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Nonlinear Programming

Lec – 03
Properties of Convex Functions-II

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So welcome to the lecture series on nonlinear programming we have already seen what convex functions are and we have seen some of the important properties of convex functions, now we will see some more properties of convex functions, so epigraph we have already defined what epigraph is epigraph is nothing but epigraph of function f.

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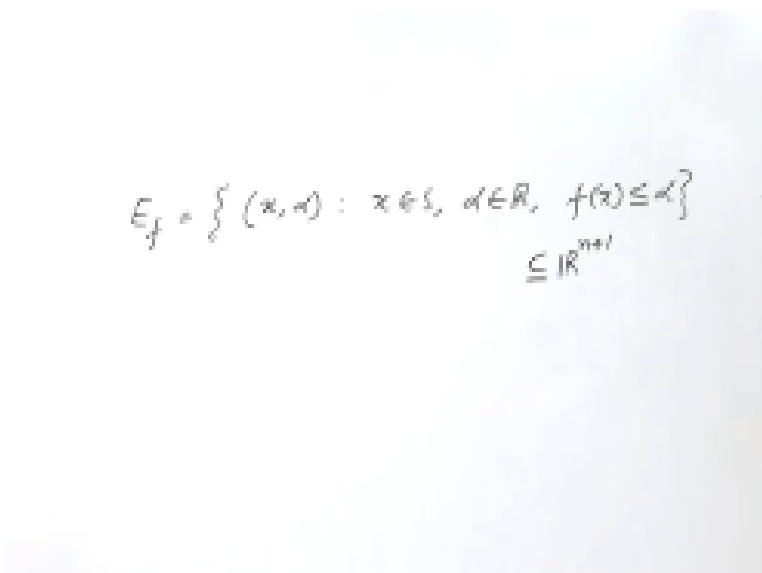

Is nothing but all those xα such that x belongs to s α is any real number and f(x) ≤ α this is what epigraph means epigraph is epigraph or function f is nothing but all those xα such that x belongs to a set s α ∈ R and f(x) ≤ α and it is nothing but a subset of RN + 1 that we have already seen because this s belongs to RN and α ∈ R so this is nothing but this tip late will belongs to this toper will ∈ RN + 1.

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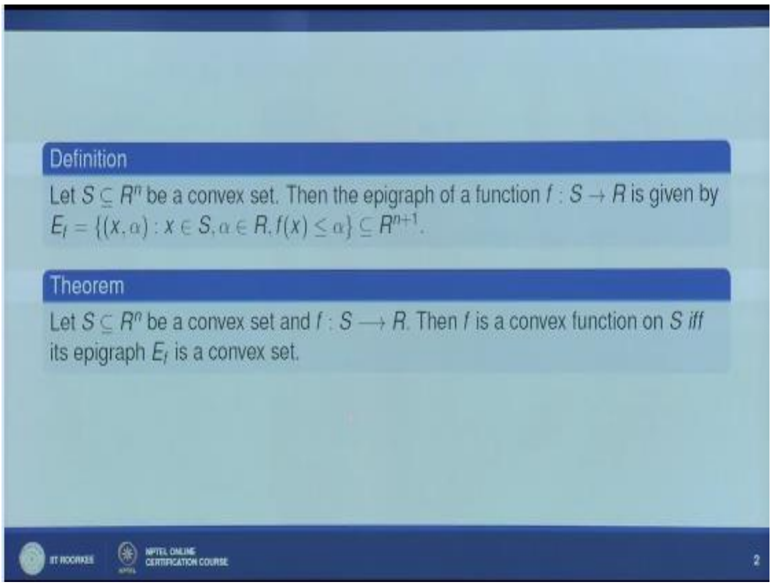

Now to show now next theorem states that that if S be a convex subset of RN and f is a function from S to R then F is a convex function on S, if and only if it is epigraph EF is a convex set. So this is the next theorem of the result that F is a convex function on S f on S if and only if its epigraph is a convex set, now how to prove this let us see okay.

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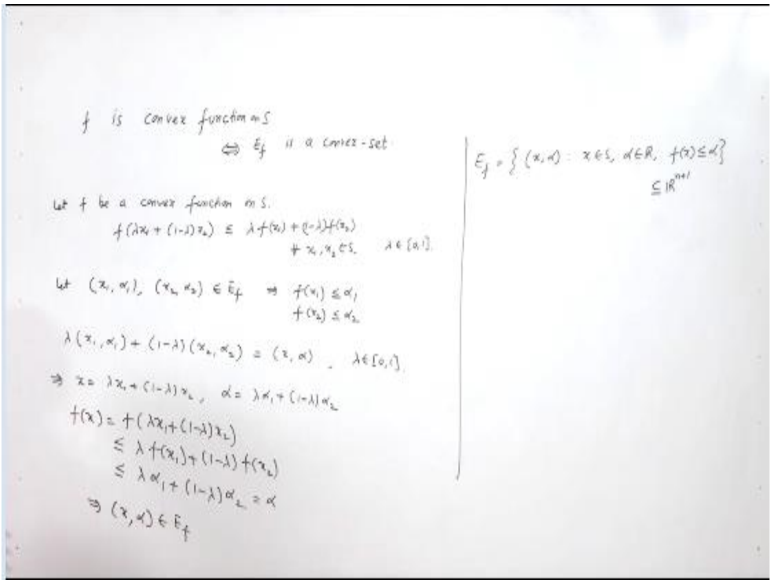

So f is a convex function on S, if and only if it is epigraph is a convex set, so this is to show now how to show this, so first we will take that f is a convex function and then we will try to show that except a graph is a convex set and then we will take that is epigraph is a convex set and try to show that this function is a convex function, now the first part is first is let f be a convex function okay on S, now since f is a convex function this means $f(\lambda x_1) + 1 - \lambda x2 \leq \lambda f(x_1) + 1 - \lambda x_2 f(x_2)$ for all x_1 x_2 in S and λ between 0 and 1 okay.

This is by the definition of the convex function that f is a convex function if $f(\lambda x_1) + 1\lambda x_2 \leq 2\lambda f(x_1) + 1 - \lambda x_2$ and this must hold for all f1 S2 in S and λ between 0 and 1, now what we have to show we have to show that if f is a convex function then it is epigraph is a convex set to prove this take two arbitrary points in the epigraph in the set that is epigraph and try to show that the convex linear combination of those two point is in s okay.

So let x_1, α_1 and x_2, α_2 are in epigraph of the function f because the points which are in epigraph are like this x , α types so it will be x_1, α_1 and x_2, α_2 let us suppose it belongs to this epigraph so it means it means that $f(x_1) \leq \alpha_1$ and $f(x_2) \leq \alpha_2$ by the definition of epigraph okay, now take the convex linear combination of these two points okay, the converse linear combination will be nothing but λ or x of first point and 1 – λ times of second point and say it is some x, α for λ between 0 and 1 okay.

So let us suppose the converse near combination we represent this by x , α so what will be x will be λx_1 $\lambda x_1 + 1 - \lambda x_2$ and what will be α will be $\lambda\alpha_1 + 1 - \lambda\alpha_2$ now to show that it is a convex set we have to simply show that x ,α belongs to epigraph that is ef okay that means we have to show that f(x) ≤ α so take f(x) and try to show that it is ≤ α so what will be f(x) will be nothing but $f(\lambda x_1) + 1 - \lambda x_2$ and by the definition of convex function it is $\leq \lambda f(x_1) + 1 - \lambda f(x_2)$ and $f(x_1) \leq \alpha_1$ and $f(x_2) \leq \alpha 2$ because these points are in the epigraph of f so this implies that is ≤ λ α 1 + 1 - λ α2 because these λ n 1 - λ are non-negative values they are lying between 0 and 1 okay.

So and this is nothing but α this is nothing but α so we have shown that f of x ≤ α, so this means if f of x ≤ α this means x , α will belongs to the epigraph, so this implies x , α will belongs to epigraph of the function f and that means epigraph is a convex set okay, so in this way we can we can say that a function is a convex function that is then its epigraph is a convex set, now we will do that we will try to prove the converse part we will take that a epigraph is a convex set and we will try to obtain this function f is a convex function.

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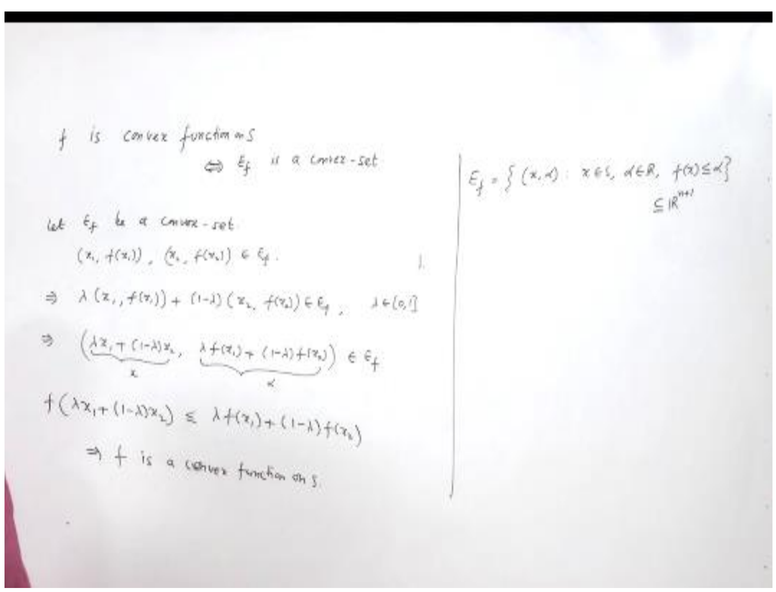

Now let us see so now for converse part let epigraph of the function f be a convex set okay, now we have to show that functions of convex function so if it is epigraphs the convex function take two arbitrary points now x_1 , $f(x_1)$ and x_2 , f(x2) will definitely belongs to epigraph of f this is because it is α it is α_1 now $f(x_1) \leq \alpha$ this is obviously true because equality holds okay if it is α $f(x_1) \leq \alpha$ α is $f(x_1)$, $f(x_2) \leq f(x_2)$ it is obviously hold.

So these point definitely belongs to a epigraph of the function f now it is given towards there the epigraph is a convex set so this means this implies λ times the first point and 1 -λ times the second point must belongs to the epigraph for λ between 0 and 1 okay so this implies $\lambda x_1 + 1 - \lambda x_2$ and $\lambda f(x_1) + 1 \lambda f(x_2)$ must belongs to the epigraph of the function f okay, now it belongs to the epigraph means what if x , α belongs to the epigraph this means $f(x) \leq \alpha$ so this is this is some x and this is α okay.

So this x ,α belongs to the epigraph means f(x≤ α so f(x) is f(x) x is this quantity this quantity this term ≤ α and α is this quantity $\lambda f(x_1)$ 1+ 1 - λ $f(x_2)$ and this implies that f is convex is the convex function on s because x_1 and x_2 or any arbitrary points this means this is it hold for any x_1 and x_2 and hence function is a convex function on this s so in this way we can say that if epigraph or function is a convex set then also we can say that a function is a convex function okay.

So if we have to show to the function is a convex function either we use our definition of a convex function which is $f(\lambda x_1) + 1 - \lambda x_2 \leq \lambda\, f(x_1) + 1- \lambda\, f(x_2)$ or we can also find the epigraph and try to show that epigraph of the function as is a convex set okay, now similarly using hypo graph I hypo graph is a hypo graph or function f.

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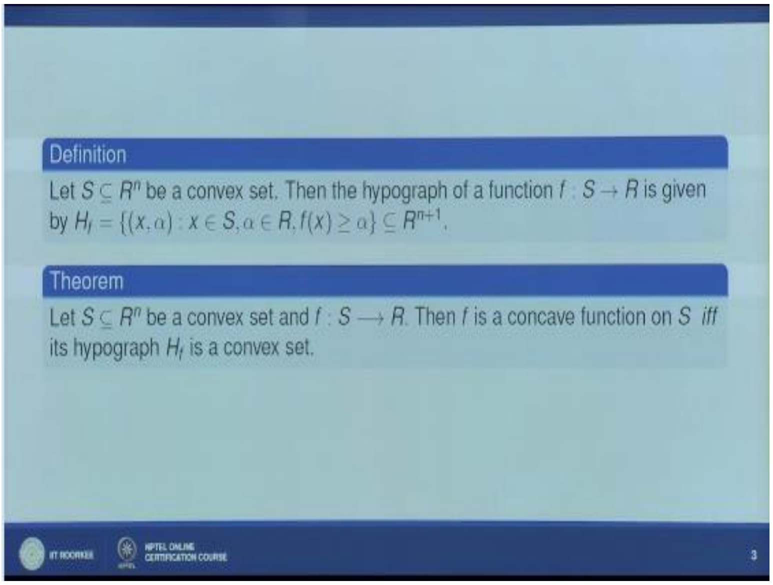

Is nothing but all those that is α so the S ∈ s α ∈ R f(x) f(x) ≤ α so this is hypo graph function f so if S is a convex set and f is a function from S to R then f is a concave function on S if and only if it is hypo graph is a convex set so the proof of this can also be obtained on the same lines on the lines of the proof which we did earlier okay so in order to prove that a function of a concave function find it is hypo graph and try to show their a hypo graph of the function f is a convex set okay.

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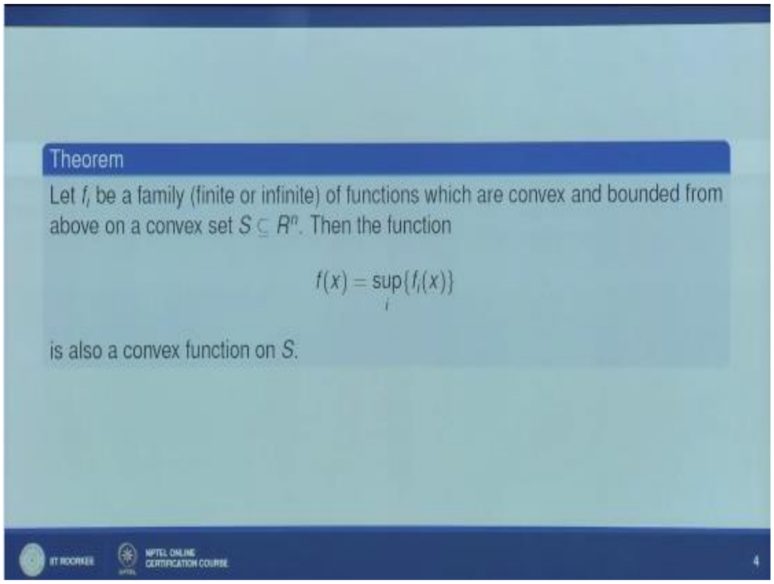

Now we have a result now let f_i be a family of functions which are convex and bounded from above on a convex set S subset of R^n then the function which is given by a supreme of f_i is also a convex function, so let us try to prove this so for example suppose you are taking maximum of x and x^2 okay.

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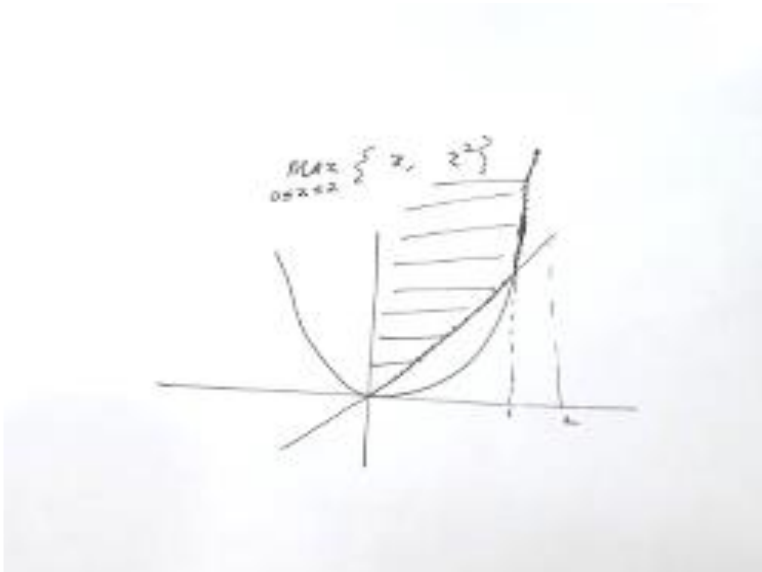

Now both functions are convex x is a linear function it is convex a function from R to R x ∈ R x^2 is also a convex function that we have also seen that x^2 is a convex function and maximum of two convex functions also convex okay that we can easily show we can easily see graphically see x, y = x is this line okay and $y = x^2$ is this intersection point okay now what is the supreme of this function supreme of this function is if it is function from say from 0 to 2, so 0 to 2 it is nothing but from here to here this is the maximum value when it is 1 okay and from 1 to 2 this is the maximum value this curve.

So this shaded line and this is the maximum or the function from 0 to 2 say a 2 is here okay, say 2 is somewhere here so the maximum of maximum of x and x^2 is this function now if we take the epigraph of this function, so f is rap of this function is nothing but this set and which is convex and since it is convex, so we can say by the depth by the theorem that the concerned function if it is a function f then this function is a convex function.

So this is the theorem that the supreme of f_i is also a convex function the proof is very simple can be obtained it is start to obtain the proof.

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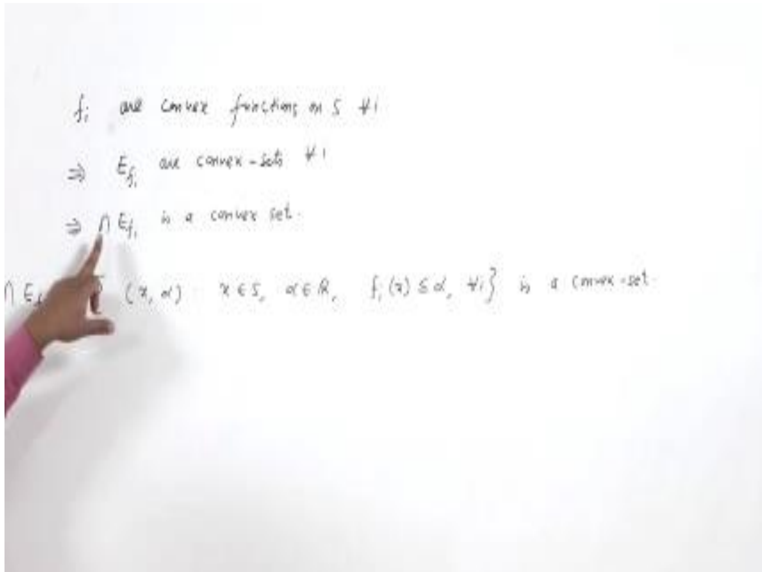

Now f_i are convex f_i are convex functions on S for all I so this implies because we know a theorem that a functions are convex then the epigraph is a convex set so this implies $e(f_i)$ are convex set for all i, and since it is a convex set, so this means intersection is also a convex set because intersection of any number of convex set is also convex okay, now let us see the intersection of $e(f_i)$ what this represent this is nothing but all those x α such that $x \in s$ $\alpha \in R$ and $f_i x \leq \alpha$ for all i, a is con is a convex set the intersection of $e(f_i)$ is a convex set.

What does intersection represent intersection of $e(f_i)$ represent all those xα so $S \in S$ $\alpha \in R$ and $f_i x \leq \alpha$ for all i, because it is the intersection that means for all i, is a convex set, now if it is true for all i, this means it will be equals to it reduced equal to supreme of i, also supreme i, does not goes to α is also a convex set okay and this is nothing but is requests to all those xα so that $x \in S$ $\alpha \in R$ and supreme of f_i is nothing but f(x).

So it is $f(x) \leq \alpha$ is a convex set, so this means this means epigraph of f where f is nothing but supreme of f_i is a convex set and since epigraph of f is a convex set this means f is a convex function convex function on S okay because we know that if epigraph of function is a convex set this means the function is a convex function and since epigraph of f is a convex set, so this implies the concern per a function the corresponding function is a convex function and what is the corresponding function it is nothing but supreme of f_i

So hence we can say that if we have a collection of a convex functions their supreme of the convex functions is also convex okay, now let us define differentiable convex functions.

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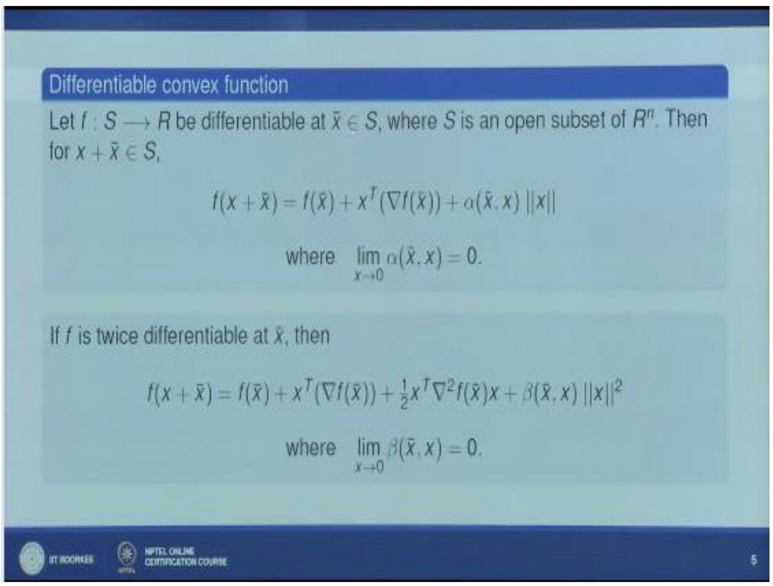

If let s be a function from S to R be differentiable at $(\bar{x}) \in S$ where S is the open subset of R^n for $x + (\bar{x}) \in S$ $f(x) + (\bar{x}) = f(\bar{x}) + x^T$ gradient of $f(\bar{x}) + \alpha$ is a function of $(\bar{x})$ and S and norm of x where this term will tend to 0 as x tending to 0, so this is how we define a function is once differentiable at $(\bar{x}) \in S$ okay what is great of f(x)bar gradient of $f(\bar{x})$ is a vector basically.

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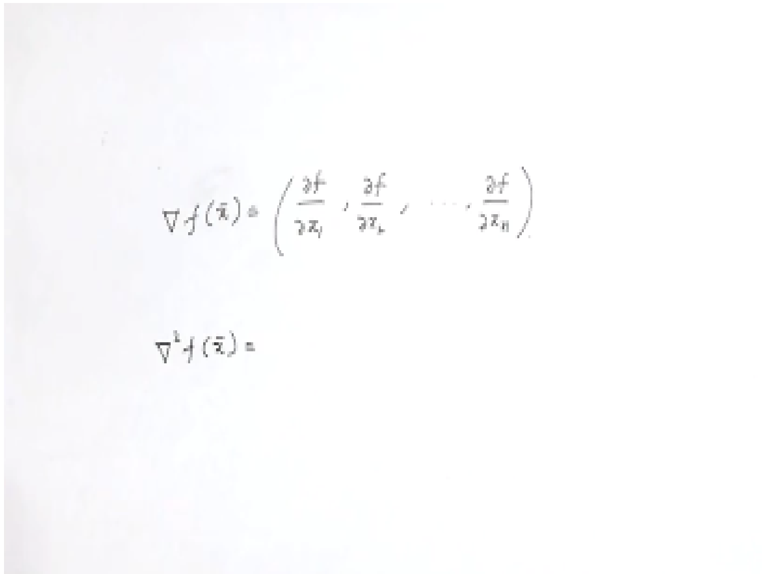

So gradient of $f(\bar{x})$ is nothing but ∇ f / δ x_1 δ f / δ x_2 and δ f/ δ x so this is greater of $f(\bar{x})$ okay now if a function is twice differentiable at $(\bar{x})$ then f (x) + $(\bar{x})$ is = f(x) + x^T gradient of $f(\bar{x})$ + half of this term + β $(\bar{x})$ x norm of x^2 we are this term0S as extending to 0, so this is how we can define the function f a function is once or twice differentiable at $(\bar{x})$ okay now what is what is gradient square of $f(\bar{x})$ that is defined this thing also gradient square of $f(\bar{x})$ is nothing but a matrix.

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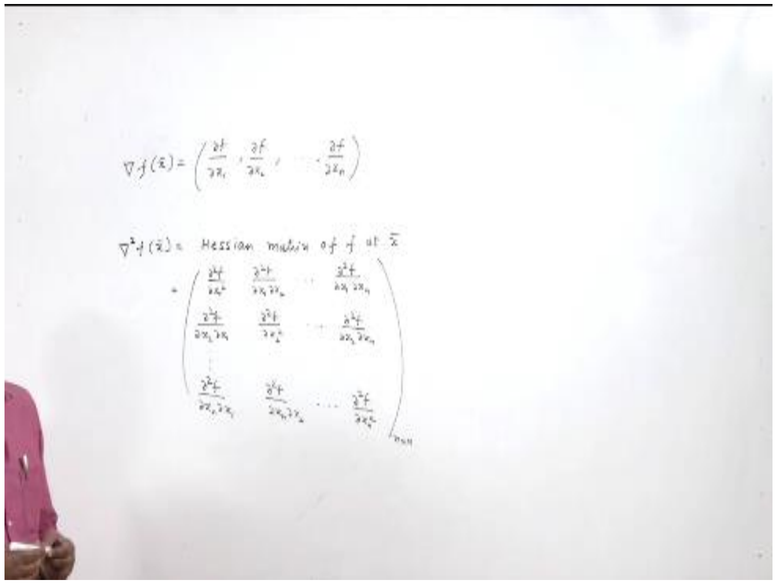

And we call it Hessian matrix of F at $(\bar{x})$ and what it is it is nothing but δ^2 of upon $\delta x1^2$ δ^2F upon $\delta x1$ δx_2 and so on $\delta 2$F upon $\delta x1$ δ xn then δ^2 F upon $\delta x2$ $\delta x1$ δ^2 F upon δx^2 and so on. Δ^2 F upon δx_2 δxn and it is $\delta 2$ F upon δxn $\delta x1$ δ^2 F upon δxn δx_2 and so on δ^2 F upon δx and squares, so this is how we can define our n cross n and cross an symmetric methods a symmetric matrix which we call as he Hessian matrix set of f at $(\bar{x})$ okay this is gradient square of F at $(\bar{x})$.

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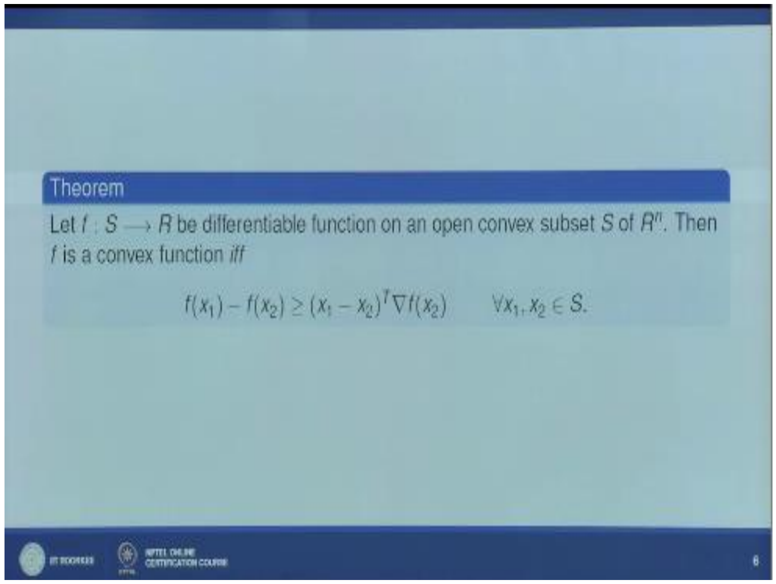

Now we have our next result for a convex function which is if f is a function from S to R which is differentiable function on a open convex subset S of RN, then the function f is a convex function if and only if f of x_1 – f x_2 is greater than equals to x_1 - x_2^T gradient of fx_2 and this should hold for all S is belongs to S, so now let us start to prove this result so what this result is basically.

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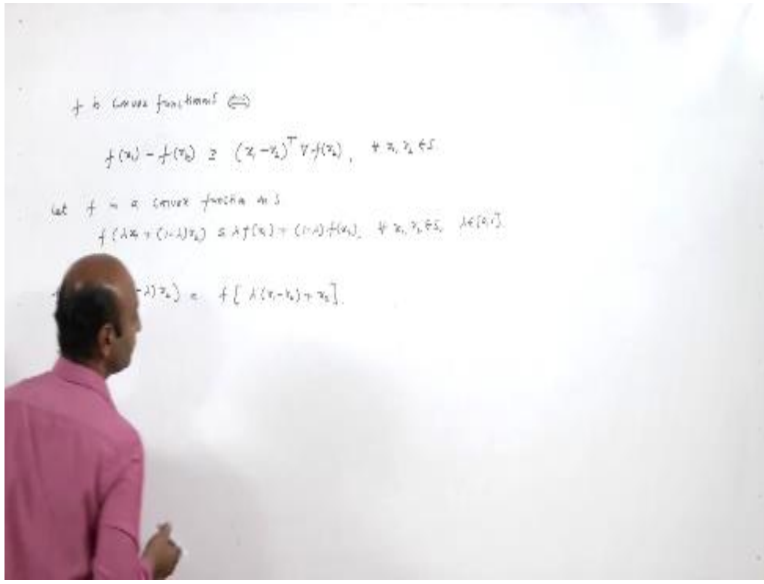

This is f($x_{1)}$ – f(x_2) is greater equal to $x_1 - x_2^T$ gradient of fx_2 and this should hold for all x_1 x_2 belongs to this, and what we have to show that f is the convex function on S if and only if this happens okay. So first let us assume that f is a convex function and we will try to show that this result hold okay, so first let f be a convex function one is okay. Let f be a convex function on S, so convex function means f of $\lambda x_1 + 1 - \lambda x_2$ must be less than equals to λ f x_1 + 1 - λ f x_2 for all S1 S2 in S and λ between 0 & 1okay.

Now what is f of $\lambda x_1 + 1 - \lambda x_2$ it is nothing but S can be written as f of $\lambda x_1 - x_2 + x_2$, you can take this λ common from these two terms it is λ of $x_1 - x_2 + x_2$, now it is given towards that function is differentiable, differentiable on an open convex set S of R and this means differentiable for all points in S okay. So if it is differentiable this means this means this means the first.

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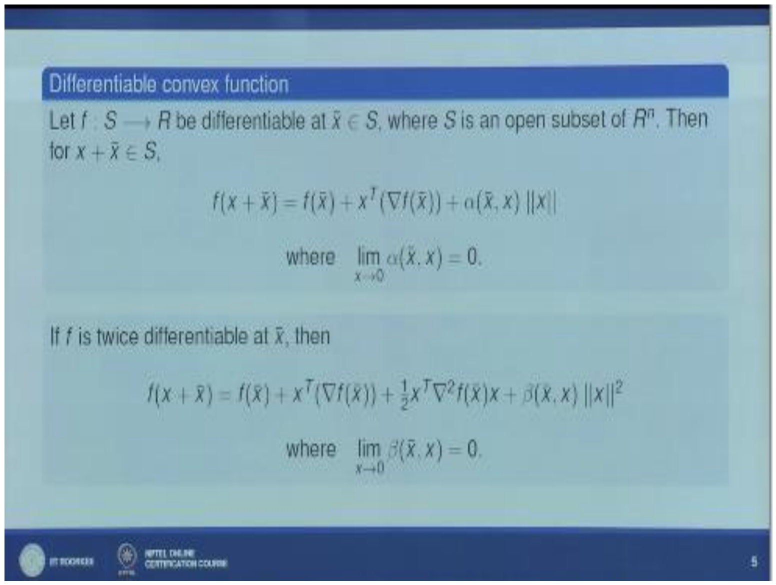

Result holds okay, so let us try to use this result.

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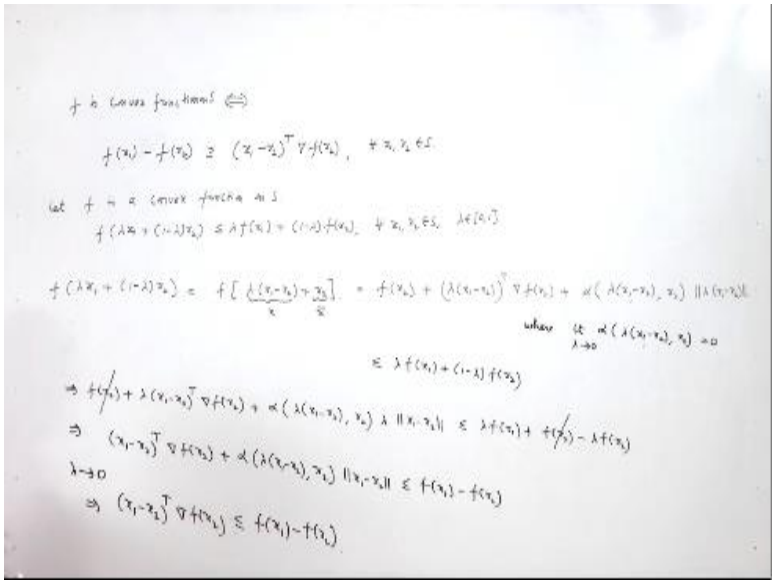

So here suppose this is x this quantity is x and this is x bar okay so you use this result it is x it is $(\bar{x})$ so it is equal to f of $(\bar{x})$ + x, x^T gradient of fx_2 + α times $(\bar{x})$ and x and norm of λx_1 - x_2. Okay this is simply by this result okay, and where this limit tending to 0 where as limit of λx_1 - x_2, x_2 will be equal to 0 as λ tending to 0 here. So f of x + $(\bar{x})$ is equal to f of $(\bar{x})$ + $(\bar{x})^T$ fx gradient of $(\bar{x})$ + α function of $(\bar{x})$ and x and norm of x okay, so now this is equals to now this quantity from this expression is greater than is less than or equal to λfx_1 + 1 – λf extreme okay.

From this expression this quantity is less than equal to means this code this is equal to this means this quantity expression is less than if I show this quantity, so what we obtain from here this implies f of x_2 + λ times $x_1 - x_2^T$ because λ is a scalar we can take it out and it is gradient of fx_2 + α of α function of λx_1 - x_2, x_2, now λ can take out because it is between 0 & 1 and norm of x_1 - x_2 which is less than or equal to λfx_1 + fx_2 - λfx_2 you multiply fx_2 here now this f (x_2) and f (x_2) cancels out from both the expressions you divide by λ throughout so what we obtain we obtain that $x_1 - x_2$.

Whole transpose gradient of fx_2 + $\alpha\lambda$x1 into x_2, x_2 norm of x_1- x_2 is less than equals to fx_1- fx_2 okay, now as now take λ 10 to 0 both the sides if you take ramp 10 to 0 both the site or this term will turn to 0 if the taking λ10 to 0 implies x1 - x_2 whole transpose gradient of fx_2 is less than or equal to fx_1 - fx_2 okay, so we have proved the first part so we have taken data to the convex function and we have obtained net f of x_1 - fx_2 Is greater than equals to $x_1 – x_2^T$ gradient fx_2 okay now we will try to show the converse part okay.

So in the converts part we will suppose at this condition hold and try to obtain if the function is a convex function.

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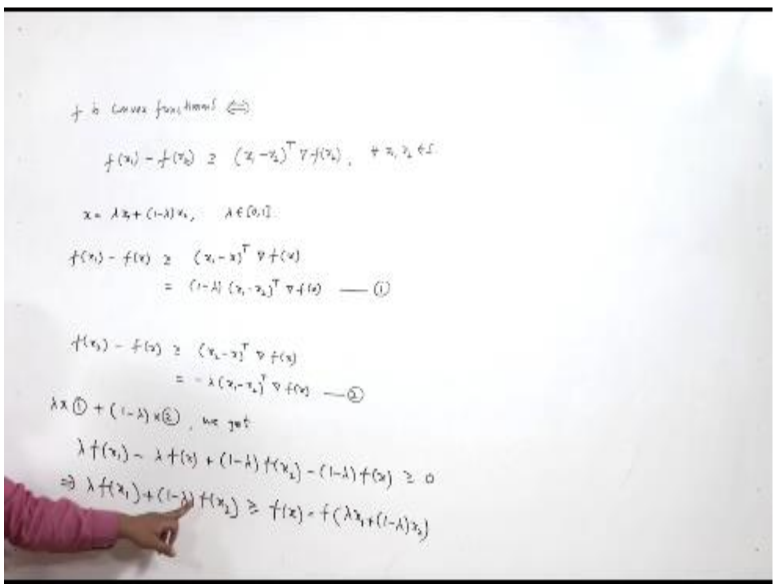

So let x = λfx x_1 + 1 – λx_2 we are λ between 0 and 1 now this inequality hold for every x1 and x_2 and x is a convex sub set of RN okay. So this will hold for x_1 and x also so if we apply for x1 for x so this will be greater request to x_1 - x^T gradient of fx okay because we have supposed that this condition hold this condition hold means this inequality hold for all fx_1 x_2 in S and we have to show that the concern the corresponding function f is the convex function okay, so we have so this result will hold for xy and x also.

So we have applied this result for xy and x so we obtain this thing so what it is equal to it is equal to it is equal to x_1 – x, x or – x will be nothing but 1 - λ times x_1- x_2^T gradient of f suppose it is a first expression, now the same result will also hold for x_2 and x okay. Because it is holding for all x_1 and x_2 in S and this is in S because S is a convex set okay, so fx_2 – fx again will be greater than equal to $(x_2 – x)^T$ gradient of f of x, now what is x_2 – x when you compute x_2 – x so it will be nothing but it will be nothing but x_2 - x will be nothing but - λ times x1 - x_2.

Whole transpose gradient of effects okay, now we have to obtain that we have to derive that function is a convex function that means f of λx_1 + 1 – x_2 is less than equals to λ fx_1 + 1 –λx_2 is fx_2 so we have to obtain that result so you multiply this with λ and this with 1 – λ okay multiply 1 with λ and 2 with 1 - λ, and add them so what we obtain what we obtain we get λ of fx_1 - λfx from this side + 1 – λfx_2 – 1 – λfx is greater than equals 2. Now when you want apply this with λ and this with this is λ this with 1 – λ and adds them, so both things will cancel out so we will get 0 on the right hand side okay.

So what we obtain from here this implies λfx_1 + 1 – λfx_2 will be greater than or equals to λfx and λfx will cancel out, so this will be written goes to fx and x is nothing but the converse near combination of x1 x2 which is nothing but is equal to f of you substitute the value of x and x as this quantity, so it is λx_1 + 1 – λ x_2. So hence we have obtained that f of λ x_1 + 1 and λx_2 is less than or equal to λ fx_1 + 1 – λfx_2 this implies f is a convex function okay. So we can easily see that if f is a convex function.

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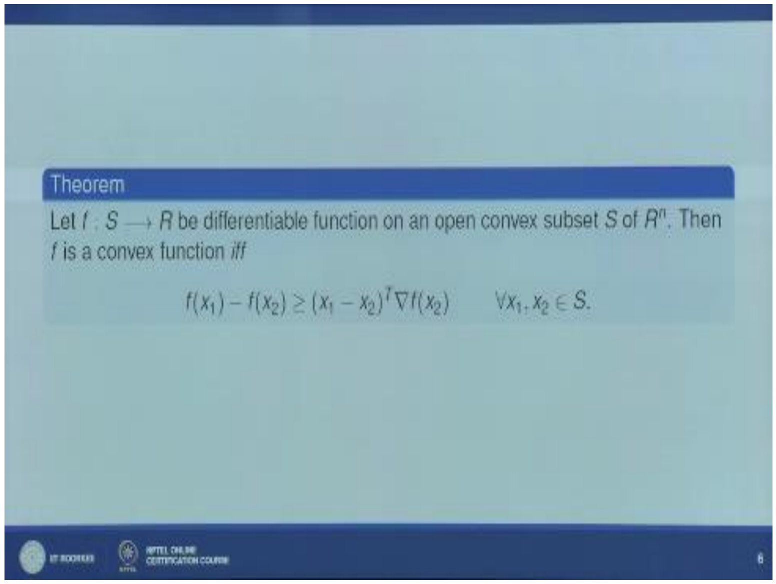

Then f of x_1 – f of x_2 will be getting close to x 1 - x_2^T gradient of fx_2, now what this expression geometrically indicates, let us see okay.

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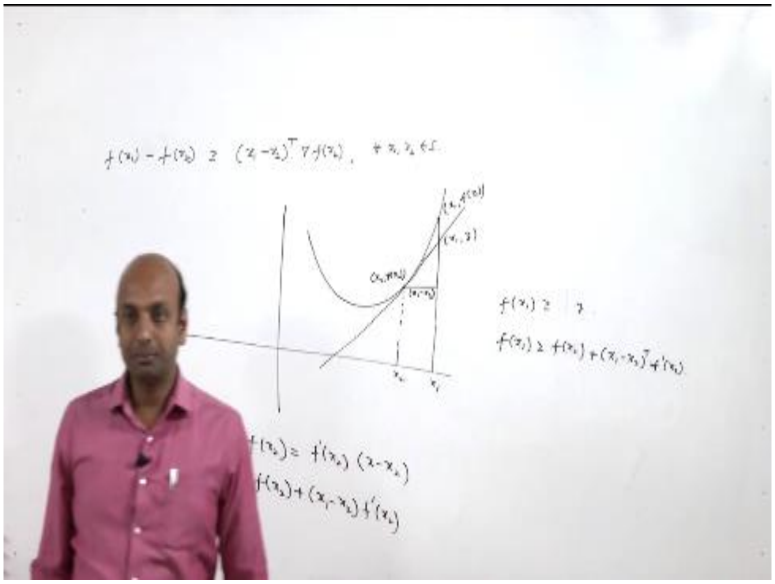

Let us see what this what this expression geometrically indicates, so you draw you draw the function f take a point x_2 suppose this is x_2 draw a tangent at this point tangent will be nothing but this go on this straight line okay, take another point say x_1 here. So what this will be this length so from here to here it is x_2 from here to here it is x_1 so it will nothing but x_1 - x_2 want it okay, so what is the equation of this line. Now at the point x_2, fx_2 the equation of the line will be nothing but y – fx_2 is equals to M, M the slope of the line slope is f' x_2 and x – x_2 okay. Now we want to find out we want to find out this point we want to find out this point so this one will nothing but the substitute x as x_1 x as x_1.

So what we obtain the y will be nothing but fx fx_2 + x_1 – x_2 f' x_2, so this if you are taking as this says x_1, y so this y is nothing but this expression and you can easily see that this point is nothing but x_1, fx_1 so x_1 is f_{x1} is this height fx1 and this is fy it difference y okay. So fx_1 and fx_1 is greater than equals to y, so fx1 is greater than equals y means this expression gradient of for the first years it is derivative this expression means this result. So that means that if you draw a tangent at any point on the convex function.

They are tangent always lies below the curve, if you take because this length is always less than at this length so that means if you have a convex function and you draw a tangent at any point on the convex function they are tangent always lies below the curve okay, so this is geometrical interpretation of this inequality. So we can prove some functions to be convex using this expression also how so let us see one example.

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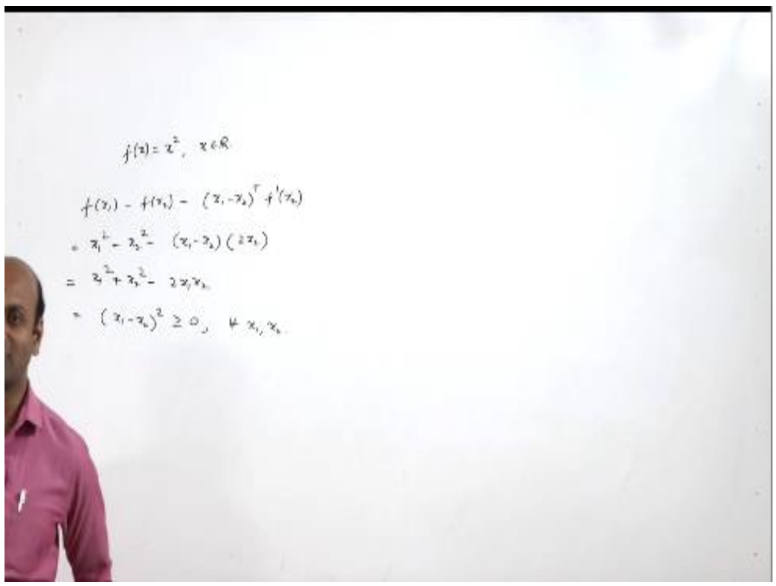

Suppose you have to show that function x^2 where x belongs to R is a convex function, so we have already seen that we can also show we can show this as a convex function simply by applying the definition of the convex function but we can also show this to be convex using this result using this result. Now let us see what is $fx_1 - fx_2 - x_1 - x_2^T$ derivative of x_2 because it is in R okay and we have to show that this quantity is greater than equal to 0 then we can say by the serum net function is the convex function.

So what is expression is for this function it is $x1^2 - x2^2 - x1 - x2$ what a derivative of this function x2 it is 2 x2, so this is nothing but $x1^2$ and this is $+x2^2$ - 2x1 x2 which is nothing but x1 – x2 whole squares and with the equal to 0 always for all x1 x2, so this implies function is a convex function because this inequality must be greater than equal to 0 and that we have shown that for this function x^2 this inequality is greater than equal to 0. Hence we can say there is a given function x^2 is a convex function, so sometimes to prove that a function is a convex function we can also use this definition okay so we will see some more popular convex function in the next class so thank you.

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