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NPTEL ONLINE CERTIFICATION COURSE

Nonlinear Programming

Lec-06
KKT Optimality Conditions

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Welcome to the lecture series on nonlinear programming. In the last lecture we have seen about convex programming problems, I told you that if we have a minimizing type of object function subject to all constraint less than equal to type, less than equal to zero. So this we call as a convex programming problem if the objective function F and all constraints are convex okay. We have also seen some examples of convex programming problem and unconstraint optimization problem we have discussed, how we can find out optimal solution of a unconstraint optimization problem.

And constraint optimization is with equality constraints using Lagrange function also we have seen in the last lecture. Now in this lecture we will see KKT conditions, what KKT conditions are and why convex programming problems are important that also we will see in this lecture. Now if we have our equally type constraints then we use Lagrange function, I mean Lagrange multiplier method to find out the optimal solution of that type of problems.

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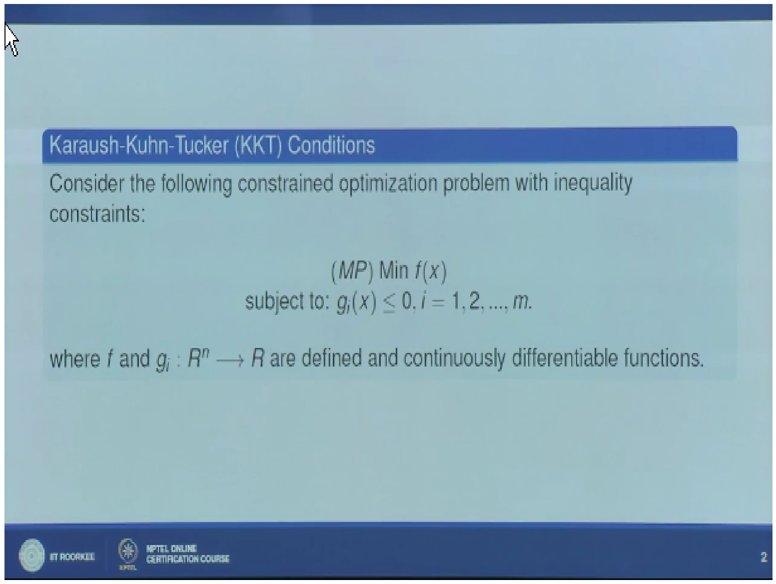

But if we have a constraint optimization with inequality type constraints, then we use Karaush-Kuhn-Tucker conditions to find optimal solutions of such problems. So consider this constraint optimization problem with inequality constraint minimizing function f(x) subject to $g_i(x)$ less than equal to 0, where f and g_i from R^n to R are defined and assumed that they are continuously differentiable functions okay. Now what a necessary condition, so what problem we are taking now.

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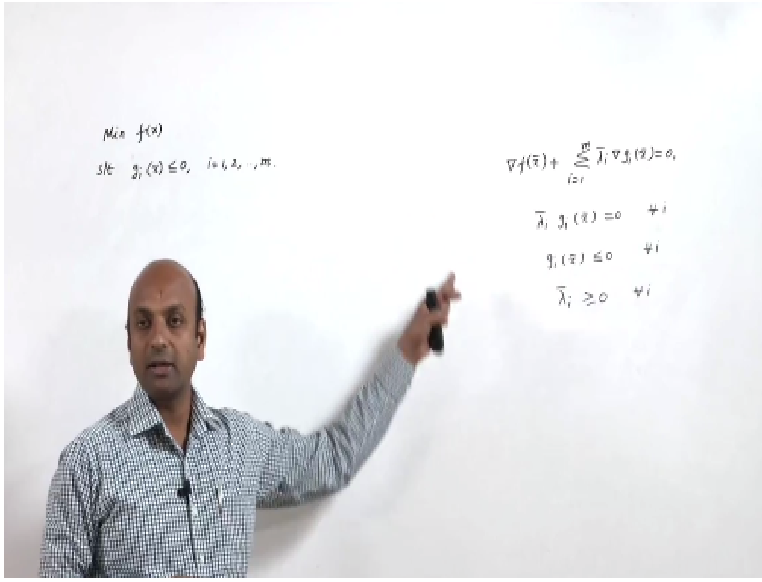

We are taking minimizing f(x) subject to $g_i(x)$ less than equal to 0_i from 1 to m okay.

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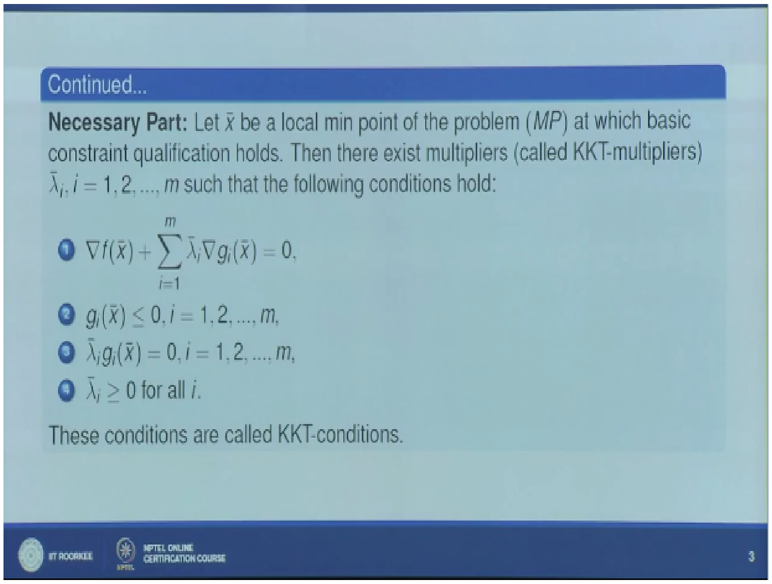

So what a necessary condition, let x bar be a point of local minima for the problem MP, MP is this problem at which the basic constraint qualification holds. Then there exist multipliers which we are calling as KKT multipliers $\lambda_i$bar I from 1 to m such that the following condition hold. So what are the conditions, number first condition is same as the Lagrange condition, for the Lagrange multiplier method.

And all these conditions are also important and we are calling these condition as KKT conditions, first condition is gradient f(x)bar+ Σ_i from 1 to m λ_i gradient of g_i at xbar must be 0 okay and λ bar okay. The second condition is $\lambda_i$bar g_i(xbar) must be 0 for all i, g_i(xbar) should be less than equal to 0 for all i, this is the feasibility condition and all multipliers which are $\lambda_i$bar must be non negative for all i.

So these conditions basically are called Karaush-Kuhn-Tucker conditions or KKT conditions okay. Here this $\lambda_i$bar are called KKT multipliers okay in Lagrange multipliers method we have an equality constraints that is why we do not require that λi bar or Lagrange multiple in those problems are non negative here we have less than equal to type constraints so here this condition is required that all multipliers must be non negative, okay so this is a necessary condition that if a point is a point of local minimum then this condition must this always satisfied. Now we come to the sufficient condition sufficient part if.

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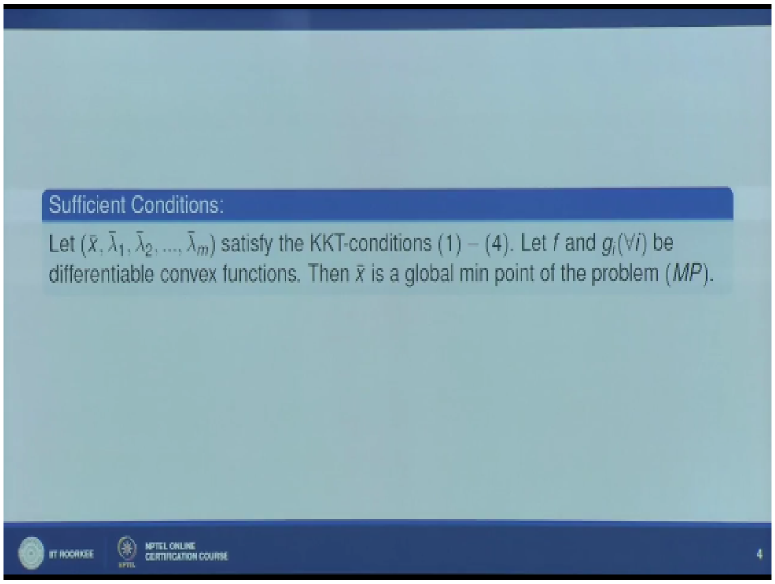

If λ bar and x̅ satisfy these 4 conditions okay and F and Gi are differentiable convex function that is a convex programming problem because if the function f and all gi is a convex then we call such as problems convex programming problem and here we are assuming that function and all constraints are convex then x̅ is a global minimum point of the problem MP so if this condition holds for a convex programming problem then the point x̅ and λ bar which we obtain solvent these conditions.

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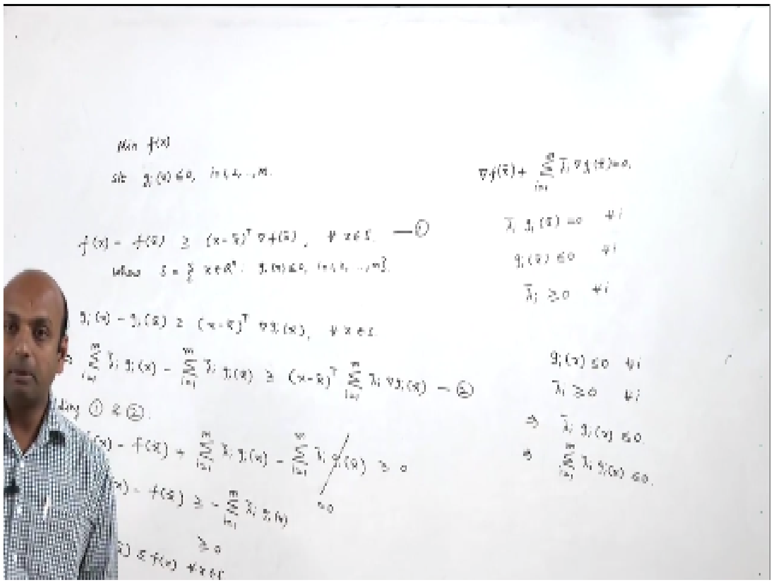

Is always a global minimum point of the problem MP, so the proof is very simple for the sufficient condition proof is very simple let us now function f is a differentiable convex function what does it mean this means f(x) is $f(x) - f(\bar{x})$ is $\geq (x - \bar{x})^T \nabla f(\bar{x})$ and this must hold for all $x \in$ s where s is all $x \in R^n$ such that $g_i(x) \leq 0$ I from 1 , 2 that is a set of that is fusible set okay as the fusible set and if f is a convex function then this means $f(x) - f(\bar{x})$ should be ≥ to this condition, this we have already discussed again g_i is also convex for all (i) this means $g_i(x) - g_i(\bar{x})$ must be $\geq (x - \bar{x})^T \nabla g_i(\bar{x})$ and it must hold for all $x \in$ s also.

And these condition hold it is given to us in the statement okay so λi bar are no negative so we can multiply by Σ over here okay it will not change the inequality because they are non negative and some from i from 1 to m so wh=at we obtain this ⇒ Σ λi bar $g_i(x)$- Σi from 1 to m λi bar $g_i(\bar{x})$ should be $\geq (x - \bar{x})^T$ Σi from 1 to m λi bar $\nabla g_i(\bar{x})$ okay because λi non negative yiu can multiply by λi and add all the constraints all the m constraints so we obtain this condition now you add 1 and 2 adding 1 and 2 it is the left hand side will be fx – f $\bar{x}$ + ∑i

From 1 to m λi bar gi of x - ∑i from 1 to m λi bar gi of $\bar{x}$ and greater than equal to, now what will be the right hand side, you see this term is common in both the equation so f – s bar transpose come out and in inside bracket we will be having this plus this and this from the first equation is 0, so this will be equal to greater is equal to 0 using the first equation okay, now again λagix, gix bar = 0 for all I so the sum will also be 0, so we can easily see that this is = 0.

Because λa bar gi $\bar{x}$ = 0 for all i then this means sum over i from 1 to m will also be 0 because each, for each 8 is 0 and now gi $\bar{x}$ is less than equal to 0 and λL > = 0 so this implies because gi $\bar{x}$ is okay it is visible okay, x so x is any point in s and s satisfying this in equality so gi of x < 0 for all i and λi bar is > = 0 for all i this implies λi bar into gi of x is < = 0 non negative and non positive the product is always non negative or non positive okay. So this implies ∑ also < = 0 because each term is < = 0 then sum is also < = 0 so from here we can say.

That fx – f $\bar{x}$ will be greater than equals to negative of ∑i from 1 to m λi bar gi of x and this quantity is less than equal to 0 this means negative will be greater than equal to 0 so this will be greater than 0, so we obtained from here we obtain that f $\bar{x}$ is < = fx for every x, because s is arbitrary point in s that means for every visible point x in s f $\bar{x}$ with always < = fx that means f bar is a global minimum point, so what we have to that of any function f and constraint gi satisfy these four conditions a long width f and all gi is the convex then the x bar which is obtain by solving these condition is always be a global minimum point of this problem okay.

So there is sufficient condition and let us discuss few examples based on this, so this is the advantage of our application of convex programming problems because convex programming problems make KKT condition sufficient it is because only because of convex programming problems that is function and all gi is are convex the KKT conditions becomes sufficient okay.

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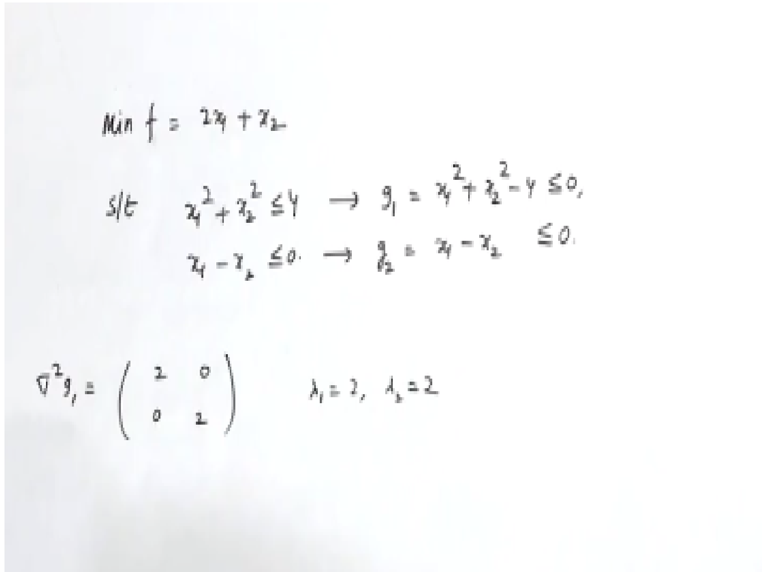

So let us discuss our first problem minimizing f = $2x_1 + x_2$ subject to $x_1^2 + x_2^2 \leq 4$ and $x_1 - x_2 \leq 0$,so how to solve this problem? So this is g1,g_1 is nothing but $x_1^2 + x_2^2 - 4 \leq 0$ and this is g_2, g_2 is nothing but $x_1 - x_2 \leq 0$, okay. So first we will see whether this problem is a convex programming problem or not because then only the KKT conditions will be sufficient, okay. Now it is a minimizing type problem and all concerns are less then equal to type.

So it will be a convex programming problem where the function and all concerns are convex, so function is linear function so it is obviously convex okay the first constraint g_1 his hessian matrix is 2, 0, 0, so λ_1 is $_2$ and λ_2 is 2 you can easily see the Eigen values of this matrix is 2 and 2 which are greater than 0, okay. λ_1 and λ 2, λ_1 >0 λ 2> 0 you read this see so it is positive definite that means functionalistic convex and hence convex, okay.

And the second concern is the linear constrain so it is obviously convex so therefore it is a convex programming problem. So let us write down all the KKT conditions and try to find out a x-bar satisfying the KKT condition that will be a global medium point of this problem by the sufficient KKT condition we can easily see, okay. So we will try to find try to write down the KKT conditions what are the KKT conditions,

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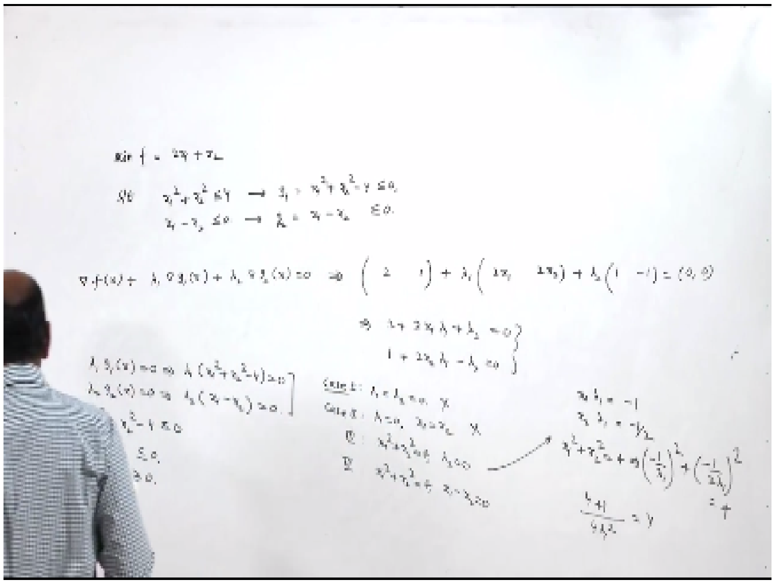

KKT conditions is ∇ + $\bar{x}$ now here we are having two constraints so $_2$ KKT multiplies will be involved it is $\lambda_1 \nabla g_1(\bar{x}) + \lambda_2 \nabla g_2(\bar{x}) = 0$, so what does it imply? It is you can take $\bar{x}$ or x it early matters okay you can take x also that x that satisfies this condition will be the global minimum point, so what is $\nabla g(x)$, it is (2, 1) + λ_1 first with respect to x_1 then respect to x_2, λ_1 gradient of g1, g_1 respective to g_1 $\nabla g1/\nabla x_1$ is 2x1.

$\nabla g1/\nabla x_2$ is $2x_2 + \lambda_2$ it is (1, -1) = (0, 0) so what we obtain from here, it is $_2 + 2x_1 \lambda_1 + \lambda_2 = 0$ and it is $_1 + 2x_2 \lambda_1 - \lambda_2 = 0$, so these are two condition we obtain from here. Now what are other conditions? $\Lambda_1 g_1 x = 0$ so this implies $\lambda 1(x_1^2 + x_2^2 - 4) = 0$ and $\lambda_2 g_2(x) = 0$
So this implies $\lambda_2(x_1\text{-}x_2)=0$, again we have a feasibility condition is $g_1x \leq 0$ and $g_2x \leq 0$ and λ_1, λ_2 both must be non negative. So these are the cyclotic conditions, now how to solve this condition let us see, now from these two conditions we see that this into this is 0 which is possible only when either this is 0 or this is 0, okay this into this is 0 this means this is 0 or this is 0, so that means we will be having four different conditions, four different cases.

So we will take case 1 okay, let us take case 1 when $\lambda_1=\lambda_2=0$ okay, case 2 when $\lambda_1=0$ λ_1 remains 0 means this condition hold and suppose this is λ_2 is not 0 this is 0 may be we cannot talk about λ_2 this is 0 suppose okay. Now case 3, when $x_1^2+x_2^2=4$ and $\lambda_2=0$ and case 4 when $x_1^2+x_2^2=4$ and x_1-$x_2=0$ so these are the only four cases which can be obtained from this condition okay, from these two conditions.

Now suppose the first condition $\lambda_1\lambda_2=0$, if λ_1 $\lambda_2=0$ then from these two condition 2=0 or 1=0 which is not possible, so this condition is not possible okay, this condition is not possible. Now suppose $\lambda_1=0$ if $\lambda_1=0$ so from here λ_2 is -2 and from here λ_1 λ_2 is 1 which is again not possible because λ_2 will be having only one value, and that must be non negative also, so this cases also omitted, this case is also not possible, okay.

Now take out the third case when $\lambda_2=0$, λ_2 is 0 and $x_1^2+x_2^2=4$ okay, so $\lambda_2=0$ and this hold then what we obtain from this condition we obtain x_1 $\lambda_1=-1$ and x_2 $\lambda_1=-1/2$ okay, now $x_1^2+x_2^2=4$ also so when you apply this condition $x_1^2+x_2^2=4$ what we obtain, we obtain $(-1/\lambda_1)^2+(-1/2\ \lambda_1)^2$ should be 4, so what will λ_1 from here it will be 4 λ_1 will be the LCM it is 4+1 will be equal to 4 and that means λ_1 will be nothing but.

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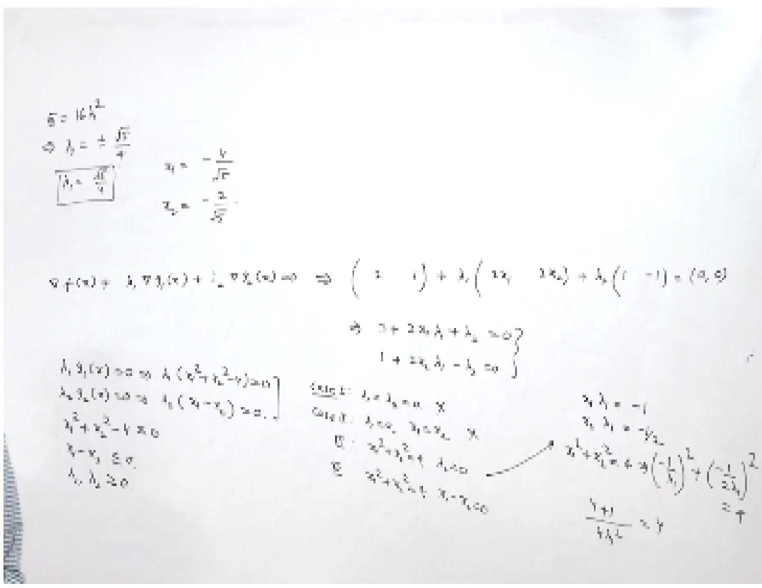

From here what we obtain it is 5=6 λ_1^2 so this implies λ_1 will be nothing but $\pm\sqrt{5}/4$ now λ_1 cannot be negative so λ_1 will be under $\sqrt{5}/4$. Now if λ 1 is $\sqrt{5}/4$ so what will be x1, x1 will be nothing but $-1/\lambda 1$ that is $-4/\sqrt{5}$ and x2 will be nothing but $-1/2\ \lambda$ 1 that is $-2/\sqrt{5}$. So these are the x_1, x_2 we obtain from here, now we have to see that all the conditions are satisfied this point or not. Now this condition obviously hold because $x1^2 + x2^2 = 4$ this condition is satisfying, so this is holding now $x_1 - x_2 \leq 4$ what is $x_1 - x_2$ is it $-2/\sqrt{5}$ which is ≤ 0.

So yes this condition also hold to hence at this point all the KKT condition hold and the problem is the convex program problem also, so any points satisfying the KKT conditions along with f and gi all convex then that point is always a global minimum point of the problem. So we can say that this point is a point of flow, global meaning of this example one. Now similarly let us start to solve the next problem what will be the f here.

F is the thing but $x_1^2 + x_2^2 - 2_x1$ and subject and this is minimizing okay minimizing a subject to $x_1^2 + x^2$ ˂ $-1 \leq 0$, so it is g1 to first constraint okay. So first we will see whether the objective function all constraints are convex or not, so what will be the gradient of ancients matrix of f it is 2002, so it is clearly convex because λ_1, λ_2 both are two $\lambda_1 = \lambda_2 = 2$ and both are stiglicutave 0 means function is means is matrix is positive definite, hence straggly convex that means function is convex okay.

And hessian matrix of g1 will be it is 2000, so λ1 = 2 which you get a 0 and λ 2 = 0 so this means hessian matrix corresponding to this g1 is positive semi definite and hence convex hence g1 is convex okay. So this problem is the convex programming problem clearly okay because objective function f and the constraint g1 only one constraint are here, and this constraint is also convex so this problem is the convex programming problem.

Now we will try to solve the KKT conditions and the point we satisfy all the KKT condition will be the global minimum point of this problems so whatever KKT condition let us see again.

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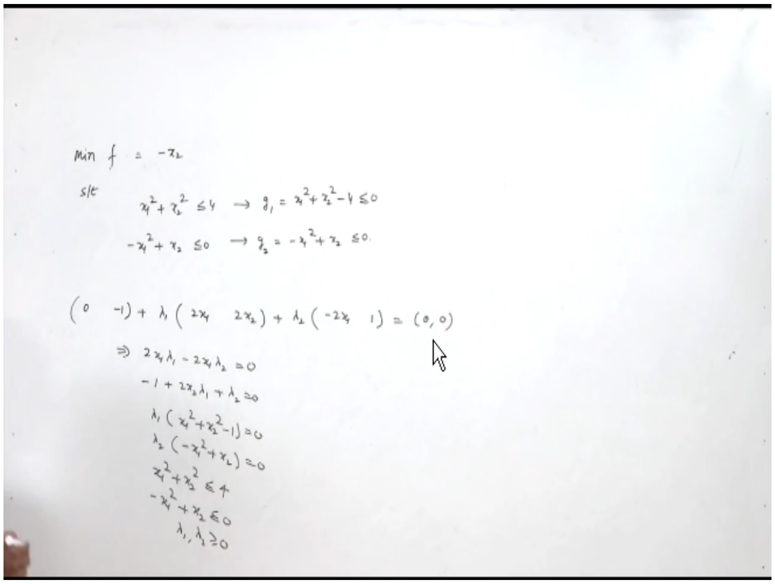

Gradient of Fx + here only one constraint is here so λ1 gradient of g1 x = 0 this implies gradient of g is to x_1 – 2 to x_2 + λ times gradient of j1 is to x1 and one which should be 0, 0 and this implies 2_{x1} – 2 + λ x 2 x 1 0 and the second implies $2x^2 + \lambda = 0$. Now the other condition are λ_1 x g_1 x should be 0 this implies λ1 x $x^{12} + x^2 - 1 = 0$, next condition is g1x must be ≤ 0 this implies $x_1^2 + x^2$ this is feasibility condition.

Now the other conditions are λ1*g_1x should be zero this implies λ1*x_1^2+x_2-1=0 next condition is g_1x must be lesser equal to zero this implies x_1^2+$x_2$it is a feasibility condition this must hold and λ1must be non negative so these are the four conditions KKT conditions okay. Now again from this condition we will having two cases so case 1 when λ1= 0 and case 2 then x_1^2+x_2-1+0 now if λ1=0 if λ1=0 or this λ1 okay this is λ1so if λ1=0 this means x_2=0 and when λ1=0 here this means x_1=1 and 1, 0 if you take 1,0 1, 0 1 and 0 is less equal to 0 which is less equal to 0 equality holds.

So this point 1, 0 set is constraints so this is the point of local minima we are interested to find out one solution satisfying all the constraints so this case is satisfying this cases giving us a one point with satisfy all the constraints okay so need to solving other cases okay so 1, 0 this is the point of global minima so let us how using KKT conditions we can easily solve find out the global minimum point of a profit.

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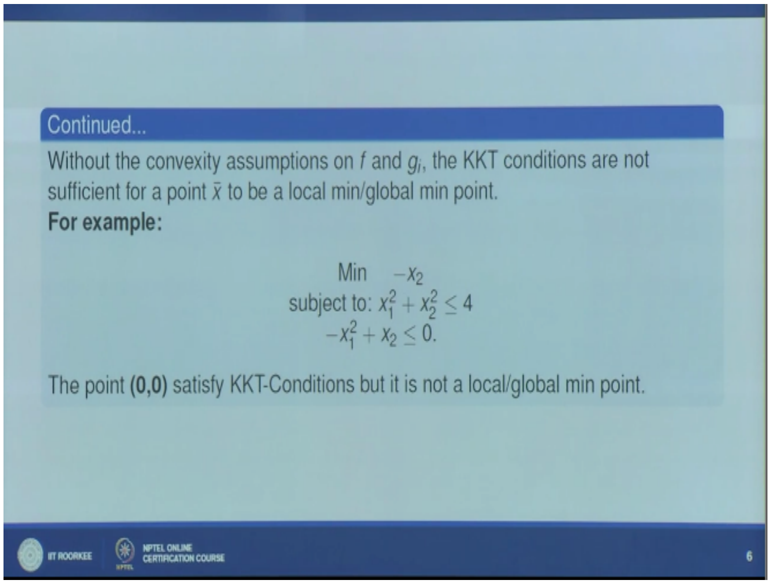

Now we doubt the convexity assumptions on f and g_i of that KKT conditions are not sufficient for point bar to will local minimum or global minimum this conditions are must convexity assumptions on f and g_i all for all i they are a sufficient conditions okay without these conditions we can't say there are point x bar to be a local minimum for example we have this problem okay

This is minimum of f= $-x_2$ subject to what are the conditions, conditions are $x_1^2+x_2^2-4$ and $-x_1^2+x_2<=0$ so the first constraint is g_1 which is $x_1^2+x_2^2-4 <=0$ and the second constraint is g_2 which is $-x_1^2 + x_2<=4$ you can easily see when you find Δ^2 of g_2 is -2, 0, 0, 0 so here λ1 is -2 for this matrix hither values is -2 which is less than 0 and for second matrix is 0

A second value is 0 or you can use tet test of principle minors to find out the to check the definite set of matrix now λ is negative and second is zero this means this matrix is negative sign definite and this is g_2 is our concave function so this problem is not a convex programming problem okay.

Now if we write the KTT conditions for this problem to see what are the KTT conditions gradient of f that is 0-1 + λ1 times gradient of g_1 it is $2x_1$ $2x_2$ + λ2 times. Gradient of g1 that is $2x_1$ $2x_2$ + γ_2 gradient of g2 - $2x_1$ 1 must be (0, 0) so this implies $2x_1$ $\lambda_1\lambda_2$ must be 0 and – 1+ $2x_2$ λ 1 + λ2 must be 0 and λ_1 times first constrains, λ_2 times second constrains must be 0 and fusibility condition $x1^2 + x2^2 \leq 4 - x1^2 + x2 \leq 0$ and multiplied must be negative. So these are the KKT conditions. Now if you see that 0.0, 0 that x1 x2 0, x1 0 satisfies this condition, x2 0 that means

$\lambda 2 = 1$ okay. Now x1 0 and x2 0 that means $\lambda_1 = 1$, $\lambda_1 = 0$ because when you put 0, 0 then this is known as 0, so this must be 0.

0 and 0 satisfying this condition 0, 0 ≤ 4, 0 0 ≤ 0 and 0 1 satisfies this constant also, so 0, 0 satisfying all the KKT conditions but still 0,0 is not the point of local or global minimum why? This we can see graphically, you see what is the, what is the constrains? The first constrains is the region inside the circle, here is the circle $x1^2 + x2^2 = 4$ this is x1 and this is x2 and when you take 0 and 0 satisfies this constrains, so shade this towards inside the circle okay.

Now the 2nd constrain is the parabola $x1^2 = x2$ that is this constrain and when you take appoint say , it is 2, 0 if you take -4 + 0 ≤ 0 it holds that means the shade is out of the parabola, so what is the common region, this is the common region these are the feasible region. Of course it is not convex that means the constant are not convex constrained okay that is clear. Now if you take this point 0,0 it satisfies all the KKT conditions but it is not a neither a global minimum point because you can see that you have to minimize the – x2 or maximize of x2 and x2 is maximum at this point or at this point, so maximum is here or here, so this point cannot be a minimum global minimum point of this problem.

So why it is not a global minimum, because the convex minimum condition is not satisfying okay, so what I want to say that if KKT condition holds at a point it does not mean that point is the point of global minimum because for that point to be global minimum beside KKT conditions convexly of f and GI is required is must okay. So that is all for this lecture thank you.

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