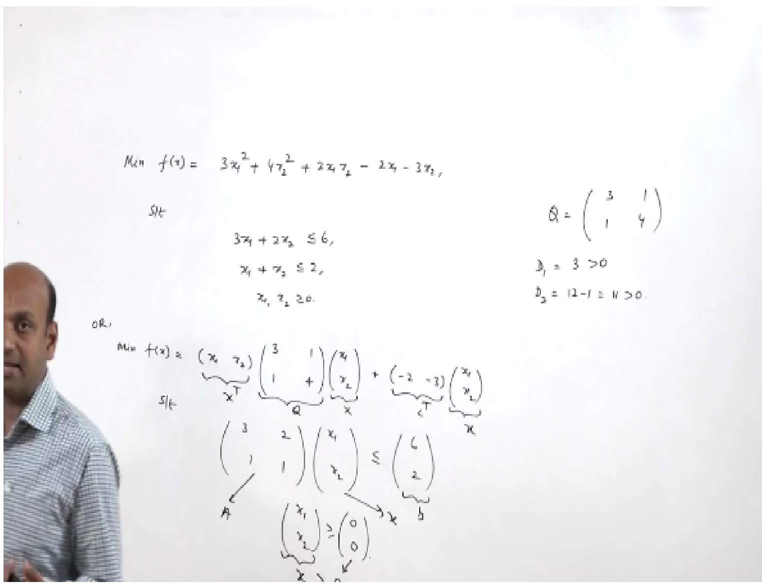

The minimizing f(x) is equals to it is $3x_1^2+4x_2^2+2x_1x_2$ – $2x_1-3x_2$ and subject to whatever conditions, conditions are $3x_1+2x_2$ less than equal to 6, and x_1+x_2 less than equal to 2, and x_1 x_2 non negative. So in this problem the objective function is quadratic because it is of degree 2. And all the involved constraints are linear, so we can say that this problem is the quadratic programming problem.

Now the main function f(x) objective function f(x) can be written as minimum f(x) which is equals to now this can be done as (x1 x2) okay. Now this 3 it is 1, you take the half of the coefficient of x1 x2 here, 1 here and 4 here x1 x2. Now when you multiply these three matrices you will get back $2x1^2+4x2^2+2x1x2$ that we can really see and +-2-3(x1 x2). So when you multiply these two it is nothing but -2x1 -3x2 which is this part of the objective function.

The linear part subject to whatever conditions, conditions are 3, 2, 1, 1 it is x1 x2 less than equals to it is 6, 2 and x1 x2 are greater than equals to 0, 0. So here if you observe this problem, so this if you take a X or x suppose this if we take at x, so it is x, if you take this as x, so this is nothing but x^T and this we can take as Q matrix Q, this is c or c^T, this is x, this is a matrix a, this is x vector x less than equals to vector b, and this x is greater than equal to 0, this is a zero vector.

So this is the same format which we discussed here, minimizing a function f(x) subject to minimize the function f(x), x^T 2x+c^Tx subject to x less than equal to b and x greater than or equal to 0. So we can say that this problem is a quadratic programming problem. Now this Q, this Q which is here, you can readily see that this Q is 3, 1, 1, 4 if you see D1 which is the leading principle minor of order 1x1, it is 3 greater than 0.

And D2 is again it is 12-1 determinant of matrix Q which is 11 is again greater than 0. So the matrix Q is positive definite and hence it is positive semi-definite okay. So we can say that this is an example of quadratic programming problem okay.

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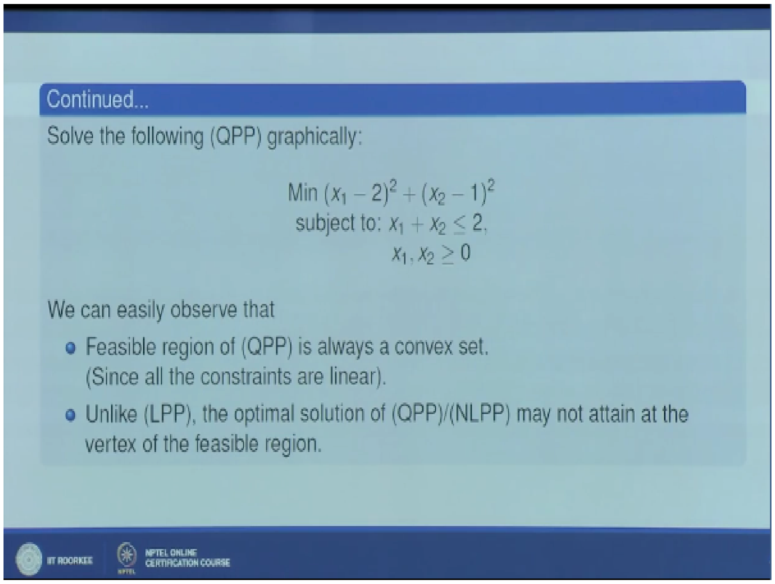

Now the first way to solve a quadratic programming problem is graphically. How can we solve a QPP graphically let us see so QPP can be solved graphically also.

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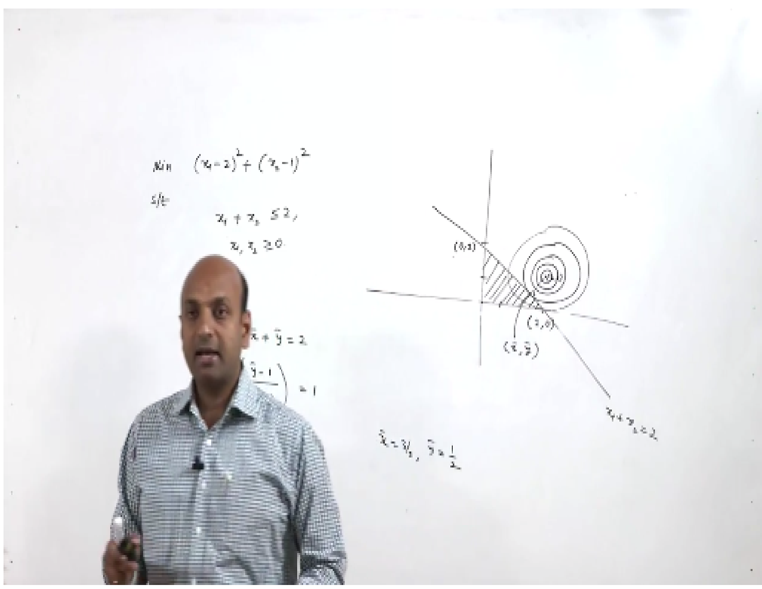

So it is minimum of $(x_1 – 2)^2 + (x_2 - 1)^2$ subject to x1 + x2 ≤ 2 and x_1, x_2 non negative so it is a QPP because objective function is quadratic in volume of degree 2 and all constrains are linear so it is a quadratic programming problem okay now how can we solve it graphically let us see so what is the feasible region, so it is 1, 2 it is 1, 2 and it is x1 + x2 = 2 because when x2 is 0, x_1 is 2 and when x_1 is 0, x_2 is 2 so this point is 2, 0 and this point is 2, 2 okay and it is ≤ 2 if you take a point on the this side of this line say 0, 0. 0 + 0 is ≤ 2 satisfying constrains.

So we shared towards origin and x1 action non negative also that is a first quadrant so this will the feasible region this is a feasible region of this problem, so we can easily see that feasible region of a quadratic programming problem, is always a convex set why because all the constrains are linear so all the constrains are convex function and we have already seen that a collection of a convex function the set constituting collection of the convex function is a convex set.

So the feasible region of QPP is always the convex set okay so this we can easily see now what to minimize we have to minimize $(x_1 – 2)^2 + (x_2 – 1)^2$ that is this is something 2, 1 this point okay but this point is not feasible we have to find that x_1, x_2 will satisfy these constrains and minimize our objective function okay, now this point will the minimum value which is 0 but this 2, 1 is feasible so we have to find that point which is closest to this feasible region that will minimize this our object function.

Now this is a equation of circle with center 2, 1 suppose radios is r okay so what we have to basically do we have to basically minimize the radius of the circle okay now we have so many circles of centre 2, 1 okay and different radius which will minimize this objective function which will touch this straight line you see if we increase the radius so that radius will be naturally higher than this radius if we increase more radius if we make bigger circle then that circle will come to the feasible region okay,

But the radius will be larger than this radius so that will not be the minimum one, so the radius which will satisfying the constraint and minimize the objective function will be the circle which touch this straight line that is this so that means this straight light will be a tangent to the circle okay now what this point is say this point is $\bar{x}$, y bar this point we have to find out this point is an optimal point okay now this point will satisfy this constraint so this constraint will be suppose $\bar{x}1$, $\bar{x}$, y bar = 2 okay.

Satisfy this constraint and now this is in perpendicular also and this centre is 2, 1 slope of this land will be nothing but y1 bar – y bar 1 upon $\bar{x}$ -2, and slope of this line is -1 and both are perpendicular so this will be noting but = 1 okay because they are negative reciprocal so from here what we obtain y bar – 1will be nothing but $\bar{x}$ - 2 and this implies $\bar{x}$ - y bar is nothing but 1, now from these two equations we can easily get $\bar{x}$ and y bar.

So what are the value of $\bar{x}$ and y bar when you add them you will get $\bar{x}$ as 3/2 and y bar as 1/ 2, so this is the optimal point $\bar{x}$ 3/2, y bar ½ that means x1= 3/2 and x2 = ½ will be the optimal solution of this quadratic programming problem okay so in this particular problem way.

We observed two things number one visible region of a quality program problem is always a convex set because it is compromising of set of linear constraints all liner constraints are convex so the set constituted by the convex set convex functions is always a convex set, so the visible region is a convex set number two the optimal solution of a quality programming problem, may not be obtained the vertexes in liner programming problems what we have a linear program problem always the optimal solution that is a minimum or the maximum the objective function if attained.

Attend only at the vertexes however in quality program problem it may attend on the it may attend other than the vertexes okay as we have already seen in this particular example, so there is a one way out to solve quality programming problem but we can easily see that only for two problems we can solve graphical approach we can apply graphic approach but not for problems more than two variables so how can we solve those problems let us see.

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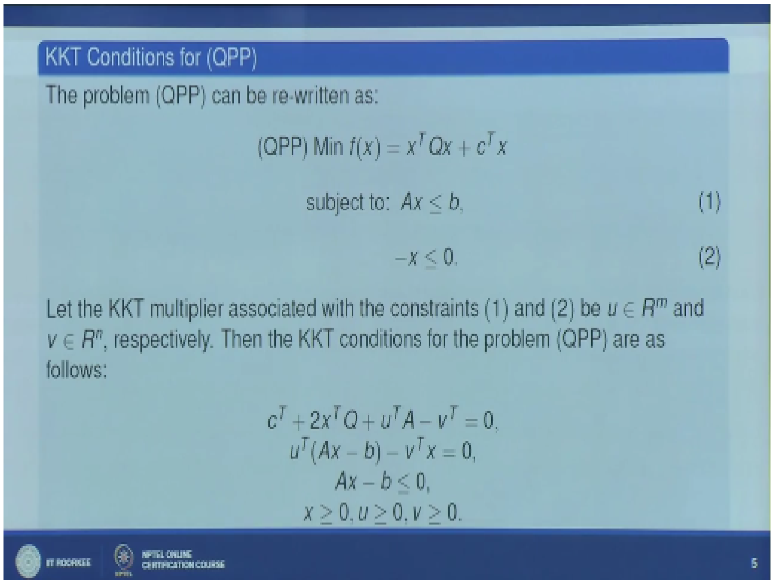

So we will write KKT conditions for QPP okay so what was the our QPP.

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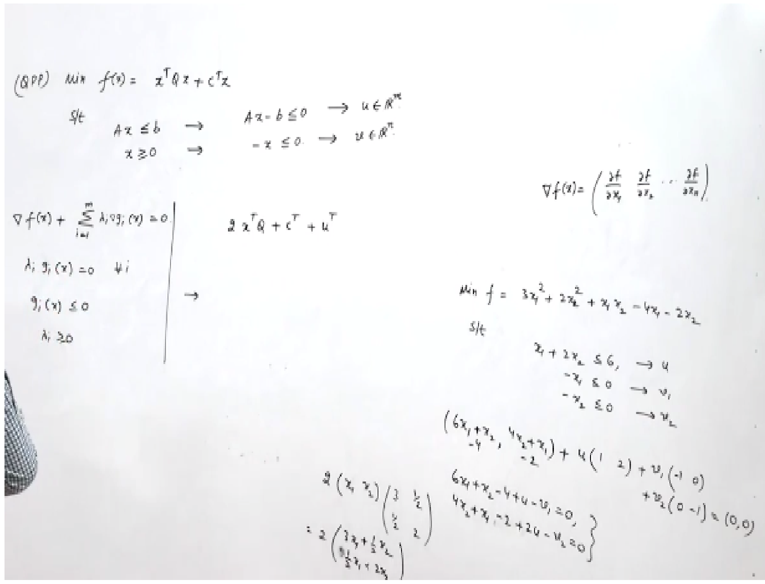

QPP was minimizing fx = x^T 2x + c^Tx subject to x < = nx non negative now this constraint can be written as ax – b < = 0 and this constraints –x < = 0 because for KKT conditions all constraint must be less than equal to 0 type okay, now suppose we take multiply corresponding to these constraints say u bar or u which belongs to Rf how many constraints we are having for this system m constraints, so we will be so we will need m number of KKT multipliers for ax – b < = 0 constraint.

Now how many constraints here we are having m so we will be we need n number of cons variables for these particular these constraints okay, so how many KKT multipliers we were having now m + n m for these constraints because these are m in number and n for these constraints because this is n an number so total number is KKT multipliers required to solve require to find out the KKT conditions, for QPP r m + n okay. Now how to write KKT conditions so we already know KKT conditions are it is gradient of a fx + ∑i from 1 to m.

λi gi of x = 0 and λi gix = 0 for all I and gix < = 0 and λi non negative now here instead of λi which are the KKT multipliers correspond to the constraints we are having u and v as KKT multipliers correspond to the constraints okay, objective function is x^T to s c^T x, so let us write down first KKT condition what is gradient of fx gradient means what is gradient of fx means what gradient of fx means ∇f / ∇x1 ∇f / ∇2 and so on ∇f / ∇x okay, now you can understand this how to write this KKT condition by an example.

Suppose objective function is minimum of x suppose it is $3x^2 + 2x2^2$ + x1 x2 – 4x1 – 2x2 and suppose it is subject of x1 + 2 x2 < = 6 and – x1 < = 0 and x2 = 0 suppose we have this QPP so what are the KKT conditions for this problem writing this KKT condition we will write the KKT condition for each other QPP okay.

So what will be gradient of fx for this problem writing this KKT conditions we will write the KKT condition for a general QPP okay. So what will be Δf(x) for this problem, it is $6x_1 + x_2$ then it is $4x_2 + x_1$, $\partial f/\partial x_1$, $\partial f/\partial x_2$ of course -4 will be here and -2 will be here, plus now we are taking u here and v_1 here and v_2 here these are the KKT multipliers ,okay. These are this is only one constraint and two these are two constraint of v_1, v_2 and this is only one constraint the u will be KKT multiplier corresponding that constraint.

Then u times gradient of this, so 1 and 2, gradient of this means ∂g/∂x1, $\partial g/\partial x_2$ + v_1 times derivative of this f_2 x_1,$-_1$ and $x_2(0)$ + v_2 (0,-1) and it is (0,0) so the first equation is nothing but so what is the first equation, so we can obtain this +u,-v1,+0 = 0 that is $6x_1 + x_2 - 4 + u - v_1 = 0$ the second equation will be $4x_2 + x_1 + 2u_1$ okay $-_2 + 2u - v_2 = 0$,okay.So these are the KKT condition across point to this.

Now if we try to write on the derivative of this or the gradient of this so what will be the gradient of this quadratic term, it will be nothing but $2x^TQ + C^T$, why it is? You is this C from here from this example what is $_2$ times x^T S_1, S_2 and what is Q? Q is, Q for this problem is 3, 2, ½, ½ okay. Now when you multiply this with this it is nothing but $6x_1$ you see it is nothing but $2(3x_1 + ½ x_2)$ and then $(1/_2 x_1 + 2x_2)$ and you multiply it with 2.

So it is $6x_1 + x_2$ okay and then $x_1 + 4x_2$ you write in a row form or column form it is upto us we have to follow the same thing in the entire problem either the entire thing we write in a row wise or entire thing we write in a column wise, okay. So the first term that is this term plus C^T, C^T here is what,-4-$_2$ so -4 -$_2$ when you add here so it will be -4 -2, okay. Then + U^T because it is some of λ with G_i that is $\lambda_1 G_1 + \lambda_2 G_2$ + and so on $\lambda_m G_m$

So it is U^T gradient of first constraint I mean these constraints so this is A gradient of this is nothing but A you can easily see if you write the KKT conditions from fro this particular problem and then minus it is V and gradient of this is nothing but –I = 0,you can see it is (-I) okay. Now so this is a first equation this equation that gradient of F(x) which is this term plus U^T we are taking transpose so that term so that we get a sum of all this things. $U_1 g_1 + U_2 g_2$ +Um gm okay now U^T gradient of this is A plus and it v^T and gradient of this that is minus sign it should be 0. Now $\lambda_i g_i$ so you multiply with λ either you take each component or you take sum of all the constraints with Lagrange multiplier or KKT multipliers, so it will be $u^T(Ax-B)-v^Tx$ that should be 0, we have for this problem m+n number of constraints okay, you multiplier with this is a $u^T(Ax-B)$ this term, sum of all like $\lambda_1 g_1+\lambda_2 g_2$ and so on and v^T into this that is the constraint.

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Then feasibility condition is $Ax-B\leq 0$ and $x\geq 0$ $u\geq 0$ and $v\geq 0$ all multipliers all λ is must be greater than equal to 0 okay, so these are the KKT condition for this problem okay, you can easily check these KKT conditions while writing one example of QPP try to write down the KKT condition for that problem and then try to see that we can, we will obtain these condition for this particular problem this QPP okay. Now these conditions can be rewritten as.

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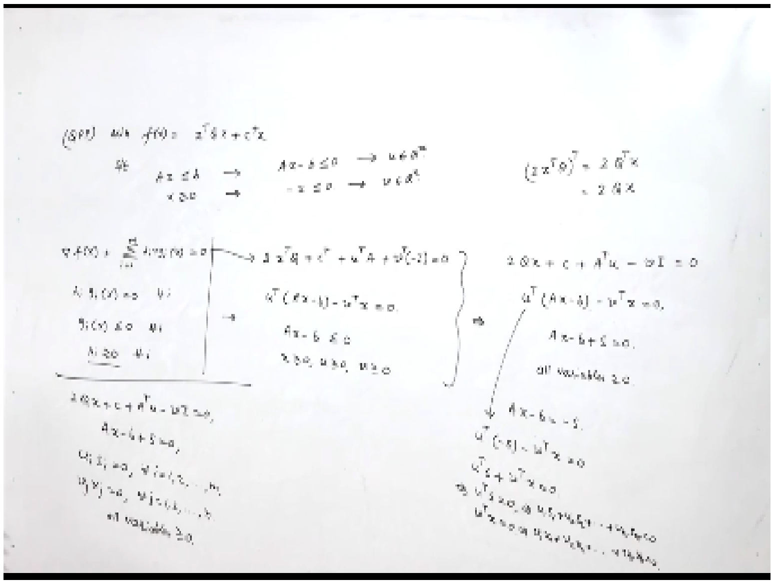

So these are the KKT conditions for this QPP okay, now these conditions can be rewritten as now you take transpose both the sides for first equation, now transpose of A+B is same as A^T+B^T so transpose of this which is nothing but $(2x^3)^T$ is nothing but Q^Tx because AxB^T, B^TA^T now Q is a symmetric matrix, so Q^T Q itself so it is 2Qx, so it is $2Qx+c+A^Tu$ you are taking transpose both the sides and a transpose of this is –vI=0 and you leave this constraint as it is $u^T(Ax-B)-V^Tx=0$ Ax-. Now this is less than equal to 0 so you can add one variable here to make it an equation, so Ax-B+S=0 we are adding slap variable we call it slap variable okay, to make it an equation.

And all variables non negative, all the involve variables are non negative. Now the same conditions can be rewrite as you see the first equation the first condition remain as it is so first condition will is nothing but $2Qx+c+A^Tu-vI=0$ the first condition, now if you see the second condition you see it is Ax-B+S=0 so what we obtain from here Ax-B is nothing but –S you substitute minuses over here.

So it is $u^T\text{-}S\text{-}v^Tx=0$, if we put this in this condition and this is nothing but $u^{Ts}+v^Tx=0$, now u is non negative S is non negative then there some $u_1s_1+u_2s_2$ up to u_ms_m will be non negative and v is non negative and x is also non negative then their sum $v_1x_1+v_2x_2$ and so on up to v_mx_m that will also be non negative and sum of two non negative values is equal to 0 this implies u^Ts is 0 and v^Tx is 0 it is possible only when both are 0 because both are non negative okay.

Now u^T as a 0 what does it mean it means u^T s is what u1 s_1 + u_2 s_2 and so on up to u_m s_m and this means what vb1 x1 + v2 x2 and so on up to vn xn. Now if now again all the components are non negative and some is 0 this means uy si = 0 for all I okay this means y1 x1 = 0 u2 x2 = 0 and so on um s_m = 0.

And again if we are plat the same concept here because each terms in this expression are non negative and some is 0 this means each term is 0 so what we have finally conclude it whether first constraint first equation in mendacities then it is ax – b + x = 0 and we have constraint which is u_i s_i = 0 for all I from 1 to m and vj xj = 0 for all j from one to m, so these are the KKT conditions for QPP okay and all variables must be non negative of course.

So they are a KKT conditions for QPP okay, so these are the KKT condition which we just obtained this is the KKT conditions for this problem.

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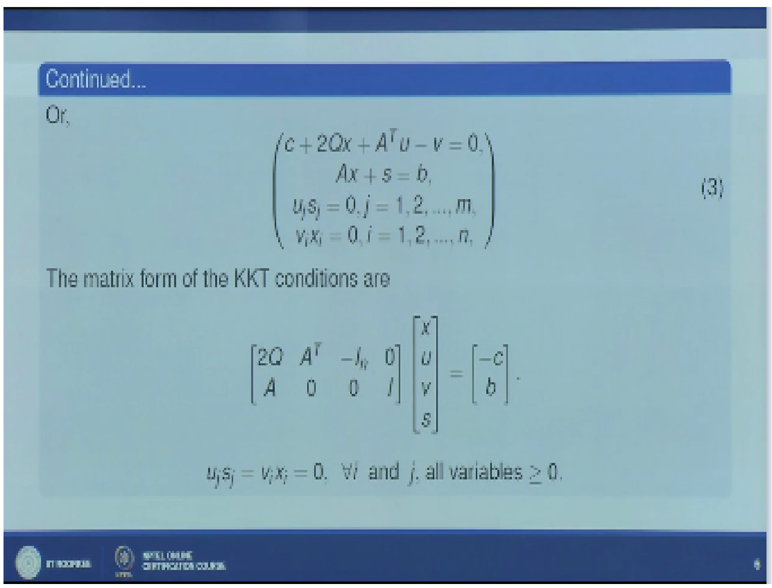

The matrix form we can easily obtain form here you see it is 2Q a^T – I and 0, 0 for s okay because s variable is also there and this is a this is 00i, i for s variables are x u v s and right hand side are –c and b with u_j s_j and vi xi = 0 for all inj and all variables non negative okay.

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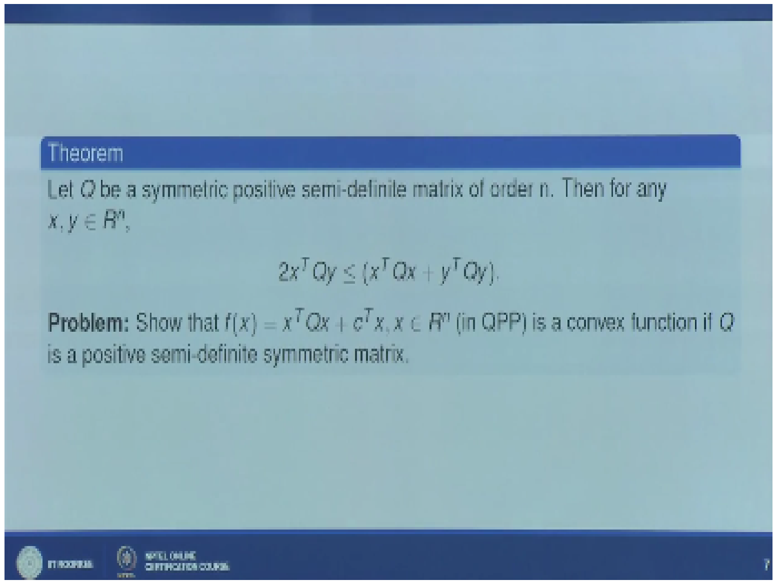

Now that KKT conditions which you obtain for QPP when it will be sufficient we have already discussed that KKT conditions are sufficient only when they involve function and the constraints all are convex if we have minimizing f_x subject to gis ≤ 0 for all I okay if we have this format of non linear programming problem that is minimizing or f_x subject to gis ≤ 0 for all I then the KKT condition is sufficient if f and g_i for all I is the convex that is if we are having convex programming problem.

Now in QPP all constraints are linear all constraints are linear means all constraints are convex okay so it is suffices to see that under which condition the involved objective function is convex if the involve objective function is convex for some particular Q, then we can say that his problem will be a convex programming problem and hence the KKT condition will become sufficient.

So now for checking for that Q we have first theorem that is let q be a symmetric positive semi definite matrix of order n then for any x y belongs to R^n, this result always hold up let us try to put this result first now here Q is a symmetric positive semi definite matrix.

So that means z^t Q_z it will be ≥ 0 for all z in R^n okay because it is a positive semi definite matrix if it is true for all z in R^n so it will be true for x – y also, it is true for any x in R^n so it will be true for z = x – y also. now if you simplify this so it is x^T Qx –x^T Q y –y^T Qx --+y^TQy =0 so this implies x^T Qy+y y^T Qx is less than equal to x^T to x+ y^T to Qy okay now this x^T Qy okay this is the scalar this is a scalar because we can see x^T is the rule vector to it will be of one row and columns x^T it is across sign and it is one cross it n cross 1 it is a collar vector.

So when you multiply to it is n cross 1 and 1 cross n that is it will be the some new matrix C which is of 1 cross 1 so it is a scalar okay now if it is a scalar since it is a scalar so x^T to Qy= ^{T} we can easily because the transpose of a scalar itself okay now this is nothing but y^T to 2^tx so 2 is a symmetric matrix so Q^T is Q so it is y^T Q x oaky.

So that based both are these, these are equal so we can easily write it implies $2x^T$to Qy will be lesser equal to x^t to Qx +y^T to Qy so in this way we have proved that first result oky now this result will be use to prove the next problem.

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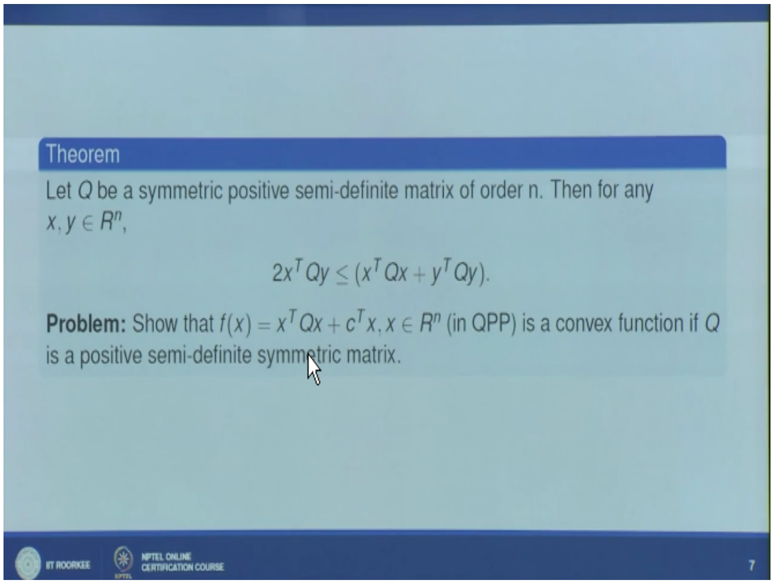

That is showed that function f we are transpose in QPP is a convex function if Q is appositive semi-definite symmetric matrix okay so how can we prove this the proof is simple .

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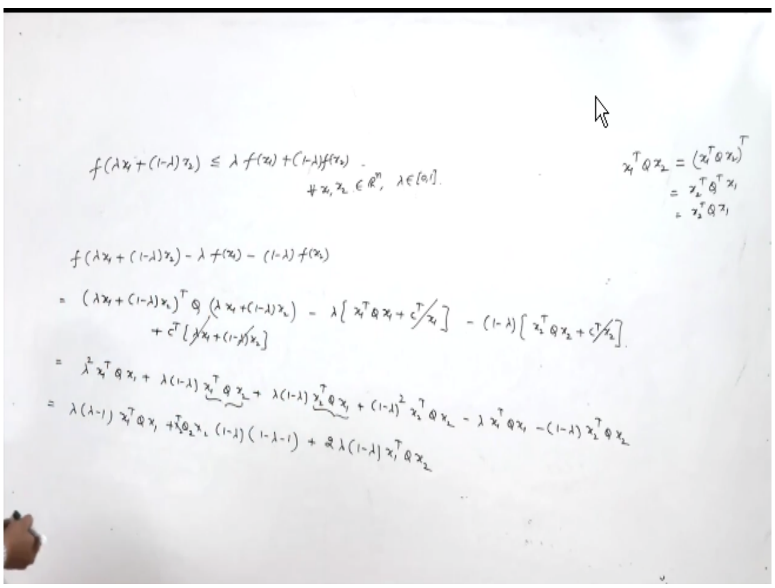

Now in how to proof this result we need only to show that f of λx_1+1-x_2 is lesser equal to $\lambda f x_1$+1-λx_2 for all x_1 and x_2 is Rn clear so λ belongs to 0 to 1 this we have to show in order to show that the involve function is a convex function so let us try to obtain this result let us take this side so this is equal to we have to show that this is less than equal to zero okay.

Then we can say the function is affix convex now what is the f , f is x^t to Qx so what will be f λx_1+1-λx_2 it replace x in this term this is λx_1+1-λx_2^T Q λx_1-x_2 –$\lambda f x_1$ x1T to Qx_1+C^T x_1 okay we left with two more terms here that is $+c^T$ λx_1 $-\lambda x_2$ and $+c^T$ here its okay, okay this is $f\lambda x_1+\lambda x_2-fx_1$ this term and – 1-λ it is x_2^T Qx^2+C^T Qx_2.

Now λ is a scalar can be taken out λ times c transposes x_1 and it is also λ time c transpose x_1 positive and negative will cancel out and 1-λ is again as a scalar can be taken out so it is 1-λ times c transpose x_2 so it is also x_1 so this term also cancelled out so what we are left with now you multiply take the transpose and multiply with this term it is λ than it is at λ $x1^T$ λ^2 it is this term + it is λ x 1- λ $x1^T$ $2x_2$ + again λ x 1- λ $x2^T$ x1+ $1-\lambda^2$ $x2^T$ to x2 and it is - λ $x1^T$- 1- λ $x2^T$ $2x_2$ okay, now let us try to simplify this okay. Now these two terms are same, are equal why? Because we have already seen that it is skilled quantity $x1^T$ so it will be equal to ^{T}, and if it is equal to ^{T} means $x2^T$ 2^T x1 and 2^T will be q because matrix, so both are equal.

So this term and this term are equal okay, so we can write this as and we can club these two terms so it is λ x 1- λ $x1^T$ x1, we can club these two terms also by taking 1- λ common okay, so + we can $x2^T$ x2 as common so again 1- λ common so it is1- λ- , okay from these two terms take λ common so this 1- λ sorry oaky, from these two terms you take 1- λ we obtain 1- λ-1 and these terms are same, so it is 2λ 1- λ $x1^T$ x1 x2. Now we apply this result that is.

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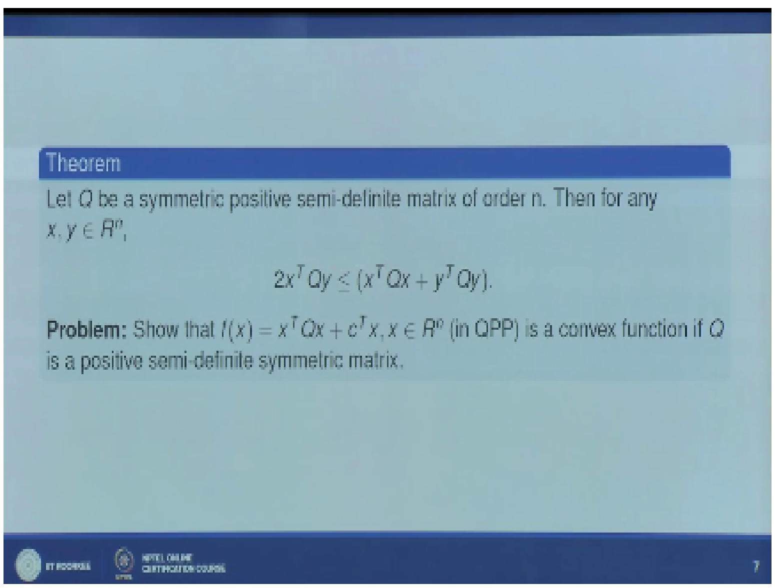

If q is the symmetric positive semi definite matrix then twice of this x ^{T} ≤ this quantity, this we have already proved , so we can apply this result over here, so this will ≤ , this will remain as it is and again 1 canceled out -1-λ x2 ^{T} x2 and it is ≤ 2 times this ≤ x1 ^{T} x2 ^{T} $2x_2$. Now this 1-λ is this and λ x 1-λ so both will cancel you call it positive negative sign. Here it is – of this and it is plus of this again both will cancel, so this is = 0.

That means that this quantity we have shown that this expression ≤0 and hence the function is a convex function. so we can easily see, we can easily say that if in QPP the involved Q if it is positive semi definite matrix then the function will be a convex function and hence it will be a convex programming problem and therefore KKT condition which we have just obtained becomes sufficient, becomes sufficient means if we solve the KKT conditions then the x which we obtained by solving those equations or conditions will global optimum or global minimum.

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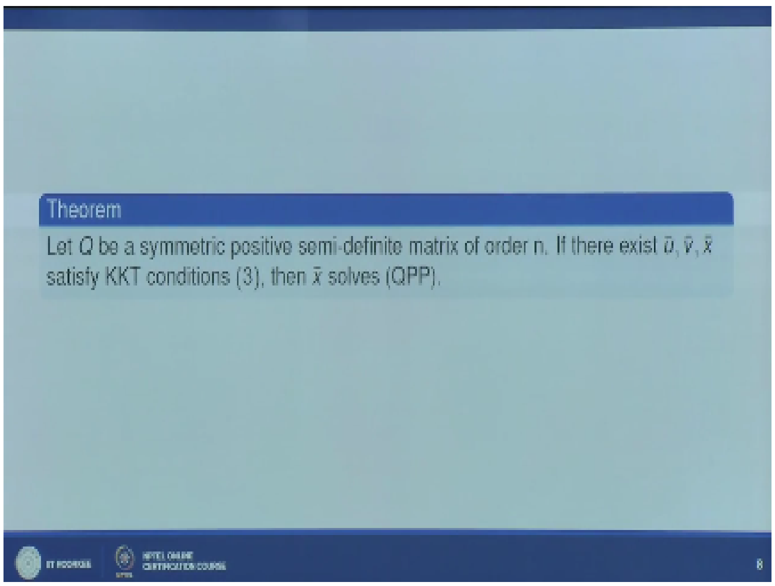

Q is the semi definite positive matrix of order n and if there is ū, v̄, x̄ satisfy the KKT condition 3 these are the KKT conditions okay then x̄ solves QPP, x̄ will be the global minimum point of QPP because then KKT condition will become sufficient okay, so in this way. So we will see ion the next class that how can solve the general QPP using KKT conditions okay, so thank you.

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