INDIAN INATITUTE OF TECHNOLOGY ROORKEE
NPTEL

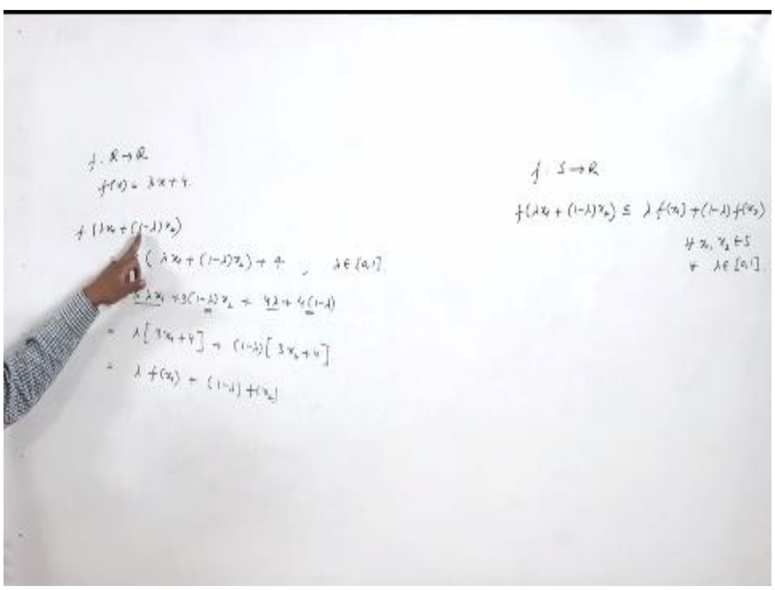
NPTEL ONLINE CERTIFICATION COURSE
Nonlinear Programming

Lec- 02
Properties of Convex Functions- I

Dr. S.K. Gupta
Department of Mathematics
Indian Institute of Technology Roorkee

So welcome to the lecture series on nonlinear programming in the last class we have seen what the convex sets are and what are their properties and also we have seen what convex functions are and we have also seen some examples on convex functions. In this lecture we will see some properties of convex functions, so let us discuss. So what convex functions are?

(Refer Slide Time: 00:49)



Function f problem a convex set to R is said to be convex if f of λ x 1 + 1 - λ h 2 is $\leq$ λ f x 1 + 1 - λ X 3 and this should hold for all x1 x2 s and for all λ between 0 so this is how we can define convex function. Now let us discuss some examples how to prove mathematically let a function f is convex geometrically we have already seen that what convex function is to a particular represent, but how can we show a function f is convex mathematically.

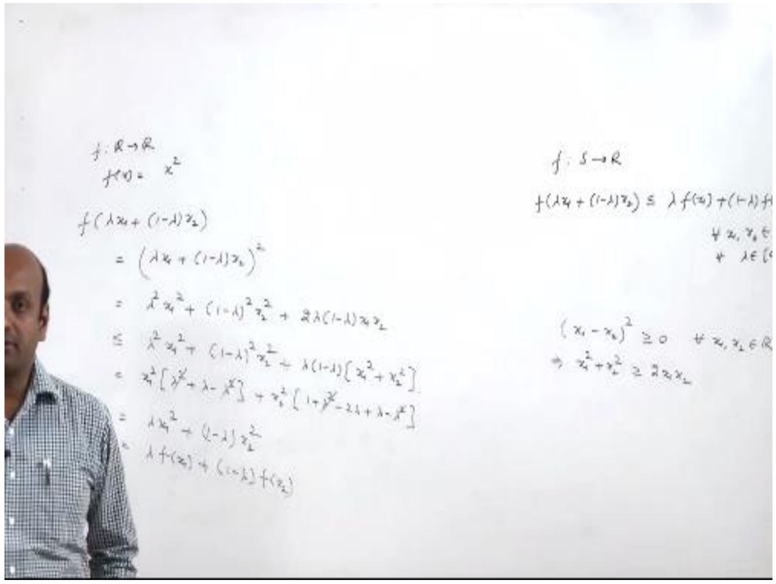
So let us discuss few examples list of this suppose function is from R to R and f is suppose mortix, so we have seen graphically that |x | is a convex function, okay how can you prove this? So you take two arbitrary points x1 and x 2 in R here s is our if for this example as = R okay, so what is f(λ x 1 + 1 - λ x 2) so this will be = by the definition is λ |λ x1 + 1 - Δ x 2 okay.

λ between 0 & 1now this is ≤ λ | λ x1 + | of 1 – λ x 2 | because | of a1 + b1 is less than or = | a1 + |1 well ok now this λ is a non-negative value so it can be taken out from the | again 1 - λ is also non-negative because λ is between 0 & 1, so this also can be take out now |x1 is nothing but x| 1 and this is x| 2, so we have shown that f of λ x 1 + 1 - λ X 2 ≤ λ of x 1 + 1 - λ x 2. So that is how we can show that |x is the convex function.

So we see some old sample based on this suppose a very simple example suppose x| = 3x+ 4 and we have to show that it is a convex function, so again you take λ x 1 + 1 - λ x 2 for any x 1x 2 in R and λ between 0 & 1, so this is = 3 times you apply this definition λx1 + 1 - λx2 + 4 for λ between 0 &1 and for any x1x2 in s SS are here this is further = 3 λ x1 + 3 times 1 - λ x 2. Now this 4 can be written as 4 λ + 4 x 1 – λ.

The sum is the sum is 4 okay, so now this can be done as λ times 3x1 + 4 you Club these two tops and Club these two terms ok so this is + 1 - λ you can take out this is 3 X 2 + 4. Now this is nothing but λ of x| 1 x| 1 is 3 x 1 + 4 and this is 1 - λ of x| 2, so we have shown that f of λ x 1 + 1 is x2 is = λ f x 1 + 1 mass m 2 f x2 that is equality holds, so equality holds that means function is convex okay.

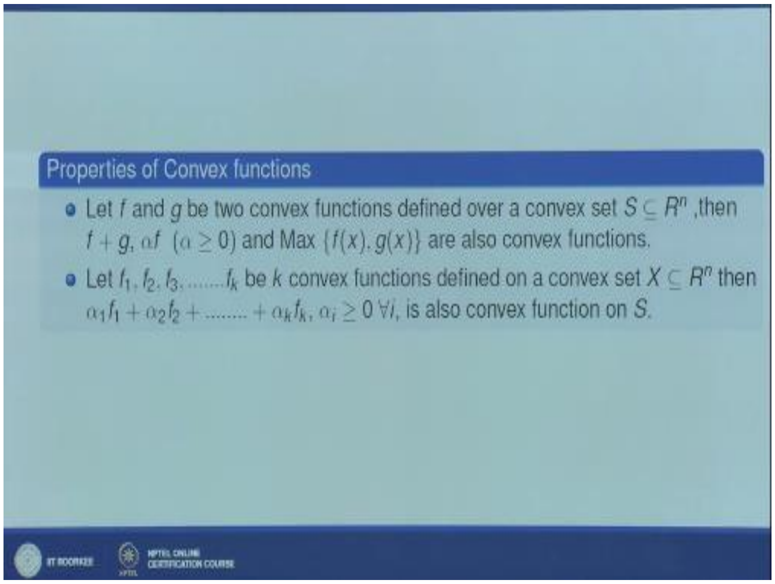
(Refer Slide Time: 05:38)



Now suppose you have to show that x| = x2 square you have to show that is a convex function again we can apply the same definition f of λ X 1 +1 - λ X 2 it is λ X 1 +1 - λ X 2^2 because x| = x square this is = λ^2X 1^2 + 1 - λ^2 x 2^2 + 2 λ 1 - λ X 1 X 2 okay. Now to solve it further we know director for NES 1 x 2 x 1 - x 2^2 > 0 for all X 1 X 2 okay, so using this inequality we can obtain that x 1^2 + $x2^2$ ≥ 2 x1 x 2, so this 2x1 x2 can be written ≤ $x1^2$ + $x2^2$ so this is ≤ λ^2 x 1 square + 1 - λ^2 x 2^2+ inequality will not change because these quantities are non-negative so this is = x 1^2 λ^2 + λ - λ^2+ $x2^2$ times 1 + λ^2 - 2 λ + λ - λ^2, so this cancel out and this cancels out, so this is nothing but λ $x1^2$ + 1 - λ x^3 or it is = λ fx1 +1 - λ x .

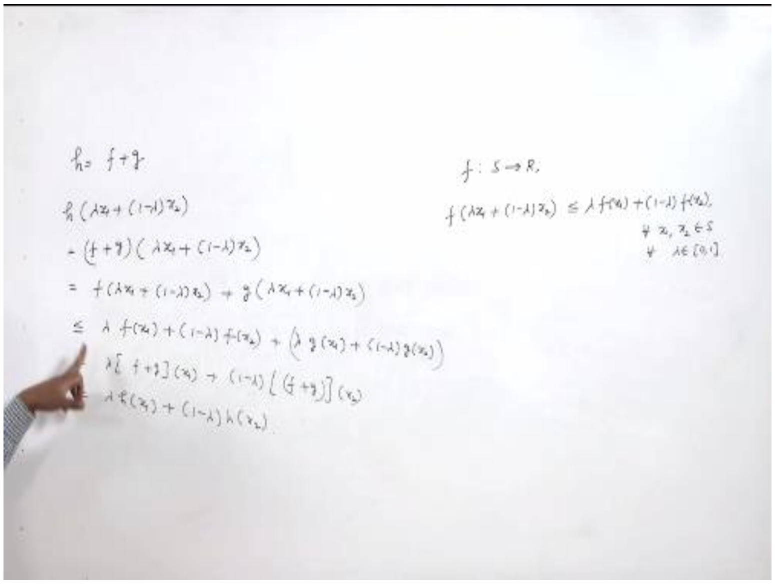
So this is how we can we can easily see that f of λ x 1+ 1 - λ x 2 is ≤ λ F X 1 + 1 - λ x 2 F x 2, so in this way we can show that a function f is a convex function I mean x^2 is a convex function. Now suppose you have to show that f is a concave function some function like – x^2 square is a concave function so in the same lines in the same lines we can show that – f I mean – x^2 is a concave function okay. Now what are properties of convex function the first property is?

(Refer Slide Time: 08:28)



If and G to convex functions defined over a convex set s than the sum is also convex α times F where α is a non-negative scalar it is also convex and maximum of two convex function is also convex, so let us try to prove the first one f + g. So then to show that f + G are upon wave function we have to show this definition okay.

(Refer Slide Time: 08:59)

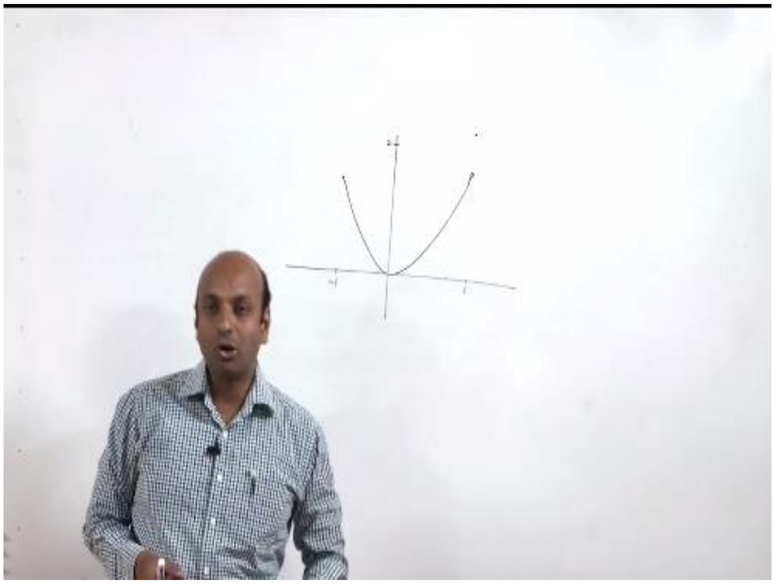


So you take you take h f x + g, so h(λ x1 + 1 - λ x 2) we have to show that this is ≤ λ s x1 + 1 - λ x2 okay, so this is =(f + g) λ x 1 + 1 – λ x 2 because h is nothing but f+g and it is again = f of λ x1 + 1- λ x 2 + G (λ x1+ 1 - λ x 2). Now f and g both are convex it is given to us, so since both are convex, so by the definition these are ≤ λ of x 1 + 1 - λ f x 2 + λ gx 1 + 1 - λ G x 2and this is = λ times f + g of x 1 + 1 - λ times f + g of x 2 and this is nothing but λ h of x 1 + 1 – λ h of x2.

So we have shown that Hof λ x 1 + 1 - λ x 2 is ≤ λ x 1 + 1- λ x 2 this means h is a convex function and it is nothing but sum of f and g, so sum of two convex function is also convex. A similarly we can show that α times F is also a convex function, now to show that maximum of two convex functions also convex this we will discuss while we will discuss epigraph of these convex functions in our next class.

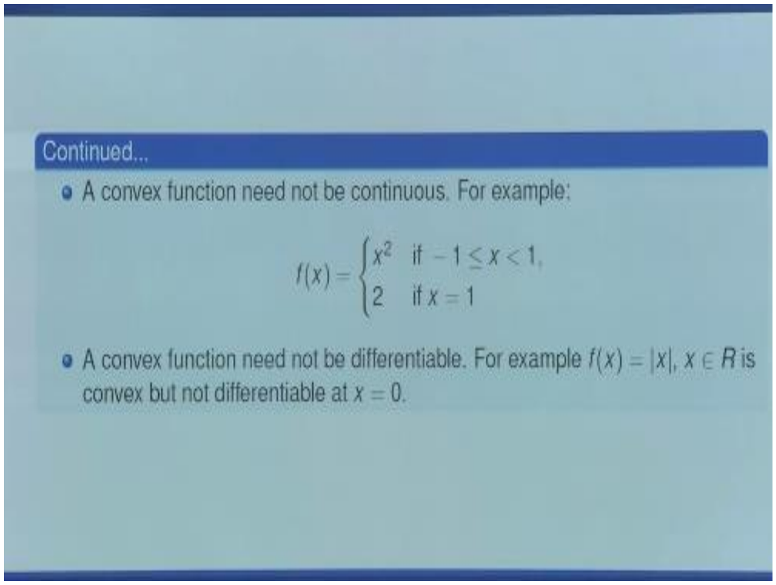
Now if we have suppose f1 f 2 f 3 up to fk, suppose we have k convex function defined on a convex set x then the linear combination of these convex function is also convex okay. This can be put very easily using the same concept okay. Next property is a convex function need not be continuous, suppose you have this example.

(Refer Slide Time: 11:46)



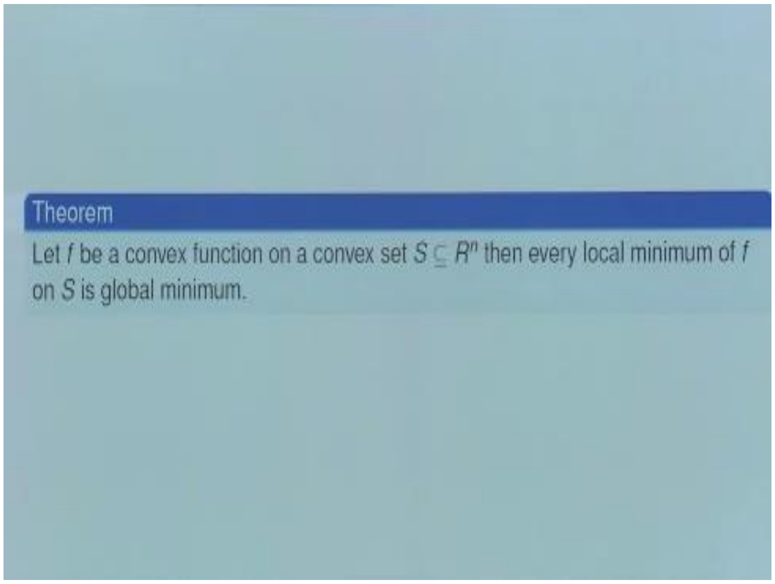
From - 1 to + 1 it is x^2, suppose –1 it is + 1 it is x^2 this is and at one at 1it is 2 so this is 1 and at 1 it is reduced to okay, clearly this function is not continuous at x =1 however the function is convex because if you take any two arbitrary points and join the line segment even with this point, so the chord is always above the curve, so a convex function need not be continuous. The next property is a convex function need not be differentiable.

(Refer Slide Time: 12:35)



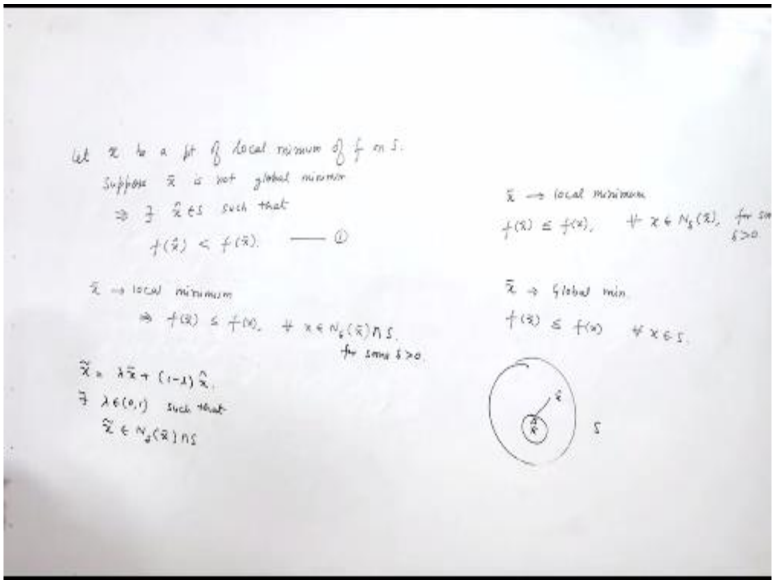
So of course if it is not continuous, so if you it will not be differentiable and another example is |x| f is not differentiable at x = 0. However we have seen that |x is convex function okay. Then that is very important property of on wave function.

(Refer Slide Time: 13:02)



Let f be a convex function on a convex set s subset of R^n then every local minimum of f on s is global minimum, so what does it mean and how we prove it let us see. So if the function is convex and you have found a local minimum on a convex set s then it will be global by this property okay. So what do you mean by local minimum? Local minimum means that suppose $\bar{x}$ is a local minimum okay.

(Refer Slide Time: 13:43)



So what does it mean it means that fx̄ will be ≤ x| for some neighborhood of x̄ you take a neighborhood of x̄, so it is for all x belongs to Δ neighborhood of x̄ for some Δ >0. This is local minimum and that means you take a neighborhood off, so you take a neighborhood of that point of x̄ this is a no matter how small Δ is if there exists some Δ neighborhood of x̄ such that for all x belongs to the Δ neighborhood of x̄ if x̄ is ≤ x| this means x̄ is a local minimum and global minimum means x̄ is global minimum.

This means fx̄ ≤ x| for all x in f for all x in the convex set s if x is ≤ x| if this holds then we say that x̄ is the a point of global minimum okay. Now how we will show that a function is convex then every local minimum is global, so suppose x̄ let x̄ be a point of local minimum of f on s okay and suppose x̄ is not global minimum, so if as what is not global minimum this means this implies there will exist some x cap belongs to s.

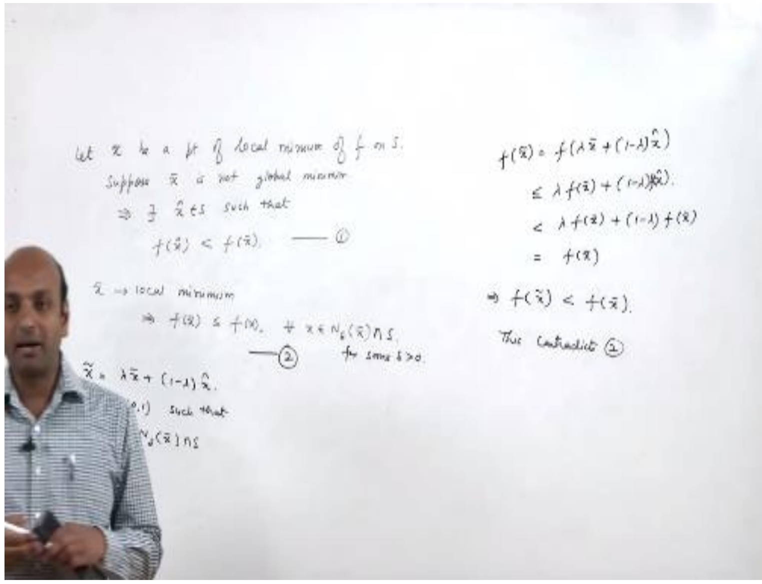
Such that such that x| cap will < fx̄ okay because x because x̄ is not global minimum is not global minimum means it this inequality is not holding for every year. This means there will exist for messiness so which is X cap such that a value of function have a smaller value than function of F at x̄ okay and fx̄ is also local minimum, so therefore as well as local minimum also so this means this means fx̄ is ≤ x| for all x belongs to Δ neighborhood of x̄ and this Δ neighborhood must be nests all so intersection with us okay.

For some Δ >0 okay, now anyhow we have to establish a contradiction then only we can say that x̄ is nothing but global minimum also okay, so take our convex linear combination of take a convex is a combination of x cap and x path okay they are convex is a combination of these two points. Now this is this is the convex set s okay which is given to us okay. This is some x̄ in it and there is some X cap in s and these are x you vary λ you will get different x okay for λ = half you will get a midpoint for λ = 1 /3 you will get another point in between x̄ and x cap.

Now this is this is some Δ neighborhood of x̄ okay there is some Δ neighborhood of x̄ of radius Δ okay of the Δ, so there will always exist some x which is in Δ neighborhood of x̄ you see if you vary λ if you vary λ, you will get different x and all are in between x̄ and s cap, so there will be some λ there will be some λ no matter how small it may be there will be some λ such that x will belong to Δ and able to first of all so okay.

So of course they will exist it some λ but some λ belongs to 0 and 1okay such that this x I mean this suppose this is the x Δ this x Δ it belongs- Δ native of x̄ intersection is no matter how small this λ is okay, so now you take you apply the convexity of f also.

(Refer Slide Time: 19:34)

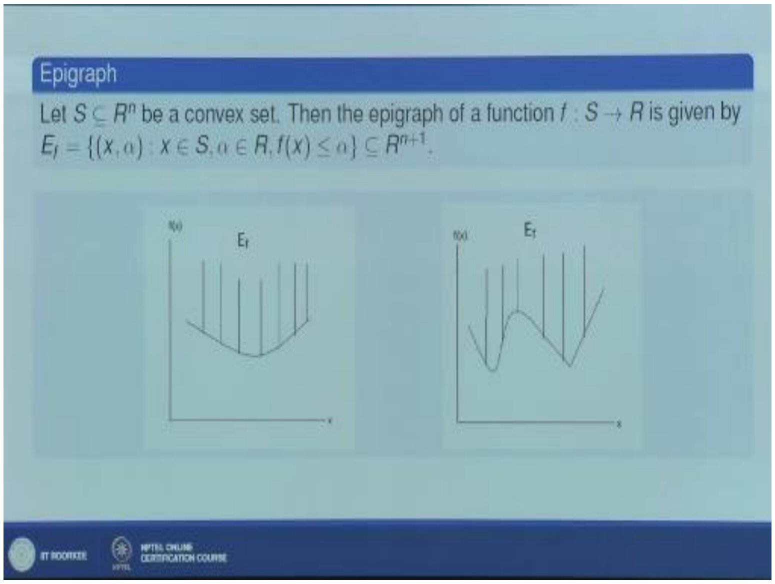


So f of f Δ which is nothing but f of λ x̄ + 1 - λ x cap ≤ λ f x̄ + 1 - λ x| gap because function is convex it is given to us and f x cap is Stickle as are f x̄, so this is strictly < λ fx̄+ 1 - λ 1- λ fx̄ o x| cap okay Rt okay. So this is = fx̄ so this implies x| cap is still < fx̄ okay, this implies now this s cap okay x Δ Y this is x Δ okay due to the x taken, so this x| f this fx Δ is stable as an x| spot and but from this must be ≤ x| for every f in Δ number of x̄, and this x cap Δ is in Δ neighborhood.

And it is in Δ neighborhood such that this in equality holds which contradicts this definition of local minimum okay. Which gives this contradicts to suppose this is 2 so this contradicts two so this implies this implies x̄ is global minimum of f okay, so hence we can say that whenever we are solving any optimization problem we are f is a minimizing type function and is convex. Then if you find any local minimum of f it will also be global.

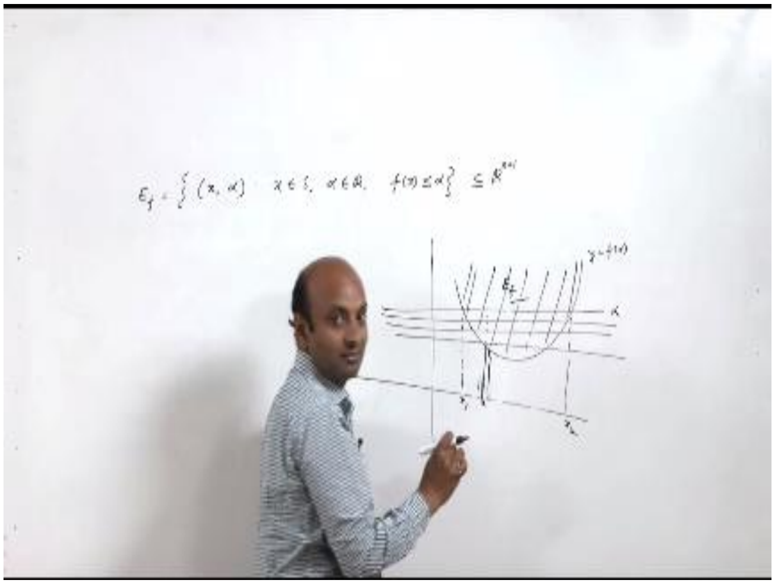
So this is a very peculiar property of convex functions okay. Now next is epigraph of a convex function what is an epigraph let us see.

(Refer Slide Time: 21:58)



So suppose s is s is the convex set then the epigraph of function f is given by this definition so what does it mean? Let us see.

(Refer Slide Time: 22:24)



So epigraph of a convex function is nothing but all x α such that x belongs to s α belongs to r and x| is ≤ α okay and definitely because f is an R^n and this is an R so this tuple will be in R^n + 1, so it will be a subset of our interest. Because this is in R^n and this is enough so this Cupid will be in R^n + 1 so it will be a subset of R^n + 1. So let us understand the definition suppose we have this function say y = x|, now suppose I take this α okay y = α and this is some x1 and this is some x2.

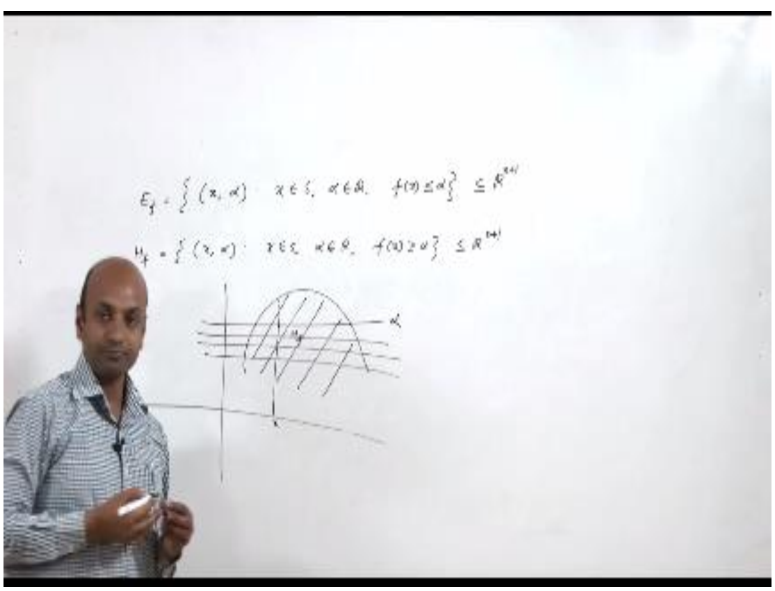
So you are interested to find out all those that is α such that x| is ≤ α suppose you take a point here so this x| is ≤α this x| ≤ α because this is this height and then this side okay. You take a point here this fs is≤ α now x comma a x form α all those points which are on this line x| I mean x comma α so the x| ≤ α. You see if you take this x for this f x| is only this height okay for this x | is only this height and this x| is ≤α, if you take a point there for this x | is only this height and α is this so x| is ≤ α, so you take if you take up all the points here on this line segment to here all other points such that x| ≤ and far.

Now very α if you vary α if you vary α so you will get you will get the line segment joining these two points as that x comma all xl fossils that episode ≤α so basically this is the entire region which represent epigraph of f, so you view very α if you vary α then you will get all this point in the shaded region which denotes the epigraph of the function f. Now you can easily observe that this function is convex very dc okay and the epigraph of this function is a convex set.

This we can easily visualize you see that if you take any two arbitrary point here and join the line segments. The line segment joining these two points is inside this region, so we can easily say that if a function is convex then the epigraph that functions is a convex set. We will discuss it later on we will prove it later on that if a function is convex then the epigraph of the convex function is always a convex set okay.

So these are the representation of epigraph now next is hypo graph, so let s be a subset convex set subset of R^n then the hypo graph of the function f will be given by all xα s belongs to s α belongs to R such that x| ≥ α you know here x| would be written = α that means switch region.

(Refer Slide Time: 26:43)



So for a hypo graph it is all x α such that x belongs to f α belongs to R and x| ≥ α, so this will subset of R^n + 1. So you can easily see suppose you have this curve like this so you take α here and all those all those x comma α so that x| ≥ α, so for this we are having here because you are if you take a point there okay. For this x x| will be this fs will be approved from here to here and α is from here to here, so that means x| ≥ α okay. So in this way you will get a bit line this chord joining this two point for this particular α, similarly if you vary α.

So you will get all those all these chords joining these two points okay, so the hypo graph of this region will be nothing but this thing and this function is concave. So we can easily see that if we have a concave function then the hypo graph of the concave function is a convex set okay. So the proof we will discuss in the next class so thank you very much.

For Further Details Contact

Coordinator, Educational Technology Cell

Indian Institute of Technology Roorkee

Roorkee – 247667

E Mail: etcell.iitrke@gmail.com. etcell@iitr.ernet.in

Website: www.iitr.ac.in/centers/ETC, www.nptel.ac.in

Camera

Jithin. K

Graphics

Binoy. V. P

Online & Video Editing

Mohan Raj. S

Production Team

Sarath Koovery

Arun. S

Pankaj Saini

Neetesh Kumar

Jitendar Kumar

Vivek Kumar

Nibedita Bisoyi

An Educational Technology cell

IIT Roorkee Production