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Nonlinear Programming - 1

Lec – 16
Dynamic Programming - III

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Hello friends so welcome to lecture series on non linear programming in the last lecture we have seen that how can we solve a network problem to find the shortest path from node 1 to node 10 or node I to node j using dynamic programming approach in that way I have seen that basically we divide problem into sub problems to find the solution the table and then what ever solution we obtain in the table will use the solution in the next table, till we got the last Table to find out the optimal solution of that larger problem okay.

Now we will see some more illustrations that how can we use dynamic programming to solve more optimization problems discrete cases so let us see more illustrations so it is one dimensional allocation problem.

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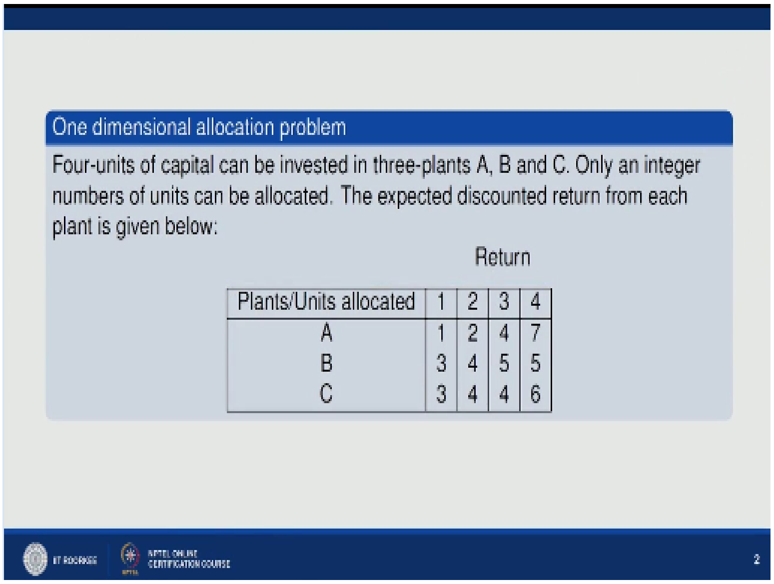

Now what it is, suppose 4 units of capital or can be invested in 3 plants, plants are A, B and C so we have 3 plants and units of capital available is 4, 4 units of capital available okay now only an integer number of units can be allocated okay that means units which can be allocated may be 0 may be 1, 2 or 2, 3 or 4 but not been fractions, the expected discounted return from each plant is given below, you see if in plant A, plant B are 3 plants, A, B and C, if in plant A the units allocated is one unit so the net return is 1 unit only.

If in plant A two units is allocated, so the next those the return is 2 units if in plant A 3 units of capital is invested so the output is or the return is 4 units similarly if in C suppose 4 units of capital the entire money as to be invested in plant C only, then from this table 6 units of return to be obtained, now the problem is how to find out how to find out that how much amount of money as to be invested in each plant.

So that then net output is maximum we have 4 units of capital available we have to invest the 4 units of capital in 3 different plants A, B and C, such that the total output or total return is maximum this is the problem so how can we find it.

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So we have to find the best allocation the second best allocation. The best allocation if only 3 units of capital is available and the best allocation if plant B is excluded to these are some of the problems which are can be solved in the same problem, so let us see how can we solve it so we divide this problem into 3 stages, stage 3 is stage 2 and is stage 1 in stage 3.

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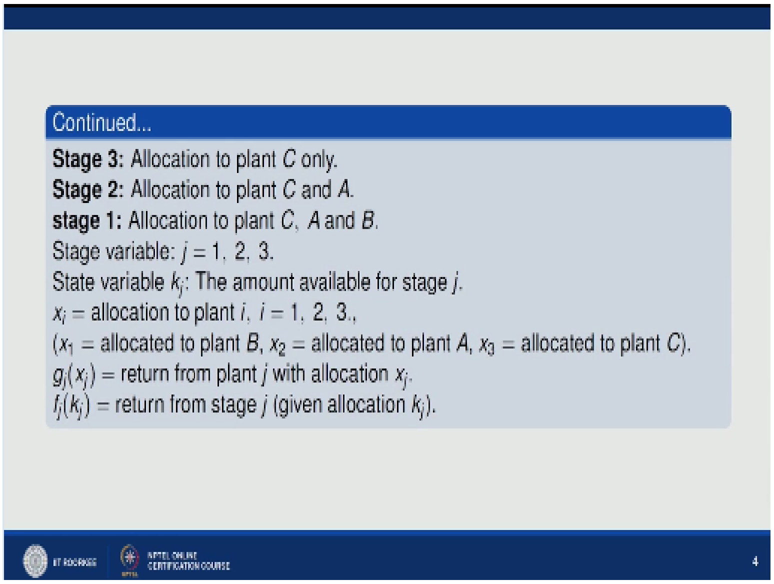

We basically deal with allocation to plant C only okay this is our choice basically that which we allocate to stage 3 which to stage 2 and 3 which to stage 1 here in this problems since it is give to us that in the 4th part it is given to the best allocation if plant B is excluded so we will keep plant B in the last stage or stage 1. So that if plant B is excluded the simply exclude the stage 1 the stage 2 will give the optimal solution so that this is the only reason I keep stage B I mean plant B in the last stage in the stage 1 okay.

Now so stage 3 will allocate to plant C only stage 2 means C and A in stage 1 means C, A and B okay so we have three stages basically we have 3 stages.

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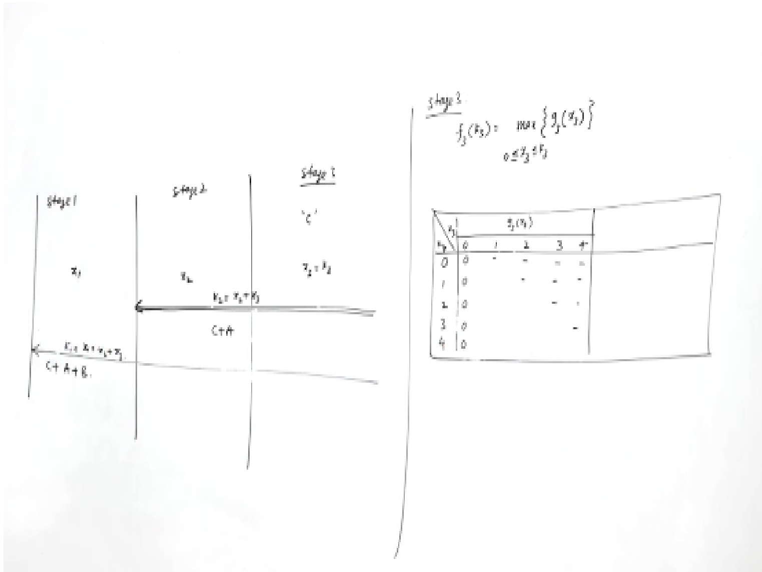

First is stage 3 which is allocation to plant C only this is stage 2 which means C and A C and A and this is stage 1, which means C A and B okay, now do you find stage variable j, j are 1 2 and 3.

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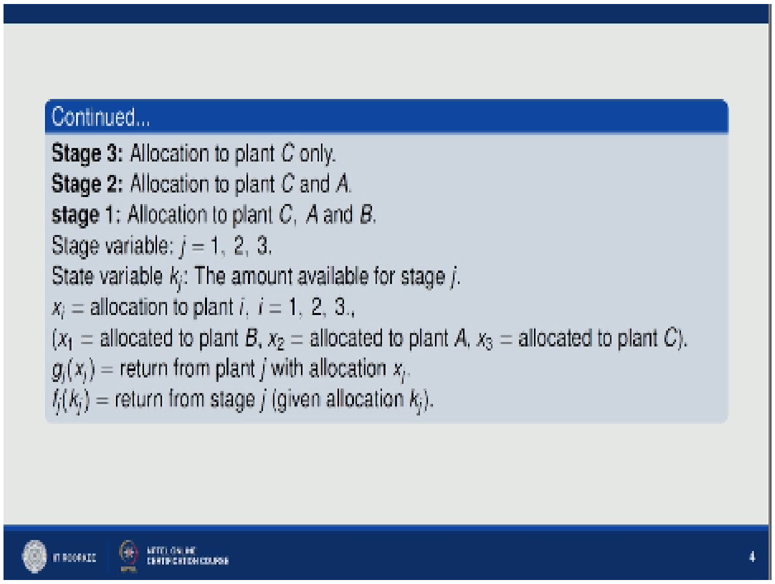

Then state variable is k_j, K_j means the amount available for a state j okay now xi is a location to plant i x_1 is allocated to plant B okay, this is plant B x_1 x_2 is located to plant A and x3 means allocation to plant C so that means x3 is here, to plant C how much is located in plant C how much located in plant C this is A and this is that means x2 and this is x1 and located to plant B okay and what is K_j, K_j is available to this stage now in this x3 in this stage we have only one variable x3.

So that will be K3 in this stage the stage 2 in this stage up to this stage we are calling it x k_2 the amount available up to stage 2 so that involve C and A so k_2 is nothing but $x_2 + x_3$ okay now stage 1 means the amount available up to stage 1, which involves B also so that means k_1 is $x_1 + x_2 + x_3$ you see x_3 means the allocation to plant C only x_1 means the location you plant A only and x_1 means the location to plant B only, but k_1 means k_3 means up to stage 3 k_2 means up to stage 2.

That is C and A collectively will K2 okay and K1 means up to stage 1 that is C A and B, okay now we find gj as written from plant j with the location xj okay, and fj xj is written from stage j.

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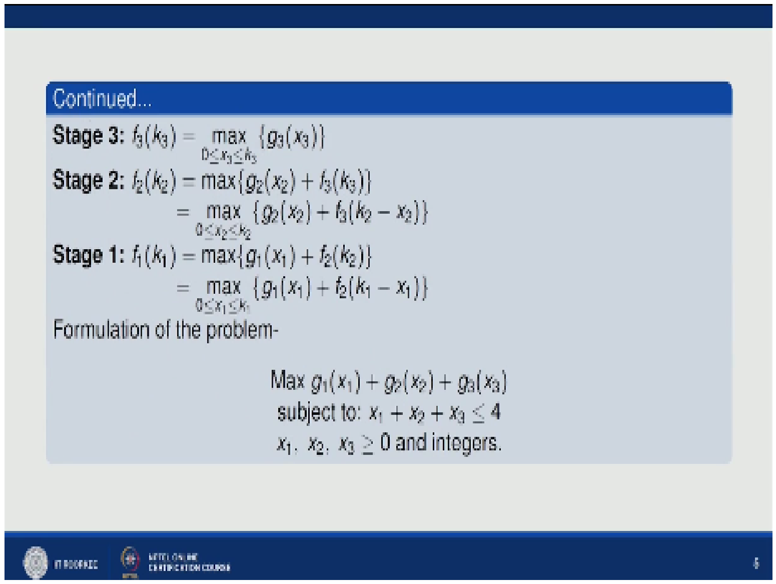

So let us discuss this over here, so first we define this into stages which we are calling it a stage j or stage 3, okay now a stage 3 if we are taking f_3 k_3, f_3 k_3 means fj kj means written from thus stage okay from stage 3 from stage 1 from stage 2 whatever we get in the maximum that will be fj kj f3kj means written I mean obtained from stage 3, now j_3 x_3 is the written which we obtained from stage 3 and the maximum of this. Where of course k_3 can be more than k x_3 can be more than k_3 this because k_3 is availability up to this stage.

And x3 is the allocation to plant C only o0kay and that all must be non negative also okay, so assuming this constraint the whatever we are obtaining the maximum of this will be the maximum output or the maximum written of this stage, stage 3 okay now how will make the table of this okay, so how we make it let us see here we will first make here it is k_2 and x_2 okay it may be a 0 1 2 3 4 you see written units available are 4 units, it may be 0 1 2 3 or 4. Similarly x3 sorry it is x_3, x_3 may be a 0 1 2 3 4 okay.

Now if no output I mean no unit available so of course the written will be 0 it is j3 k3 g3 x3 = 0 if no capital is available so return will be 0, now x3 can be more than or less than equal to k3 it is k3 okay x3 will all will be less equal to k3 so if it is if k3 is 0 and x3 is 1 so this entry is not possible because x3 must be less than equal to k3, so this entry is not possible similarly this entry is not possible similarly these three entries are not possible and similarly these three entries are not possible, so it will be lower triangular matrix type, okay because x3 will always be less than equals to k3. Now if x3 is 1 so the return can be obtained from the table you see, x3 is that location from plant C.

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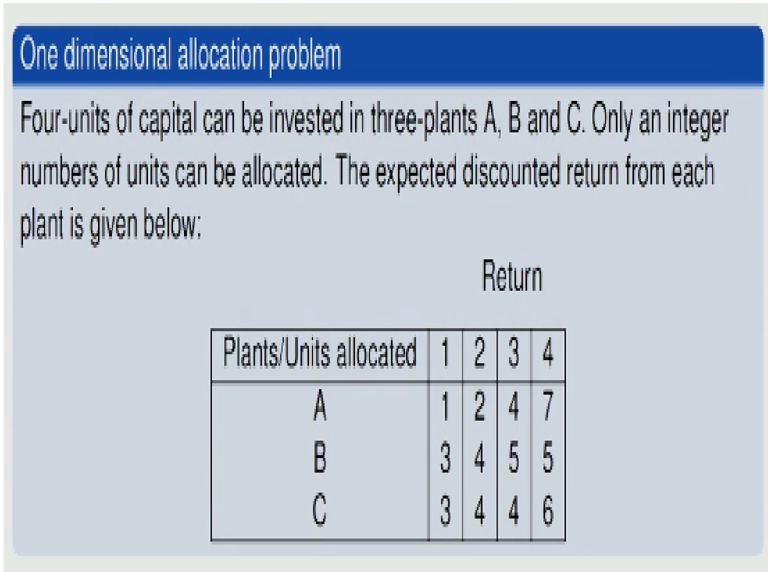
One dimensional allocation problem

Four-units of capital can be invested in three-plants A, B and C. Only an integer numbers of units can be allocated. The expected discounted return from each plant is given below:

Return

Plants/Units allocated	1	2	3	4
A	1	2	4	7
B	3	4	5	5
C	3	4	4	6

And from plant C it is if one unit are capital available so it is 3, it is 3 units, so it is 3, okay.

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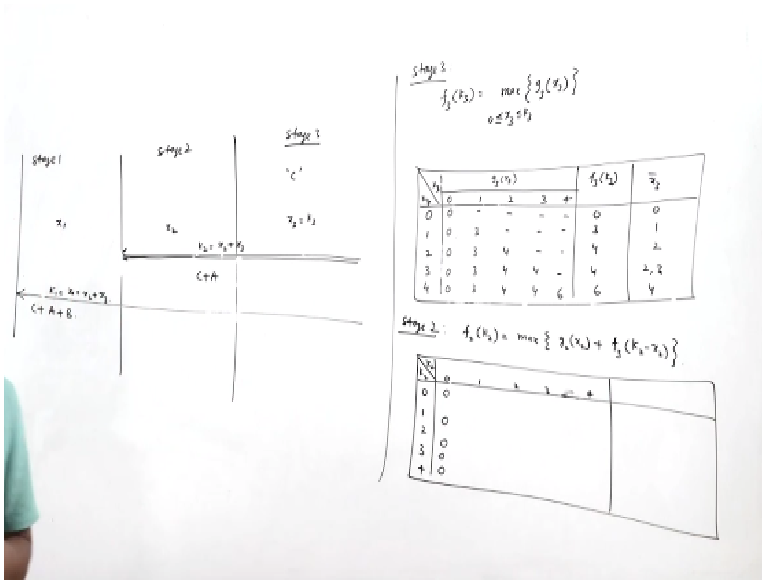

Now if 2 units of capital available for plant C so it is 4 and again 4 and 6 so it is 4, 4, 4 then it is 4,4 and 6.So what is f3(k3), f3(k3) is a maximum of all this maximum of this is 0 and which for which x3 this maximum of obtaining at 0 so it is 0,now this maximum obtaining at 3 at which s3 at 1 so we are calling s3- bar at which s3 this maximum is 4 and which s3 at 2 this is maximum is 4 at 2 and 3.

This maximum is 6 at 4, so this is the maximum written I mean return obtained from the stage 3 it is 3 means when all the money is to be located for plant c only, now we come to stage 2, stage 2 means CAN that means we have to find f2 (k2), f2 (k2) will be maximum of g2(x2) you see g2(x2) is what, g2(x2) is a return from the plant A, okay. Return from the plant A and plus whatever return we are obtaining from a stage 3 that will be CNA.

You see in this stage up to this stage two plants are available C and A the net output of these two stage these two plants will be the maximum return obtain from stage 2,now g2(x2) is a return obtained is a maximum return obtained I mean return obtained from stage A I mean plant A sorry, okay. If we have a plant A the return obtained from plant A obtaining from a different values of x2 will be g2(x2) and plus whatever we obtain from a stage 3 that will give C and A that will give our return false stage 2,okay. So here we add f3(k3) whatever we obtain for the stage 2 now you see k3 is s3 up to this stage is k3 up to this stage is k2,okay.

So k2 – x2 will be the net available for a stage 3 so that will be k2 – x2. Now see k2 is the amount available up to a stage 2 okay and x2 is the availability in this stage I mean for plant A so the difference will give the availability for a stage 3 so for a stage 3 so this to find out now, so again we make a next table based on this, this will be say it is k2 it is x2, k2 have 0, 1, 2, 3, 4 x2 are 0, 1, 2, 3, 4, okay. Again for 0 the returns are 0,okay and if it is 1 and it is A it is 1 you can see here it is 1.

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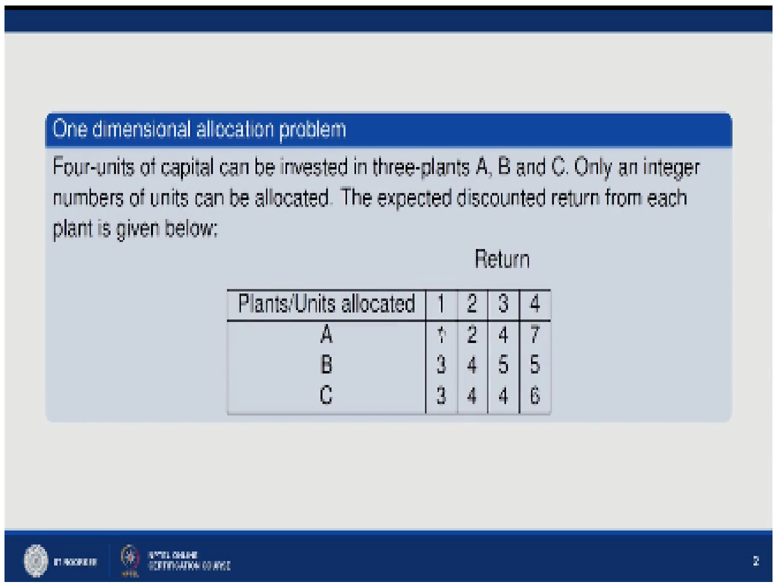

If, two units available 2 if 3 then 4 if 4 then 7 it is 1, 2, 4, 7.

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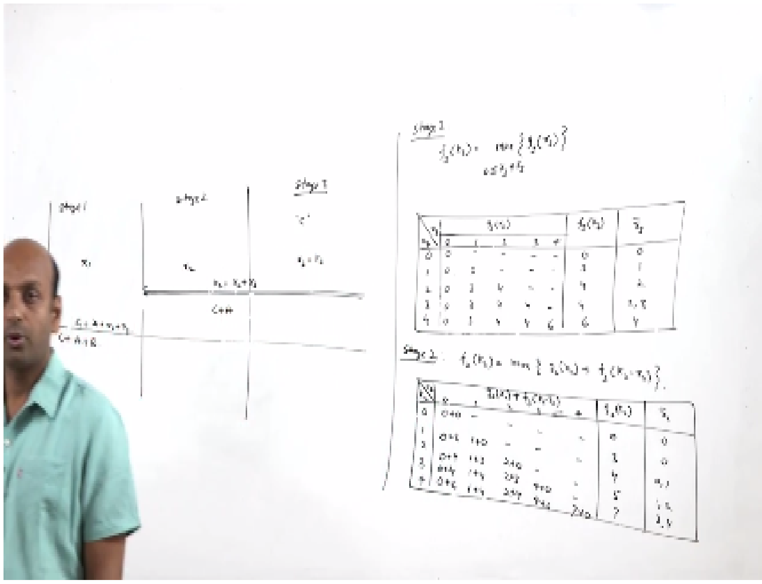

So it is 1,2,4 and 7 so that is g2x2 now we have to add f3k2-x2 also okay, so this minus this 0-0 is 0 and f3 at 0 is 0, then 1-0 is 1 k2-x2 1-0 is 1, and f31, f3 at 1 is 3 then 2-0 is 2 and f3 at 2 is 4 okay, so we have to add 0 then 3 then 4 then 4 then 6 again here it is 1-1 is 0 then f3 at 0 and f3 at 0 is 0 it is 2-1 is 1 and f3 at 1 is 3 so it is 3, so it is +0+3+4+4 again it is 0+3+4, again it is 0+3 again it is 0 and what will be f2k2 so maximum of all this, maximum here is only 1 and 3, so maximum is 0 at $\bar{x}2$ at which x2 and x2 is 0.

Here the maximum is 3 at which x2 and x2 equal to again 0, here maximum is 4 it is 4, it is 4 maximum is 4 at which x2 0 and 1, here it is 4,5,5,4 so 5 is maximum and 5 maximum we are getting at this point and this point it is 1 and 2 so it is 1 and 2. Here we are getting maximum as 7 so 7 we are obtaining at 3 and 4, so this is maximum output what we are obtaining in a stage 2, that means if we are locate the entire money for C and A only so the maximum output will be 7 units, that means you plant B the screwed, okay.

If we screwed to the plant B and spend the entire unit of capital on C and A only so the maximum output will be 7 units at x2=3 or 4 okay. Now for C, A and B we will make the last stage to stage 1 and how can we do that.

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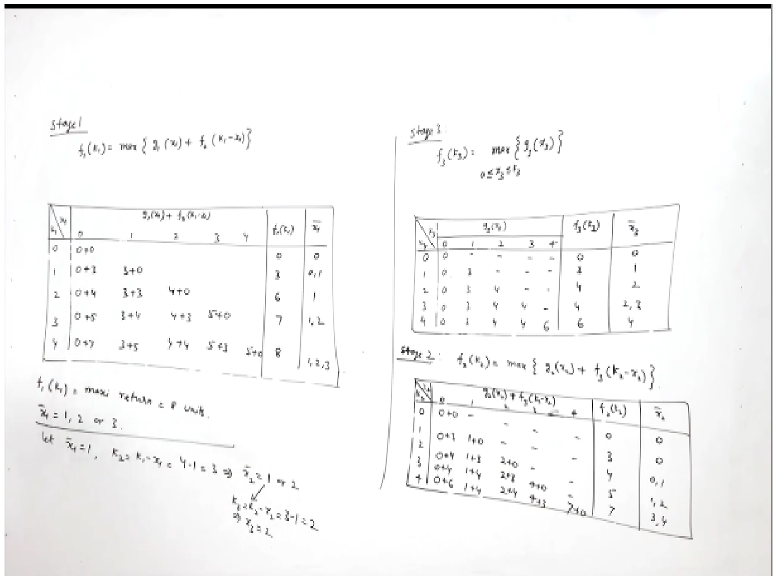

Stage 1 again where we can formulate problem it is f1x1 which is maximum of it is g1x1+f2k1-x1 whatever it will label up to their stage –x1 will give C and A that is up to stage 2, okay. So this is a last stage or the it is k1x1 it is 0,1,2,3,4 it is 0,1,2,3,4 and 0 all values are 0 now it is at B it is for B, for B outputs are returns are 3,4,5,5 so 3,4,5,5 okay, so it is these are g1x1 and plus we have to do f2k1-x1.

So if it is 0 and if it is 0 so 0-0 is 0 so f20 f2 when k2 is 0 is 0 so it is +0 again when it is 1-0 so f21, f2 at 1 is 3 okay, so it is 3+3 2-0 it is +4 again +5+7 it is 1-1, 1-1 is 0 so f21 it is f2 and k2 is 1 is here okay, 1-1 = 0 so f2 0 f2 0 is 0 so it is +0 +3+4+5 again +0+3+4 +0+3+0 so what will be f1k1 f1k1 maximum, maximum is 0 at which s1 bar at s1 bar = 0 this is 3 maximum at 0 and 1okay. now this is 4 64 so 6 is maximum at one 5885 so 8 is maximum at 1 and 2 and 7888 so 8 is maximum at 123 so if f is spend 8 4 units of capital so the net output is 8 units you can see 8, okay it is 8 it is 4 + 7 so it is 7 okay.

At one and 2 okay so the net output the maximum return to f1 k1 which is the maximum return will be 8 units and what is allocation that is on which plant how much a money is to spend so for that so we have a solution optimal solution sm bar =m1 2 or 3 okay suppose s1 is 1, if s1 is 1 so k2 will be k1 – x1 and k1 is 4 because 4 capital is available and x1 is one so it is -1 it is 3 when k2 is 3 x2 is 1 or 2 so this implies x1 bar 1 or 2 okay.

So for x2 = 1 for x2 bar = 1 k3 will be k2 – x2 and k2 is 3, 3 – 1 = 2, and when k3 is 2 x3 is 2 so x3 is this implied x3 is 2 okay. This is the one solution x11 x1 is for plant b x2 is for plant a and x3 is for plant c okay, similarly we can find for x2 = 2 also and similarly we can find when x2x1 is what 2 or 3 okay. So this is how we can find out the optimal solutions of such type problems. Now we have other parts of also the second best allocation now in this row this is 8 and after 8 we have 7 units with the second best so second best allocation is 7 units okay.

That means the second maximum return is of 7 units and what is the optimal solution what is allocation, allocation is this 7 that means x1 bar is 0 okay, for second best allocation so second best allocation return is 7 units maximum return and for that x1 is 0 okay again we find k2 which k1-x1 and k1 is 4-x1 is 0 which is 4 and from the stage 2 we already seen in the table from stage 2.

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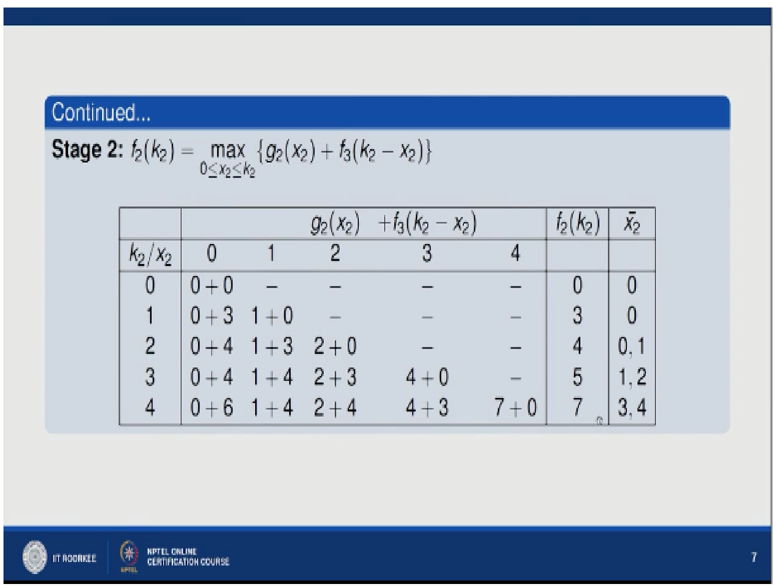

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Stage 2: $f_2(k_2) = \max_{0 \le x_2 \le k_2} \{g_2(x_2) + f_3(k_2 - x_2)\}$

	$g_2(x_2)$	$+f_3(k_2 - x_2)$				$f_2(k_2)$	$\bar{x}_2$
k_2/x_2	0	1	2	3	4		
0	0+0	–	–	–	–	0	0
1	0+3	1+0	–	–	–	3	0
2	0+4	1+3	2+0	–	–	4	0, 1
3	0+4	1+4	2+3	4+0	–	5	1, 2
4	0+6	1+4	2+4	4+3	7+0	7	3, 4

When x2 is 4, when x2 is 4 x2 will be 3 or 4 okay so from here x2 bar is 3 or 4 again we will find solution from this or this okay so that will be the second best allocation.

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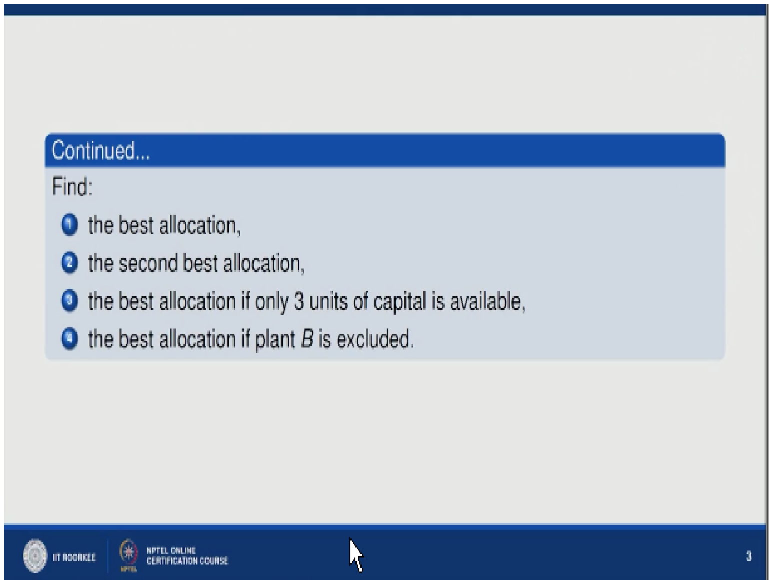

Now third part of the problem is the best allocation is only 3 units of capital is available so if only 3 units of capital is available this means you simply delay delete this row and column because you are having only 3 units of capital okay now if you delete this row and this column so the return is 7 units which you obtain when x1 is 1 and 2.

And now k1 is 3 because 3 units of capital is available so we will use the same algorithm same steps to find out the optimal solution okay now the next is best allocation you plan be is squirt it for plan b is excluded so you have to delete the stage 1 stage 2 will give maximum return because now you have two plants C and A so stage 2 exclude C and A both.

So you simply exclude stage 3 up to the stage 2 that will give the optimal solution okay if plant B is exclude it so this stage will give the optimal return if plant B is excluded that is 7 units and x2 bar is 3 and 4 so we can find the optimal solution when x2 is 3 or 4 so this have been discussed over here okay.

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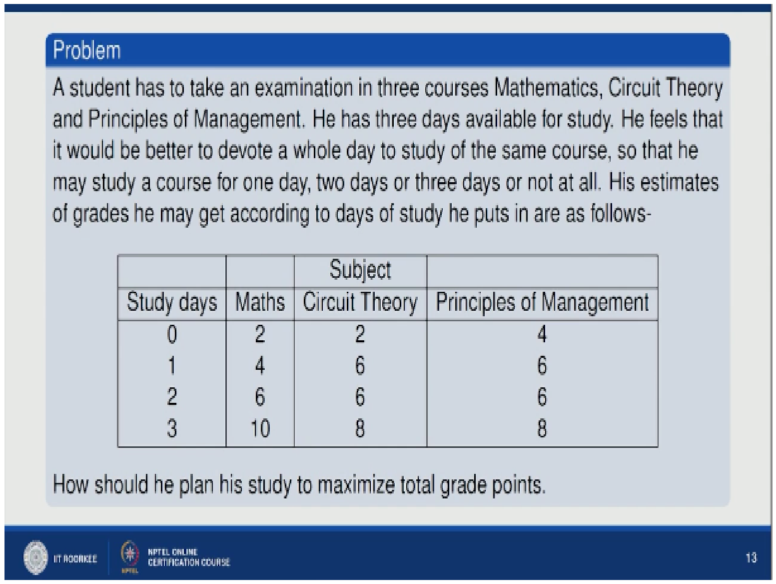

Now we have one more problem based on this let us quickly see how can we solve this problem on same lines a student has to take examination in 3 courses mathematics, Circuit theory and principles of management okay he has three days available for the study he feels that it would be better to devote the whole day to study of the same course.

So that he may study a course for one day two day or three days or not at all so he has decided to study one course for tare one day or two day or three day or not at all that means it has be 0, 1,2 or 3 he estimates the grades he may get a study of days of study he puts in area s follows suppose he gives 0 days to study I means no time no day for mathematics.

So he will on 2 credits in mathematics similarly 2 in circuit theory and management suppose he give 2 day in mathematics only than tear 2 day he will on 6 credit units of mathematics 6 units of circuit theory and 6 receipt principles of management, if the entire 2 days they voted in mathematics so only then 6, if the entire 2days voted in the circuit then theory is 6 and similarly we have the entire table. Now how should we plan to study maximize the total grade points. How he should plan his 3 days so that the total credit become maximum. So how we will do that? This is again a discrete optimization problem that will be used by programming approach, so how we will do that? Suppose that y1, y2, y3.

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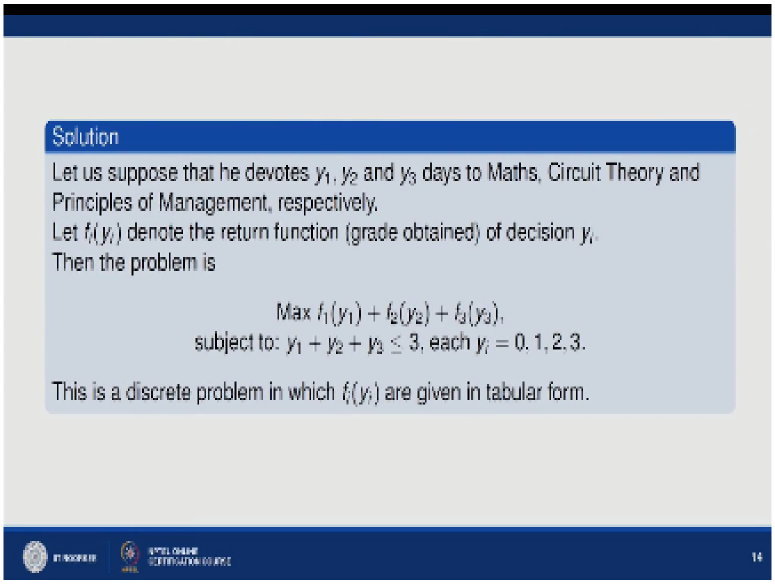

Are it mathematics circuit theory and principle management f_i y_i below denote the return function correspond to variable y_i , so the problem will be maximizing f_1 y_1 + f_2 y_2 + f_3 y_3 subject to y1 + y2+ y3≤ 3, where y_i will be 0,1,2, or 3.

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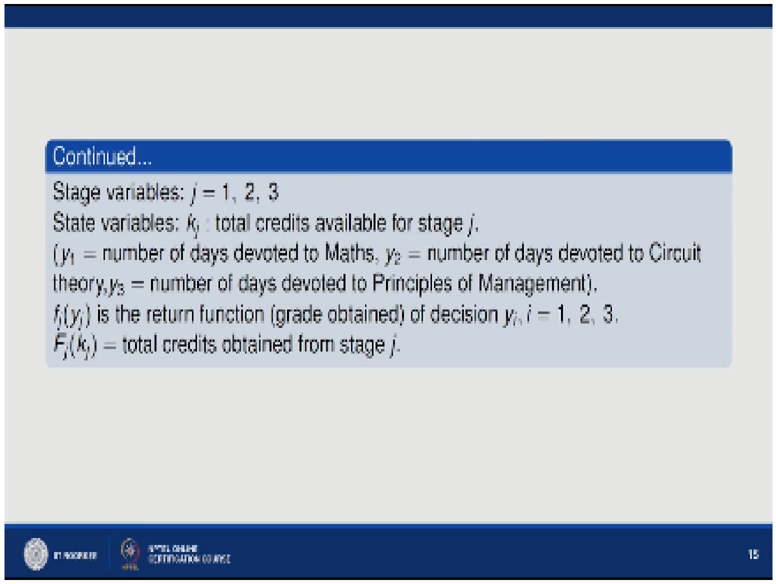

So the stage variable here are which is 1,2 and 3 y1 denotes number of days devoted to math's, y2 denotes number of days devoted to circuit theory, y3 denotes number of days devoted to principles of management, like this as the same problem that we discussed for this like plan a, plan b, and plan c. here we have math's , circuit theory and principles of management okay, f_i, y_j is the return function of the desirable variable and F_j k_j total credits obtained from stage j.

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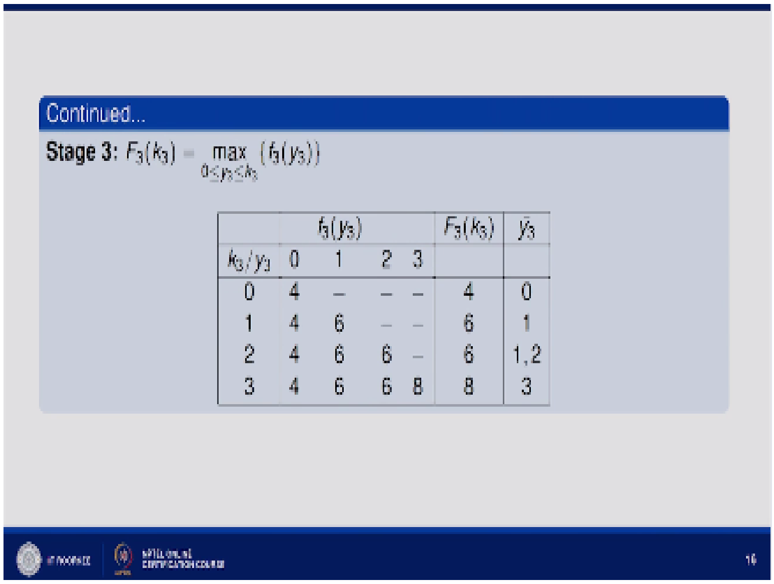

So again we start from stage 3 go to stage 1.

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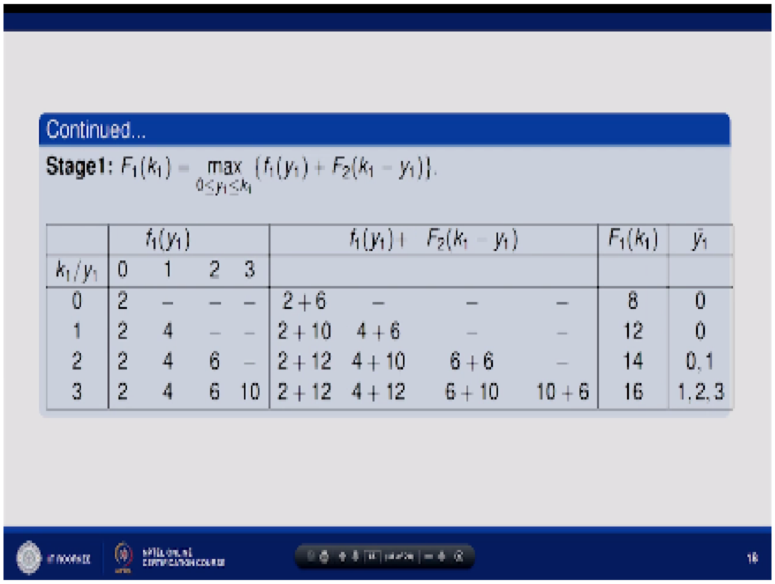

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Stage1: $F_1(k_1) = \max_{0 \le y_1 \le k_1} \{f_1(y_1) + F_2(k_1 - y_1)\}.$

	$f_1(y_1)$				$f_1(y_1) + F_2(k_1 - y_1)$				$F_1(k_1)$	$\bar{y}_1$
k_1/y_1	0	1	2	3						
0	2	–	–	–	$2+6$	–	–	–	8	0
1	2	4	–	–	$2+10$	$4+6$	–	–	12	0
2	2	4	6	–	$2+12$	$4+10$	$6+6$	–	14	0, 1
3	2	4	6	10	$2+12$	$4+12$	$6+10$	$10+6$	16	1, 2, 3

In the last example we will obtain that the optimal, the maximum credits is 16 units and that can be obtained y1 is 1,2 or 3 okay.

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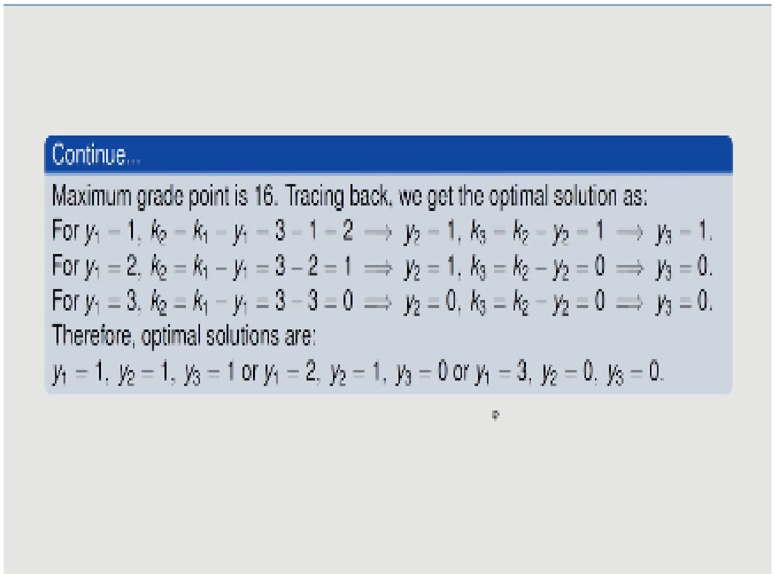

So that maximum grade point 16 unit, tracking back we get the optimal solution as y1 = 1 we get y2 = 1, and y3 =1, if y1 = 2 then we get y2 = 1and y3 = 0 and y1 =3, y2 = 0and y3 =0, so these are 3 optimal solution for which the maximum grade point is 16 units. Now you can see the table also for example our 1st solution that is 1,1,1 okay if you see the table, and see 1,1 and , now 1+1+1 that is 3days okay, 1 day for mathematics, 1day for circuit theory and 1 day for principle management, that is 6 + 6 = 12, 12 + 4 is 16.

Which is the maximum output again the 2nd optimal solution is 2,1,0 , 2 days for mathematics, 1 day for circuit theory and 0 day for principle management, so if 2 days for mathematics is 6 okay and 1day here is 6, 6+6 12, 12 + 4 is 16 okay. So in this way if you plan and study in this way you will get the maximum credits so in this way we can solve some problem some discrete optimization problem using dynamic programming approach, so thank you very much.

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