INDIAN INSTITUTE OF TECHNOLOGY ROORKEE

NPTEL
NPTEL ONLINE CERTIFICATION COURSE

Nonlinear Programming

Lec – 01
Convex Sets and functions

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So welcome to the lecture series on nonlinear programming, so what nonlinear programmings are and why they are important this we will see in this course. You see that whenever they solve any engineering or science problem we frequently in quainter various optimization problems which may be nonlinearly nature. So in this course we will see that what basically nonlinear problems are and how to solve such problems okay.

So the first lecture is on convex sets and functions now first what is OR nonlinear programming is a part of operation research, so what is the work first we will see this.

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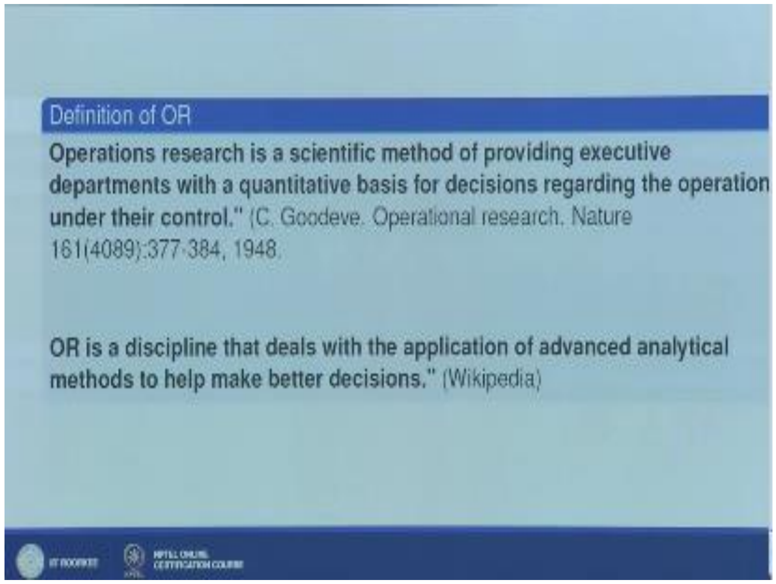

OR or operation research is a scientific method of providing executive department with a quantitative basis for decisions regarding the operations under their control or by Wikipedia it is a discipline that deals with the application of advance analytical methods to help make better decisions. So OR is the basically part of the mathematics due by which we can take better decisions.

Now what is an optimization problem you see that in optimization problem they have two components an objective function a function which we have to maximize or minimize subject two some constrains which we saw which say constrain set. So like here we have a problem P which is minimization type minimization of effects subject to conditions are X belongs to C.

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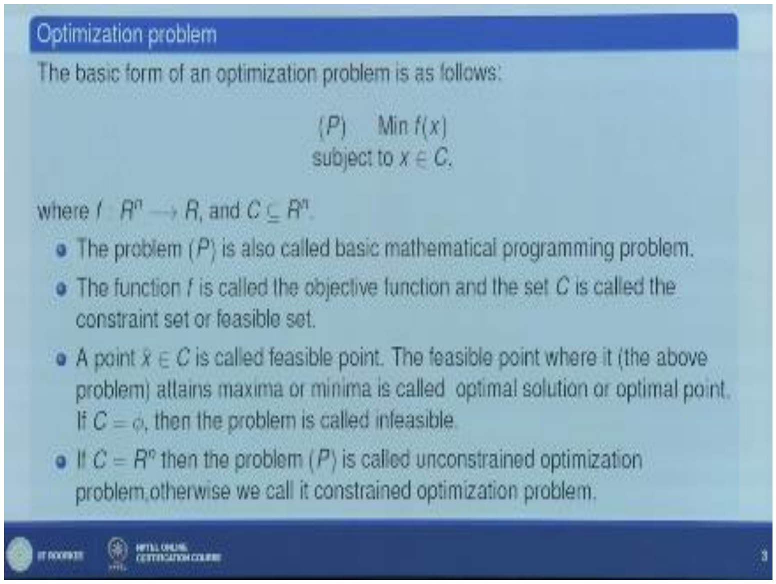

We are axis the function from R^n to R and C is the subset of R^n, now this problem we are call in this problem as an optimization problem because here we have to minimize a function F okay, subject to conditions that x belongs to some set C. So this function F here which we have to minimize is called objective function and the set C is called set of constrains or constrain set. Now this problem is also called basic mathematical programming problem okay.

The function axis called objective function and a set C is called the constrain set or the feasible set. Now nay point $\bar{x}$ which belongs to C, C means that constrains set which belongs to C we are calling that point as feasible point or feasible solution okay. And that collection of all the feasible solution we are calling a feasible set or C. if $C = \varphi$, φ means C is empty though then the problem is called infeasible, infeasible means no solution problem has no solution okay.

Now if $C = R^n$ suppose C is the entire N dimension Euclidian space $C = R^n$ then the problem p is called unconstrained optimization problem there is no constrain that is we have to minimize this function or whole R^n okay, otherwise we call it constrain optimization problem if C is a proper subset of R^n then we called this problem as constrained optimization problem. Now let us consider this problem.

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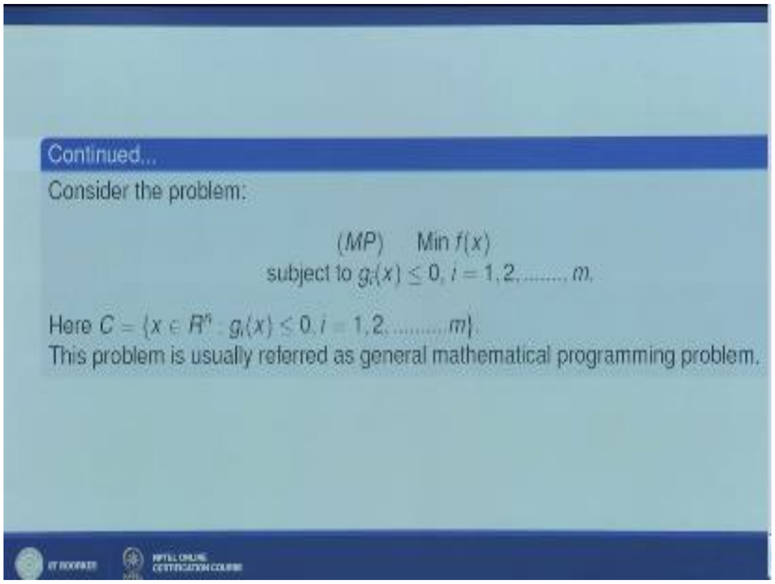

What this problem is this problem is minimize.

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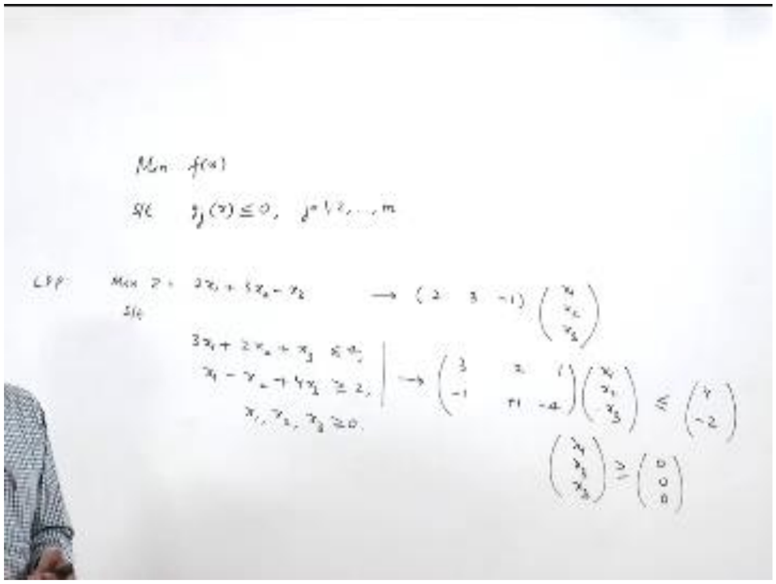

Effects subject to Gjx ≤ 0 and j = 1,2, ….m okay. Now here in this problem this function F is to minimize, so this function we are calling as objective function and what are the constrains Gj ≤ 0 are the constrains. How many constrains m number of constrain so here we have to minimize this function F subject to m number of constrains which is Gj ≤ 0 okay. Now this optimization problem can be broadly classified in to two categories.

First is linear programming problems and second is nonlinear programming problems what are linear programming problems if objective function the function which we have to minimize ort maximize is linear okay and all constrains are linear so such problem are called linear programming problems. And otherwise we call it non linear programming problem so this is in this slide if objective function and all the constrain GJ are linear then the problem Mp is called linear programming problem which mathematically can be express in this form and if Mp is not linear then we call as non linear programming problems.

Like example of LLP is something suppose we have to minimize this function minimize z = say $2x_1 + 3x_2 – x_3$ subject to suppose it is $3x_1 + 2x_2 + x_3 \leq 4$ and $x_1 – x_2 + 4x_3 \geq 2$ and x_1, x_2, x_3 non negative so this is the LPP, this is basically a linear programming problem okay. Now in this

LPP this can be rewritten as 2 3 -1 and this can be written as x_1 x_2 x3 this constrains can be written as 321, this can be written is 1 -1 4 and it is x_1 x_2 x_3 it is one is less than other is greater.

So you first multiply this by -1 okay to make it less than equal to so all those constrains all the sign will change and it is ≤ 4 -1 and x_1 x_2 x_3 non negative. So basically in this formulation if we compare with this formulation though this is C^T, and this is x this is a matrix A of orders 2x3 this is $x \leq b$ this is a vector b and this $x \geq 0$ okay. So we can always expressed any linear programming problem in this way that is minimizing $C^T x$ subject to $x \leq p$ and $x \geq 0$ okay.

Now if any with the objective function or any at least one of the constraints is non linear then such problems are called non linear programming problems. Now there is a particular case of non linear programming problem which is quadratic programming problem if the objective function that is effects is of degree 2 that is quadratic and all constrains are linear then such problems are called quadratic programming problems. We will discuss about quadratic programming problems further lectures. So next is convex set.

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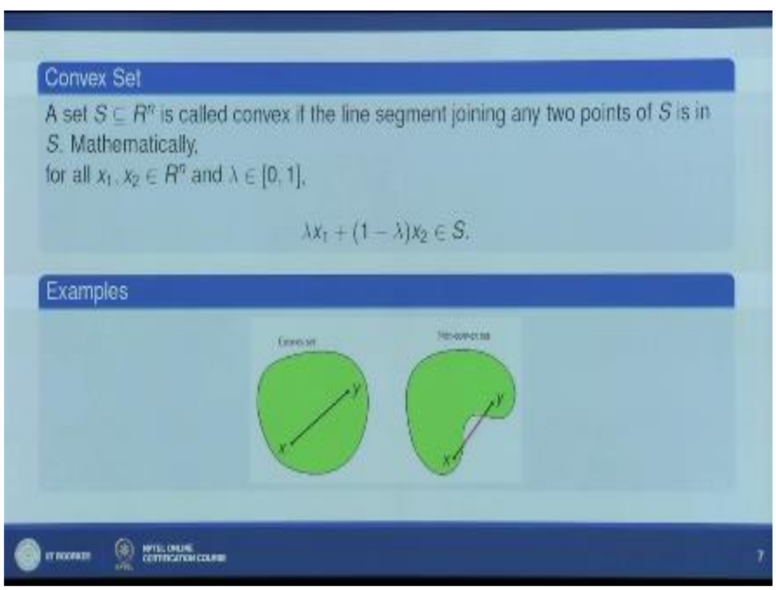

Now a set S is called convex if the line segment joining any two points of S is in S, so first what does it meaning.

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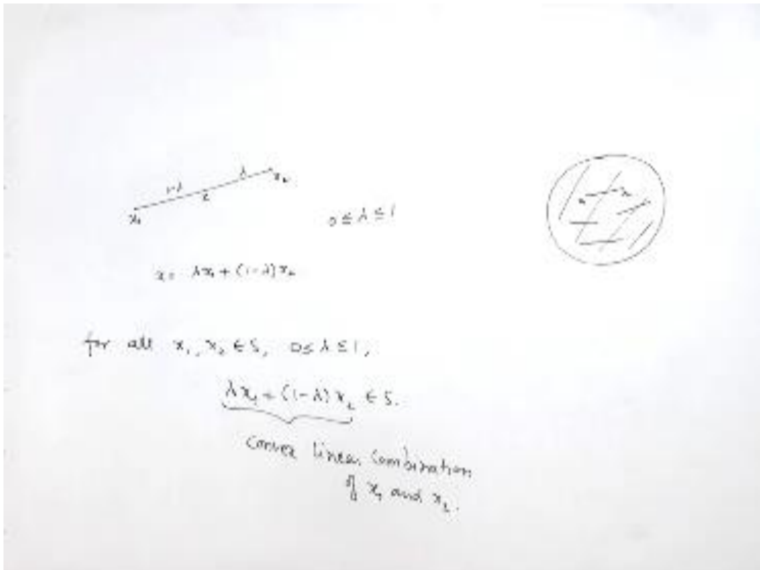

Suppose we have two points x_1 and x_2, so this is the line segment joining x_1 x_2. Now you take any arbitrary point x in between x_1 x_2 suppose this point divide this line segment in the ratio suppose $1 - \lambda$ and λ, now you want this point x in between x_1x_2 therefore these ratios 1- λ and λ must be non negative that means λ should lying between 1 and 0 okay. So now what will be x, so x will be nothing but $\lambda x_1 + 1 - \lambda\ x_2$ so this will be x.

Now for a convex set suppose we have region inside the circle we take any two arbitrary point in this set suppose these are two arbitrary points, the line segment joining these two point must be inside the set that means take any arbitrary point on the lying segment joining these two point for any λ between 0 and 1 the point x which is $\lambda x_1 + 1 - \lambda\ x_2$ must be in S, you take any two point here you take two point here join the line segment the line segment must be in this set itself you take two point here join the line segment .

The line segment joining these two points must be inside the region you take any two point here, so line segment joining these two points must be inside the region that means for all x_1 x_2 and s and λ between 0 and 1 the $\lambda x_1 + 1 - \lambda\ x_2$ must be in S, for all $x_1 x_2$ in S okay. So if this result hold for all $x_1 x_2$ be in S λ between 0 and 1 this means the set is convex okay. Now this sometimes we call it convex linear combination of $x_1$and $x_{2,}$ so sometimes we call it convex linear combination or x_1 and x_2.

So we can also defined convex set like this it take any two arbitrary point in S the convex linear combination of points must be less. If this property holds then we see the set is convex like we have two pictures here in the first picture if you take any two arbitrary points in the S and join the line segment the line segment is totally contained in the green portion of the set okay.

Now if you take this second example in a second example if you take these two points x and y the line segment joining these two point is not totally inside the set so this is not convex okay. Now we have some properties of convex set the first property is intersection of any collection of convex sets is always convex.

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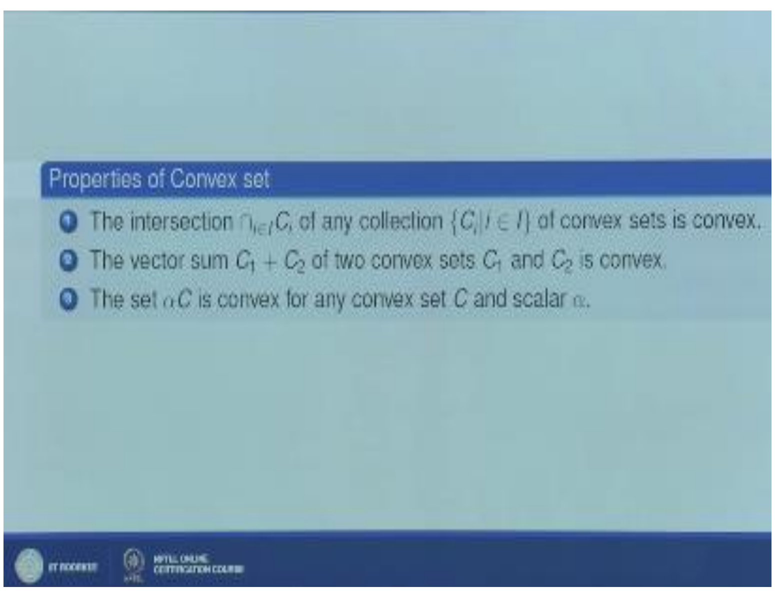

The proof is very simple let us see.

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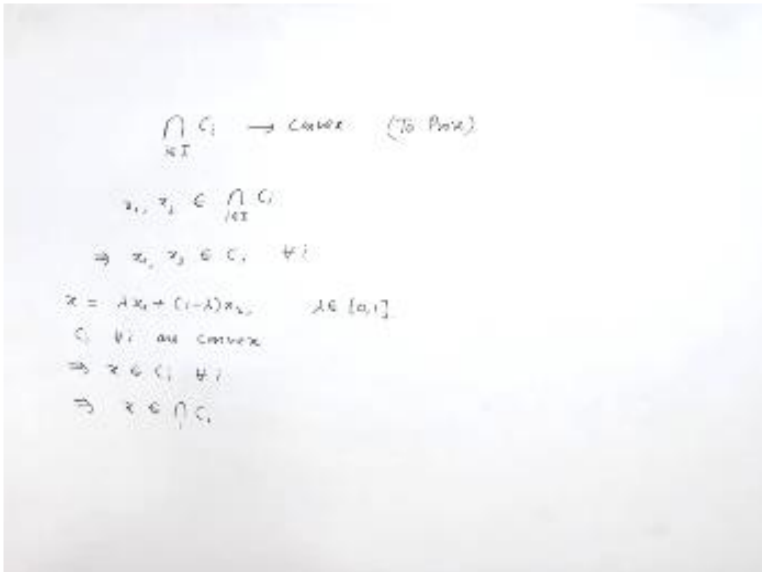

So we have to showed that the intersection of any collection of convex set. So this we have to show this to prove okay and it is given to us that all C I are convex so whenever we have to show set of convex take any two arbitrary point in that set and try to show that the convex linear combination of those two points is in S okay, and suppose you want to show the set is not convex so try to show a counter example try to show that there are some point there exist the point x1x2 and some λ between 0 and 1 such that the convex linear combination of those two point is not in S.

So that means S is not convex okay, so here we have to show that this is convex so take any two arbitrary point in this set okay, now this implies x1x2 belongs to Ci for All I okay and now you take x is convex linear combination of these two points for λ between 0 and 1 and since Ci for all I is are convex. So this implies x belongs to Ci for all I because Ci is convex for all I and x is nothing but convex linear combination of x_1x_2.

So this x will be in Ci for all I and if it is in Ci for all I this means x will be in intersection also. So that means this inter section of Ci is a convex set because what we have shown that he convex linear combination of any two arbitrary point of the set is in S is in the set. So that mean a set is convex okay. Now the second property is the vector sum $C_1 + C_2$ of two convex set C_1 and C_2 is also convex.

So the proof is again simple let us see.

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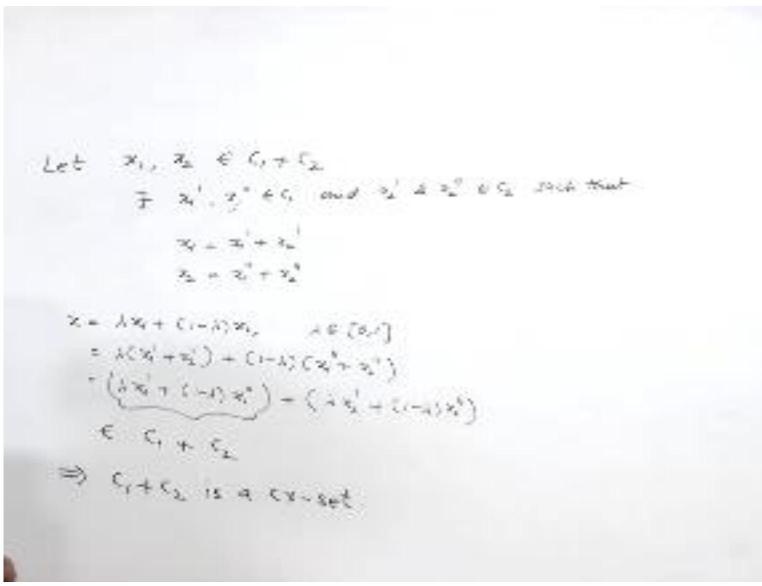

So you take any two arbitrary point say x_1 and x_2 in $C_1 + C_2$, so we have to show that this set is convex so we have to prove that the convex linear combination of x_1 and x_2 is in $C_1 + C_2$. So let this holds this implies since x_1 is in $C_1 + C_2$ this means they are exists some x_1 direction x_1'' in C_1 and x_2' and x_2'' in C_2 such that x_1 will be equals to $X_1' + X_2'$ and x_2 will be post to $x_1'' + x_2''$ okay.

Because these are in C_1+ C_2, so that means x_1 can be expressed as some element of C_1+ some element of C_2 similarly for X_2. Now you take convex linear combination of these two points for λ between 0 and 1 so this is nothing but λ $(x_1' + x_2') + (1 - \lambda)$ $(x_1''+x_2'') = (\lambda x_1' + 1\text{-}\lambda\ x_1'')$ +

($\lambda x_2' + 1 - \lambda x_2''$). Now this element is a convex linear combination of two points in C_1 and it is given to a C1 is the convex set.

So this will belongs to C_1 again this x_2' and x_2'' are in C_2 and C_2 is the convex set so this point will be in C_2 so this point is in C_1 this point is in C_2 so this will belongs $C_1 + C_2$. So that means this x belongs to $C_1 + C_2$ and X is nothing but convex linear combination of these two points, so this implies $C_1 + C_2$ is are convex set oaky is a convex set. So in this way we can show that some of two convex set is also convex.

Now in the next is the third property that α times C is also convex for any convex set C and for s for a scalar α, so this also can be prove very easily. You take two arbitrary points in α C.

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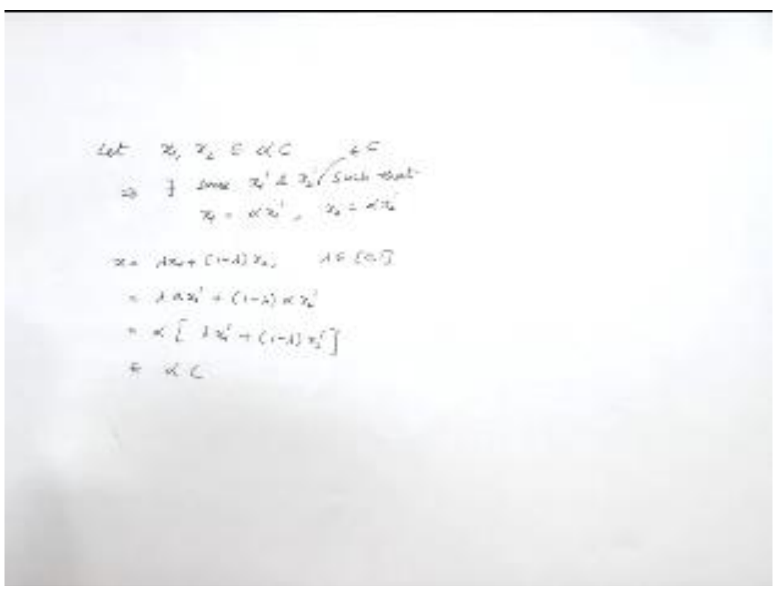

Let x_1 x_2 belongs to α times C this implies they are exist some x_1' and x_2' such that $x_1 = \alpha x_1'$ and $x_2 = \alpha x_2'$, now take convex linear combination of these two points will λ belongs to 0 and 1 okay take convex linear combination of these two points and we have to show that this x belongs to α times C okay. Now this is nothing but α times $\alpha x_1' + 1 - \lambda \alpha x_2'$ and this is α times $\lambda x_1' + 1 - \lambda x_2'$ and this belongs to α C.

Why this belongs to α C because these are the point say in C these are the points belongs to C okay, and C is the convex set so this belongs C. So α times C that means α C is also a convex set okay. So now we have convex function what convex functions are?

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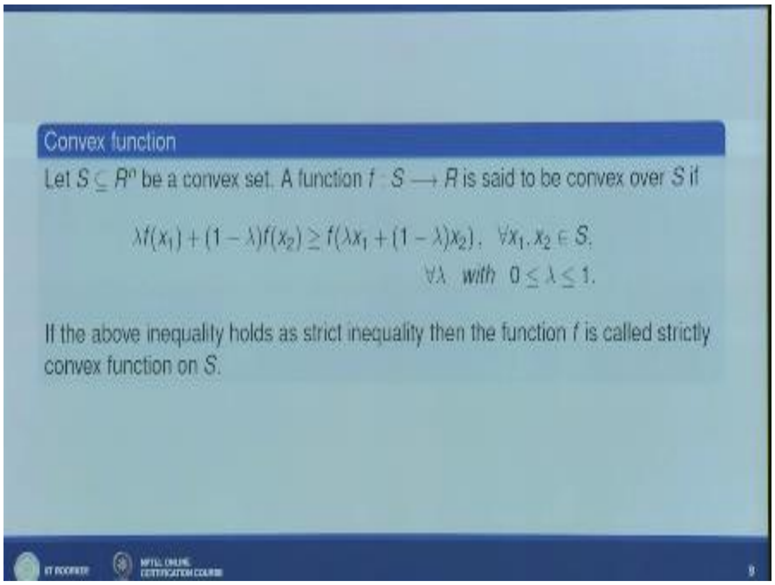

So let us be a subset of R^n be a convex set okay and a function F from S to R is set to be convex over S if this condition hold for all X_1 X_2 to be in S λ between 0 and 1 okay. So what does it means let us see, so this definition means.

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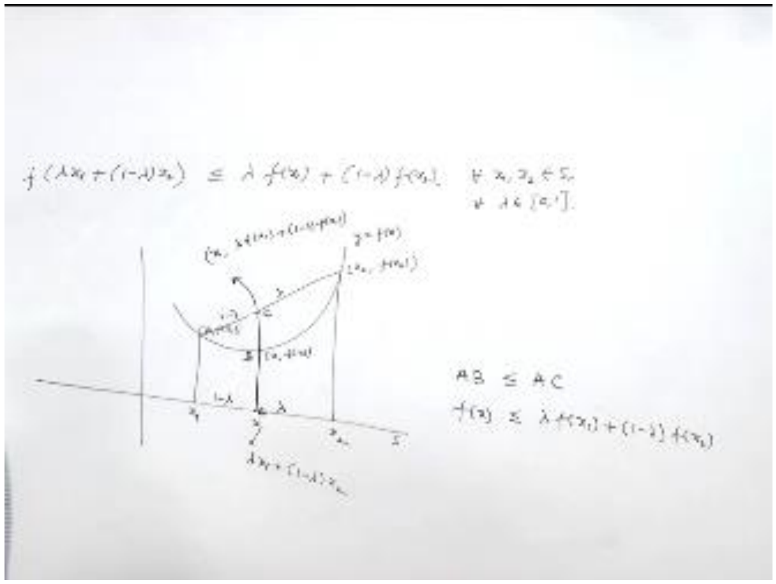

F (λx_1 + 1 – λ x2) ≤ λ F (x_1 + 1 – λ f x_2) and for all $x_1 x_2$ in S and λ between 0 and 1 okay. So here s is the convex set okay, so if of convex linear combination of any two point of S ≤ λ fx_1 + 1 – λ fx_2 then we say that the function F is a convex function. So what is the geometrical inter protection of this definition let us see okay.

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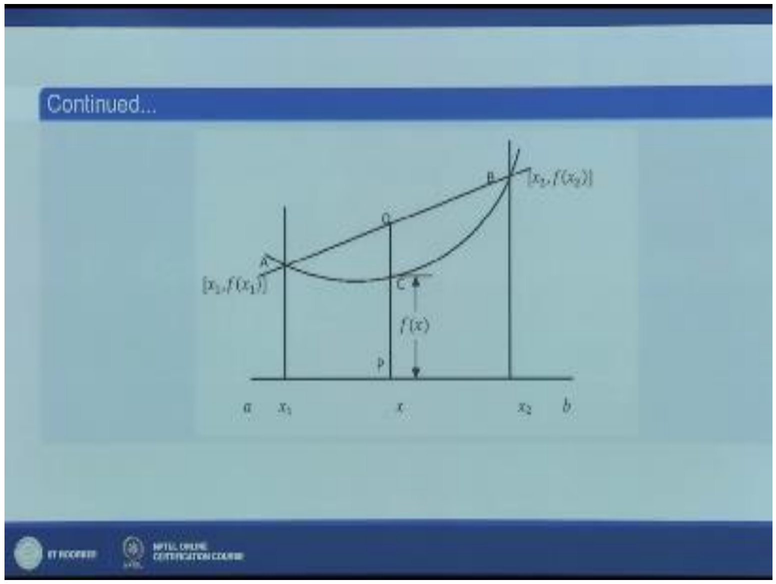

Let us see the geometrical inter protection of this definition, now suppose we have a function of this type okay suppose this is the point x_1 and this is some point x_2 okay any arbitrary point this is s okay suppose this is S. Now this point this is the function y = fx so this point will be nothing but x_1, fx_1 and this point is nothing but x_2, fx_2. Now join the code of joining these two points now take any x in between x_1 and x_2 that is this x is the convex linear combination of these two points so this will be nothing but λ x_1 + 1- λ x_2 for λ between 0 and 1.

So this ratio will be nothing but 1 – λ and λ, now you join this code here so this point will be nothing but x, fx okay, what is fx f (λx_1 + 1 – λ x_2) and what this point will be the ratio will be here 1 – λ n λ because these three lines are parallel, so whatever ratio we are having here the same ratio will be here. So what will be the quadrate of this point? This point will be nothing but x, and x, it is λ fx_1 + 1 – λ fx_2.

So if we are calling as point A it is B it is C so AC is AB is always $\leq$ AC clear because this height is $\leq$ this height and this is nothing but F_x so F(x) $\leq$ and Ac is nothing but this height which is λ Fx_1 + λ Fx_2 this height. So it is λ Fx_1 + 1- λ Fx_2, and what is x, x is λx_1 + 1 – λx_2, so this height is always $\leq$ this height so what is it mean? That if a function is convex then you take any two arbitrary points on the curve join the code the code joining those two point always lies above or on the curve okay.

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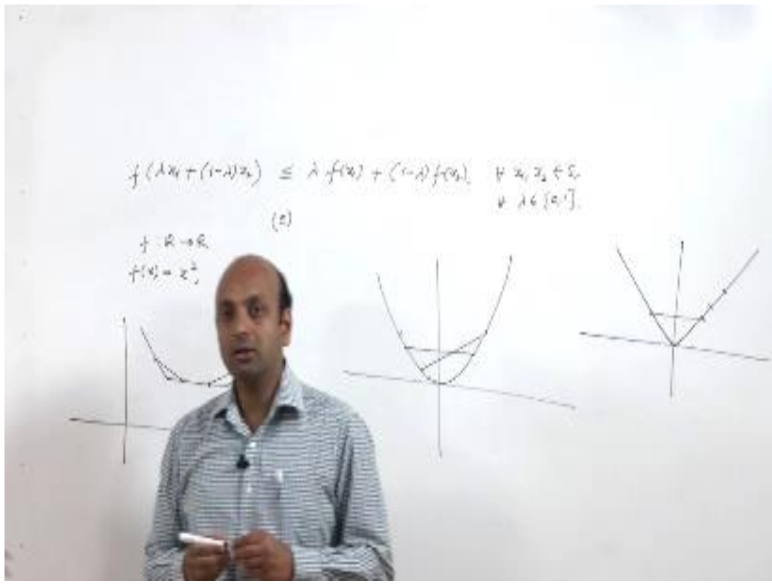

That means suppose we have this function suppose you have to Fx = X^2 okay f is from R to R okay. F is from R to R and F = X^2, now what is the graph of this function graph is something like this okay. Now you take any two arbitrary points on the curve any two arbitrary point join the code the code joining these two points always lies above the curve okay you take two arbitrary points here join the code the code joining those two point always lies above the curve you take two points here.

So if it is the function is convex so we can say that is the function X^2 is a convex function okay, now you take a function like this. suppose you take a function like this a boat shape function you take two arbitrary points here join the code the code joining those two point always lies above the curve you take two points here join the code the code joining these two points lies above the curve.

Now you take two points here the code joining these two points lies on the curve then the code joining these two points any two points on this curve either lies above the curve or on the curve, so this means there the function is a convex function. Now if the inequalities reversed if these inequalities is reversed that is we have written equal to then the function is called concave function that means a function is concave a function acts is concave.

If an only if negative of f is convex okay, so how we can interoperated geometrically for a concave function we can say that take any two arbitrary point on the curve join the code the code joining those two points always lies below or on the curve that means this is the geometrical interpretation of the concave function on the similar lines, there are some examples.

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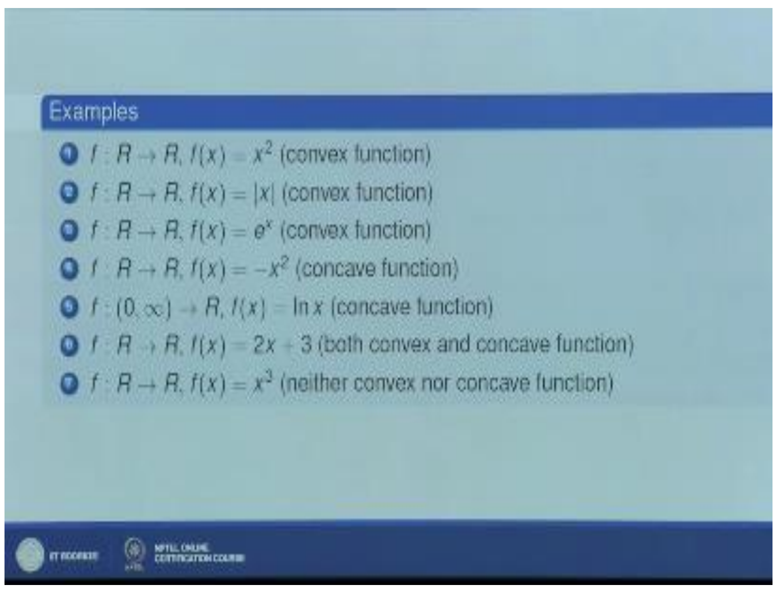

X^2 is a convex function which we are already seen this is x^2 is a convex function because if you take any two arbitrary point join the code the code joining these two point always lies above or on the curve, mode x, mode x is also convex function because when you brought mode x, so mode x is something like this you take any two arbitrary point on the curve the code joining those two points always lies above the curve okay.

You take two points here it is one the curve though the code joining any two points on the lies either lies above the curve or on the curve so that means it is a convex function. E^x is again a convex function that you can easily see graphically.

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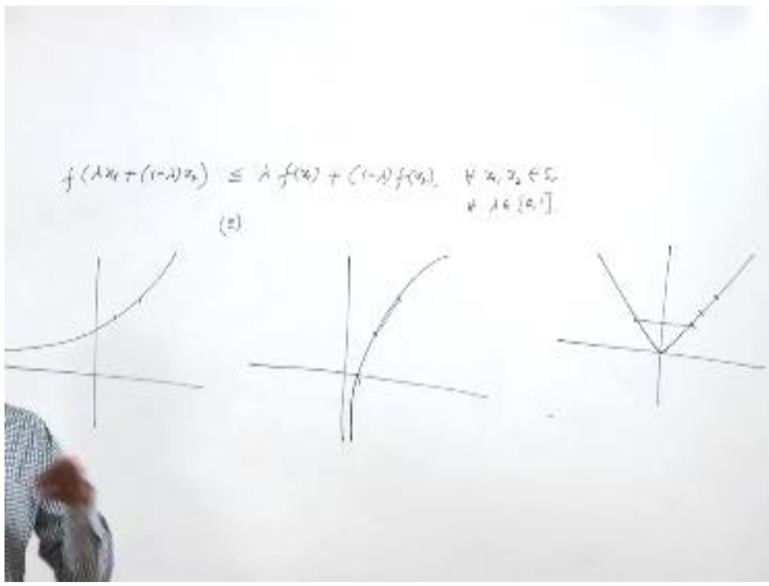

If you draw the graph of E^x so this is E^x, so when you take any two arbitrary point on the curve the code joining those two points always lies above the curve okay. If it is a line segment joining those two points lies above the curve so that means it is a convex function, negative X^2 is of course concave function because it is negative which is X^2 is convex, lnx is also a concave function we can easily see graphically, if you draw the graph of lnx.

Lnx is something like this when you draw the graph of lnx; lnx is something like this okay. This is one something like this so when you take ant two point the code join those two points' lies below the curve so it is a concave function. Now linear function 2x + 3 is both convex and concave we can easily see because equality holds and the function X^3 is neither convex nor concave okay that we can visualized geometrically also. Suppose I have sign x.

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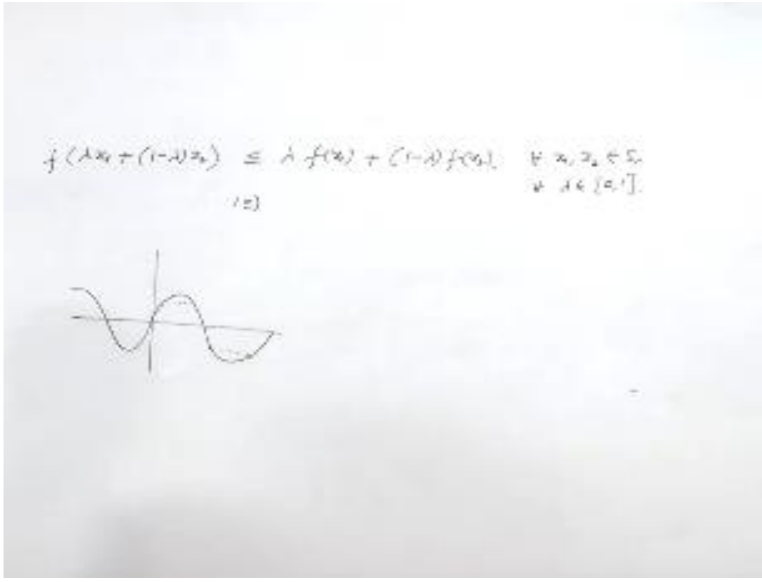

Sign x is something like this okay, now when you take two points here so the code joining these two points lies below the curve and here it is above the curve so it the neither convex nor concave because in some portion the code is below the curve and some portion the code is above the curve so the such functions are neither convex nor concave okay. What convex set and convex functions are that we have seen in this lecture that we will discuss in the next lectures. Thank you.

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