Integral Equations, Calculus of Variations and their Applications
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Lecture 09
Solution of integral equations by successive approximations

Hello friends, in today's lecture we will discuss the method of successive approximation for solving Fredholm integral equation, so if you remember in previous class we have discuss method of successive substitution to solve a Fredholm integral equation and Volterra integral equation of second type, so today we will discuss the method of successive approximation, so in the beginning if you remember in ordinary differential equation.

When we begin we always start with the Picard's theorem and we take Picard's method of successive approximation to solve the ordinary differential equation.

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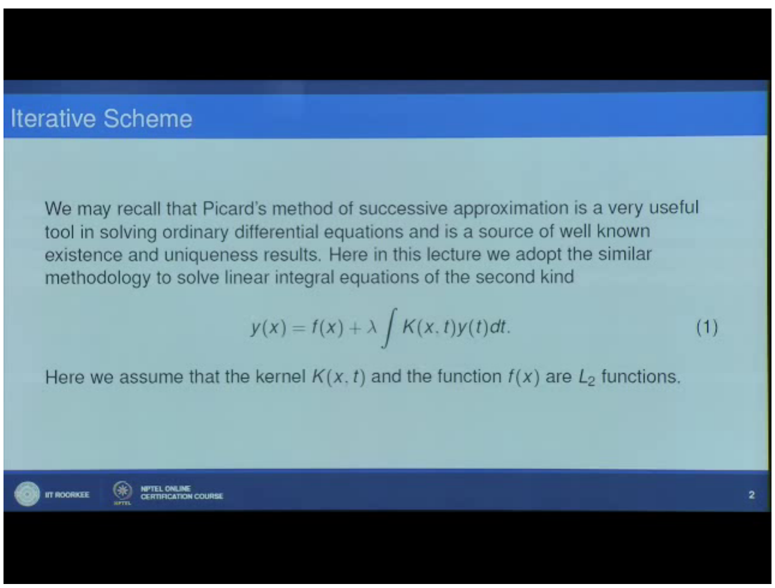

Right from there we can try to have existence and uniqueness results depending on the condition on the non linear function presented in that differential equation, so we are taking that idea try to solve our Fredholm integral equation and Volterra integral equation of second kind in a similar manner.

So for that we consider this y x equal to f x plus lambda k x t y(t) dt, here I am not putting any kind of limit because the methodology for Fredholm integral equation and Volterra integral equation is almost the same, they are only few point where it is different, I will tell you where it is different, rest it is quite similar, so that's why I am not putting any limit here and I am just considering y x equal to f of x plus lambda k x t y(t) dt.

So it is integral equation of the second kind and here for this particular analysis we are using very weak condition on kernel k n f, here we just assuming that k x t and function f x are L 2 function, L 2 function means it's quite integral function, so that we can define in a following way, we can say that if function f is called a L 2 function in the interval a b, if in this interval a to b square of this f (t) dt is going to be less than infinity.

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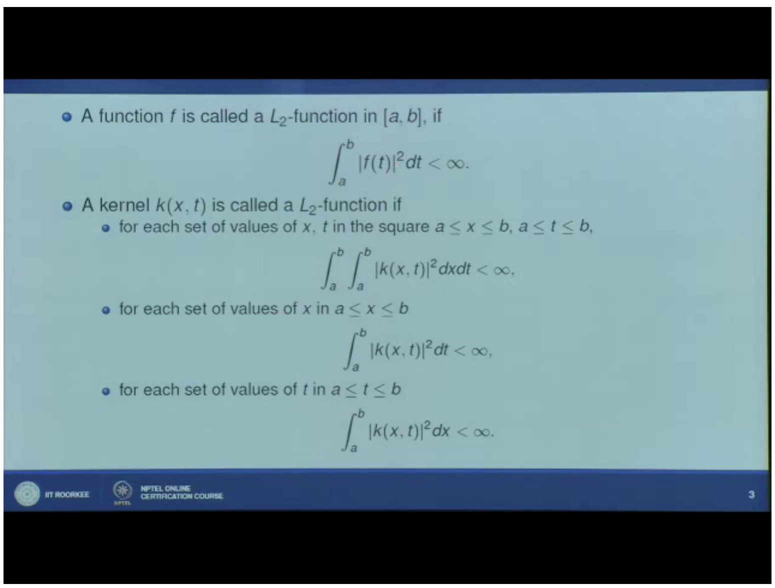

So here that's why we called it square integral function means the square of the function is basically an integral function, in a same way we can define that a kernel k x t is called a L 2 function if for each value of x and t in this square a to b cross a to b, this integral a to b, double integral a to b a to b, k x t square d x d t is less than infinity and a to b k x t square dt is less than infinity, if we take the integration with respect to t.

And if we take the integration with respect to x then also it is less than infinity, so if a function f satisfy this condition we call it L 2 function, if kernel satisfy these three condition we say that kernel is a L 2 function with respect to variable x and t, now here what we try to do if you remember there we,

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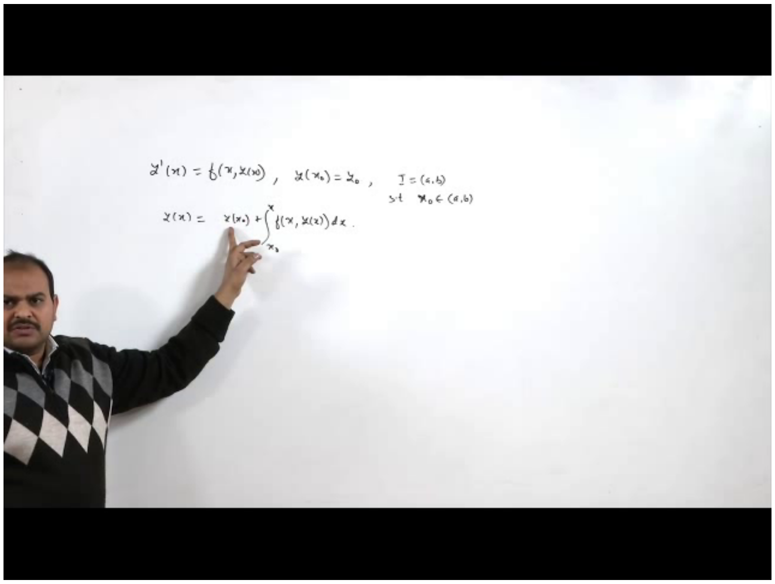

Let me write it here we have y dash x equal to f of x, y of x here, I am talking about Picard iteration method for solving a ordinary differential equation. And here we take y of x not is equal to y of not and here we are considering any kind of interval a to b such that your x not is lying inside your interval a b right,

So here we start with the, we write the corresponding integral equation and we say that it is your y of x not plus x not to x and f of x y of x, d of x, so what we start, we try to solve by saying that since this y is unknown and we cannot integrate like this.

So what we try to do is we replace this y of x by approximation of this, so what we start with we always start with the initial condition provided here, so we say that y of x is replaced by y of x not and we say that this is the first approximation of the solution y of x and so on we can get n iteration and we try to show that this n iteration y n, we call it y n it is converging to element and we later on we say that this limit is basically converge into the solution of this integral equation.

And hence it is the solution of the given ordinary differential equation, the same idea we are using here so for that I am using,

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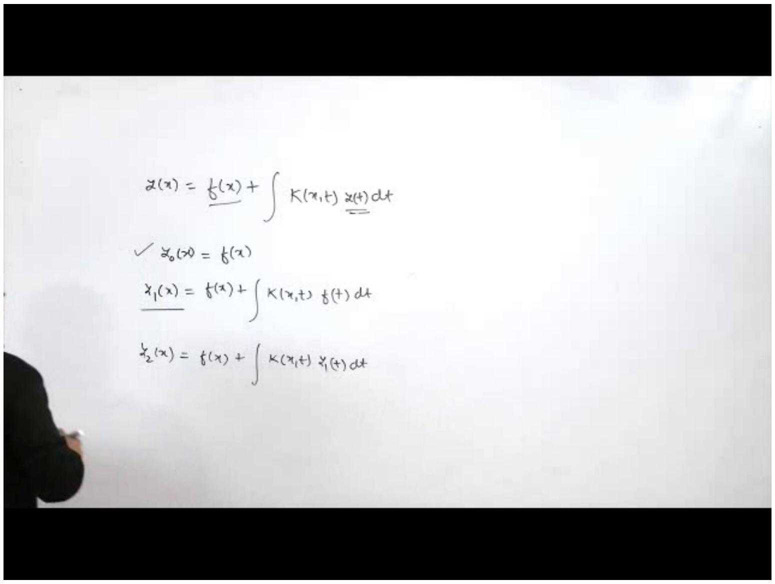

So here we have this y of x equal to f of x plus k x t and y(t) dt, so here we all know that this y t is unknown function so we cannot perform this integration in a precise manner so what we try to do here we start that let us say consider the first approximation of the solution y of x and first approximation we are taking this as function f of x.

So we say that your first approximation is the non homogenous term presented here, so is y not t is equal to, it is not t it is x here, so first approximation we are saying that it is y not x equal to f of x and then we try to show that how to get by one by two and so on, so to get your first approximation what we do we use this zeroth approximation and we say y one x is basically f of x plus k of x of t and y t I am writing f of (t) dt.

And in this way we say that it is a zeroth approximation and the first approximation we are calling by this, so here we are replacing y of t by its zeroth approximation and we say that it is first approximation and similarly once we have first approximation with us then we find out the second approximation y to x as f of x plus k of xt, now here y t I am using your y one x, so it is y one (t) dt and so on.

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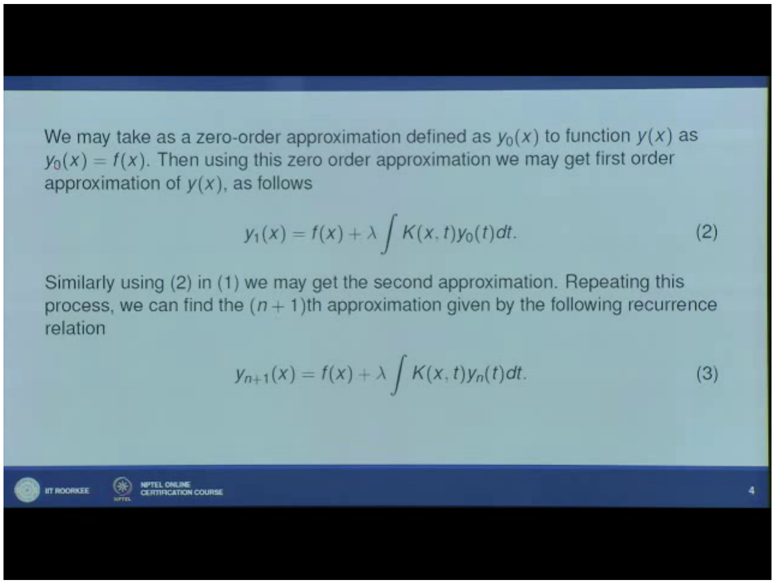

So that we are going to put it here and we are saying that our zeroth, zeroth order approximation is given by y not x equal to f of x and using this we can define our first order approximation y one x as this.

So here in place of y I am using zeroth order approximation of y of x and similarly we can keep on doing this and we can say that n plus oneth approximation of y of x can be given by in the sense that y n plus one x equal to f of x plus lambda k x t y n t.

So using nth approximation we can get n plus oneth approximation, now idea is that if these approximation will tend to some kind of limit then we try to show that, that limit is précised with the solution of the integral equation, so if y n x converges uniformly to a limit as n 10 into infnity then we can check that this limit satisfy the given integral equation and is the required solution, this we can see it from equation number three that if this convergence is uniform.

Then you can take limit n 10 into infinity then this will tend to y of x equal to f of x plus lambda, limit can go inside because convergence is uniform and you can write it that this is also k x t.

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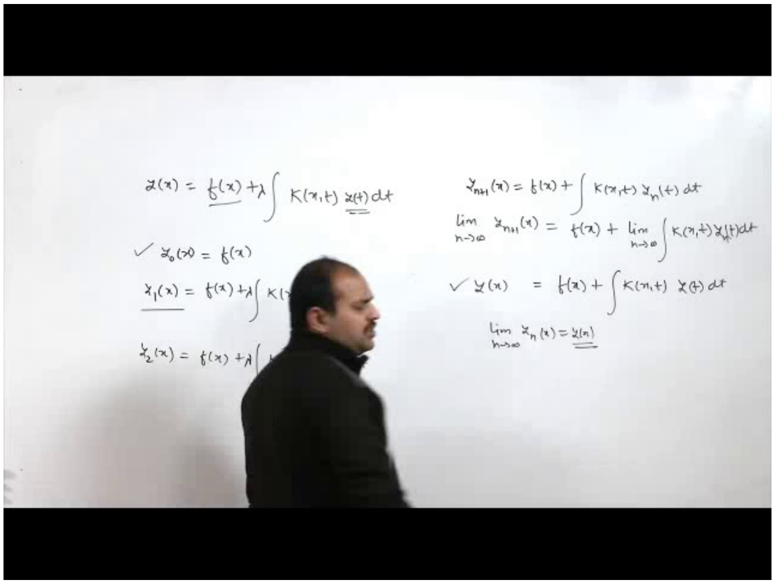

Let me write it here, so here it is what y n plus one x is equal to f of x plus k of x t and y n(t) dt here. Then if you take the limit, limit n 10 into infinity here, then y of n plus one x equal to limit n 10 into infinity.

Since it is independent of n so I am keeping it as f of x plus limit and tending to infinity, k of x t and y n(t)dt, now since convergence is uniform then this can be interchangeable and you can write it that f of x plus this limit will go inside then it is k of xt and let us call this limit as y of t here, so and this is your y of x then we can say that if your limit n 10 into infinity you defied y n x as y of x then this limit y of x satisfy this equation. And which is the required solution of the integral equation.

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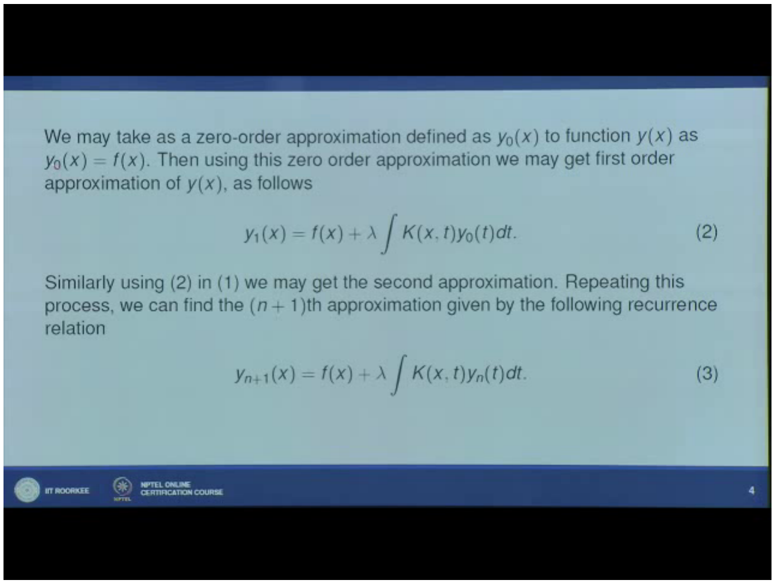

so what we want is we want to find out say this uniform limit this limit and try to check the conditions such that this convergence is uniform, so for this we want to look back our arbitrary procedure here.

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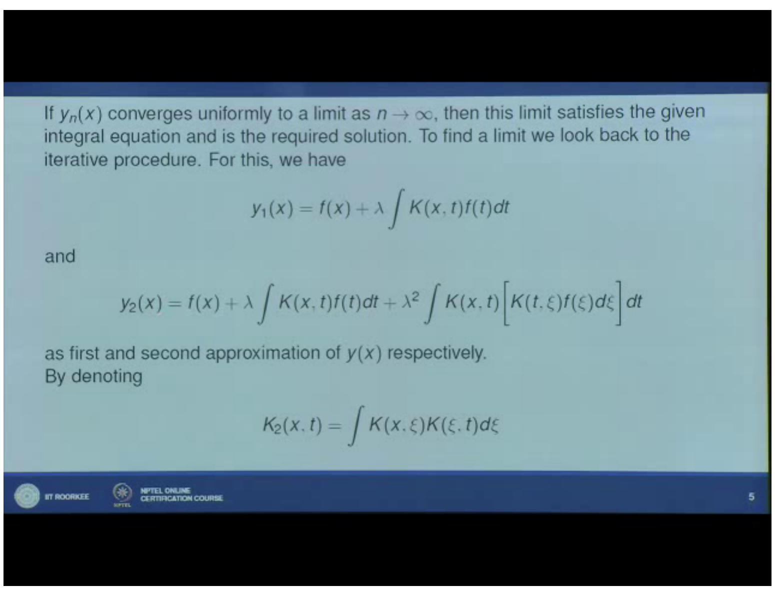

So here y one x is define as this and y two x is define like this. So how this y to x is define like this that we are going to check, so here we have already defined what is y one x here and y two x here.

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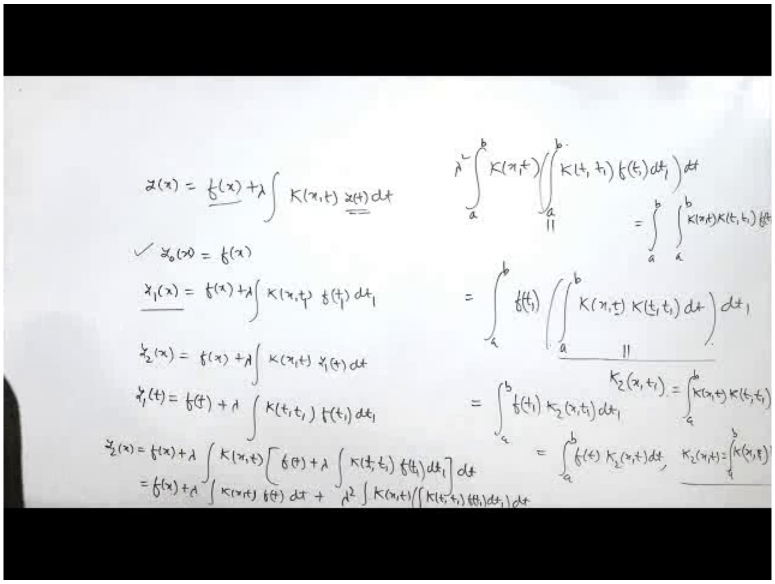

And there is a small correction here, this is a lambda here right, please take it lambda here ok, now it is ok, now here we want to put the value of y one t, so to find out y one t here we are going to use the previous integral equation, so if we want to write y one t here then this integral variable I have to change so let me write it here that y one t equal to f of t here plus lambda times, I am writing this variable t as.

Say call it t one right and then you change so it is k of t one f of t one d of t one, so here what we say that integration with respect to t one variable, now I am putting here, so y two x is basically what f of x as it is plus lambda times here I am writing k of x t and inside you have this thing that is f of t plus lambda times k of t t one t t one f of t one and d of t one and the whole integration with respect to d of t, so here if you expand it you will get f of x plus lambda times k of x t.

F of(t) dt plus lambda square now if you multiply here you will have lambda square, this is k x t here and then there is one more integration here k of t t one f of t one d t one and d t here, so what we try to do here we try to simplify the last expression, now how we can simplify this last expression, so here if you look at last expression is this lambda square k of x t integration k t t one f of t one d t one and d t what we try to write it here.

We try to write it this as lambda square k two x of t f of (t) dt kind of thing, so this we want to write it here, so to find out this k two fo x t if you compare here, your k two of x t will be what, k two of x t we need to find out right, so for that if you look at here, we try to do the change of order here so that this f t one is come out of the inner integral right, for that I need to do change of integration.

So if I look at your one particular case that is Fredholm integral equation case then I am just putting the limit here a to b and a to b and then I can change the order of the integration, if you change the order of integration you will be what, you will get a to b here and a to b here and I am taking inside here so it is k of x t k of t t one and f of t one d, now this order is reverse so it is d t and d t one, so when you take inner integral with respect to t then you can take your f t one out.

So I can write it this as a to b, now f t one is out so let me write it f of t one and then inner one is a to b, I am writing k of xt and k of t t one d of t and this is d of t one, now here I will call this expression as k two x and t one here, if you look at here integral is, we are doing integral with respect to t, so k two x t one we are denoting this whole expression, so I am just writing what is it formula for k two x t, it is a to b, k of x t, k of t t one d t.

You can identify like this, it is x t one, so it is started from k x t and t is going to be variable for this integration k t t one right, and we call this as k to x t one, so this term is going to be a to b, f of t one k to x t one and d t one, since t one is just a dummy variable I can write this as a to b, f of t k two x of t d of t, is it ok, and you can easily find out what is the value of k two x t, so k two x t is basically what using the same formula.

It is what a to b starting from the variable k x and you can write it zi here, zi is the variable for this integration and this will start from zi and first variable is zi and other variable as t here and d zi, so your k two xt is defined as this and with the help of this I can write the second term as a to b f t k two x(t) dt, so it means that I can write our y to x in this particular form y to x is equal to f of x plus lambda k x t f(t) dt plus lambda square k x t k t zi f zi d zi.

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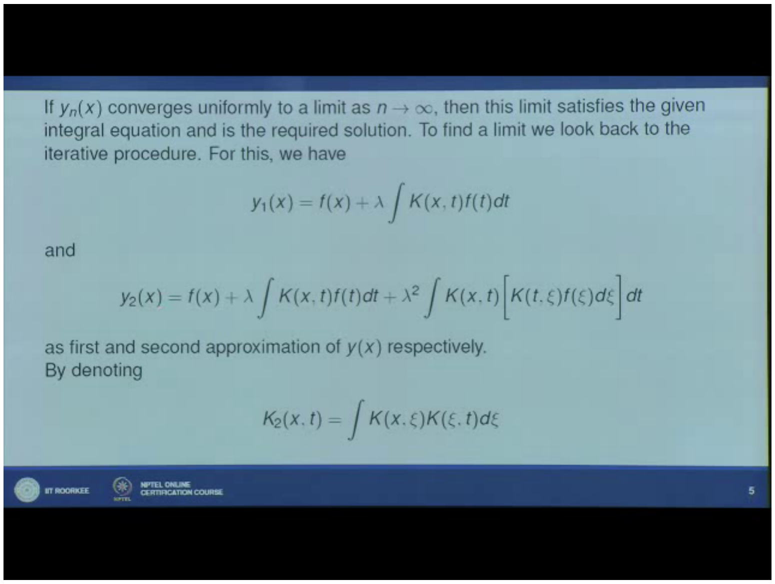

So this whole thing I am writing this as k two x two, so that we have explain now that how we can write k two x t as this, so k x zi k zi t d zi ok, now with this notation I can write it y two x equal to f of x plus lambda k x t, f t d t plus lambda square k two x t x, t f(t) dt.

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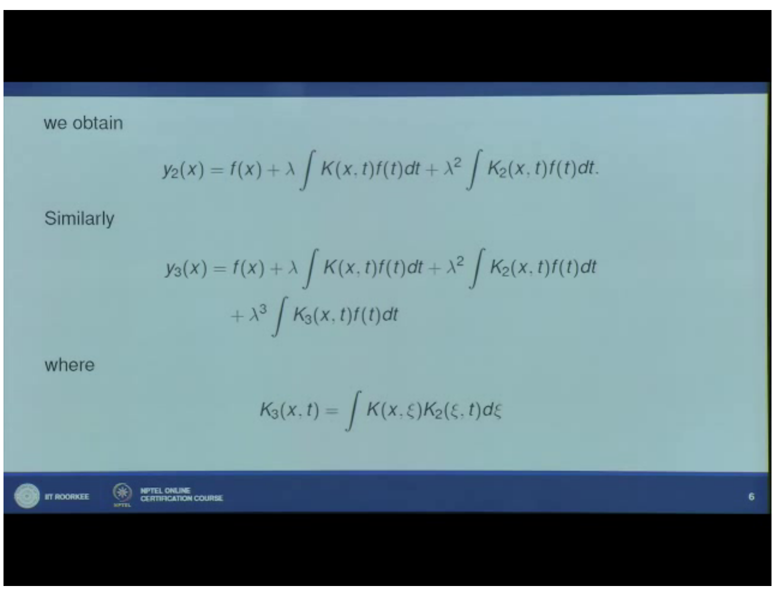

So we keep on doing this procedure and in this way we can y three x and similarly you can define k three x t as this k x zi k two zi t d zi and we can continue this process.

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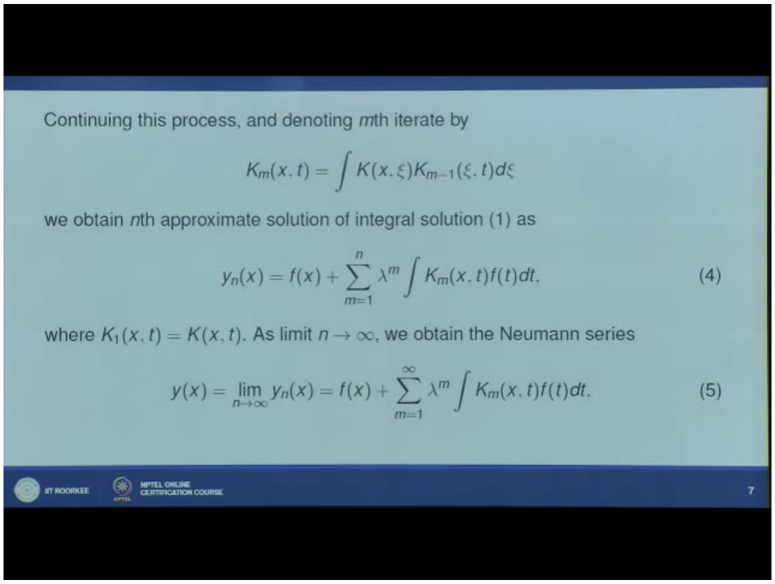

And if we denote similarly as we denoted k three and k two we can denote k m x t as k x zi and k minus one zi t d zi here this zi and this zi is your variable for integration

And with this notation I can write nth approximation like this y n x equal to f of x plus m equal to one to n lambda to power m, k m x t f(t) dt, so here what we have achieved here is the nth approximation to solution y of x, so it means what we want is that if we keep on doing this.

Then this y n x is tending to some kind of limit, so here if I define as limit ending into infinity we get the following series which we call as Neumann series, now this we can write it only when the convergence is guaranteed, so now we try to show that of course this convergence is going to be guaranteed and we want to find out c why it is guaranteed and if it is convergence is there we can write this limit as y of x, which we claim that it is the solution of the integral equation.

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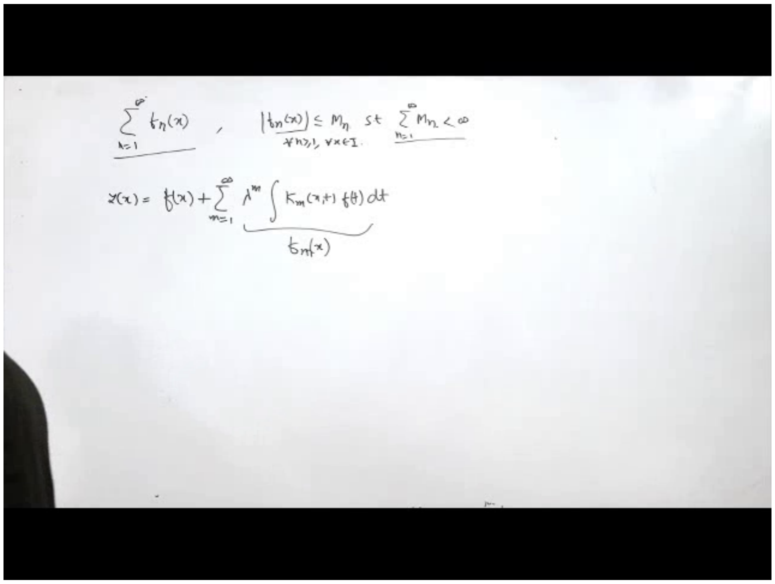

So we now want to find out the convergence criteria of a this Neumann series, for this we want to use the test rishoffm test, for that we remember just recall that suppose we have series of functions f n x here and such that n is from say one to infinity and here with the property that your mod of f of n x is less than r equal to say m n, such that this m n is basically what, summation m n this is for every and greater than equal to one.

And for every x in the interval where we are looking at the convergence's part, you can write it I here, such that this summation m n n is from one to infinity this is less than infinity, so if we can find out m n say bound for every term of this series such that this series infinite series conversion then we say that this series is absolutely and uniformly conversion, so we are using this fact and for that we need to find out this m n and we want to show that this series.

This infinite series will converge for that, so here our series is what, here our series is y of x equal to f of x plus summation m equal to one to infinity lambda to power m and integral k m x t and f of (t) dt, so here this is your f m, f m x so we want to show that this f m x is bounded by something,

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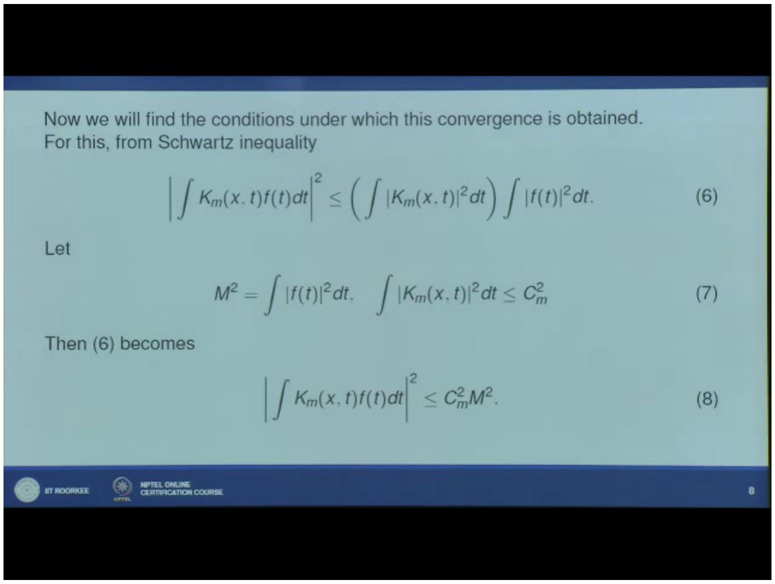

So here look at here f m x is basically what, this term is k m x t f (t) dt, so here we want to find out the bound of this, so for that we are using the Cauchy Schwarz inequality.

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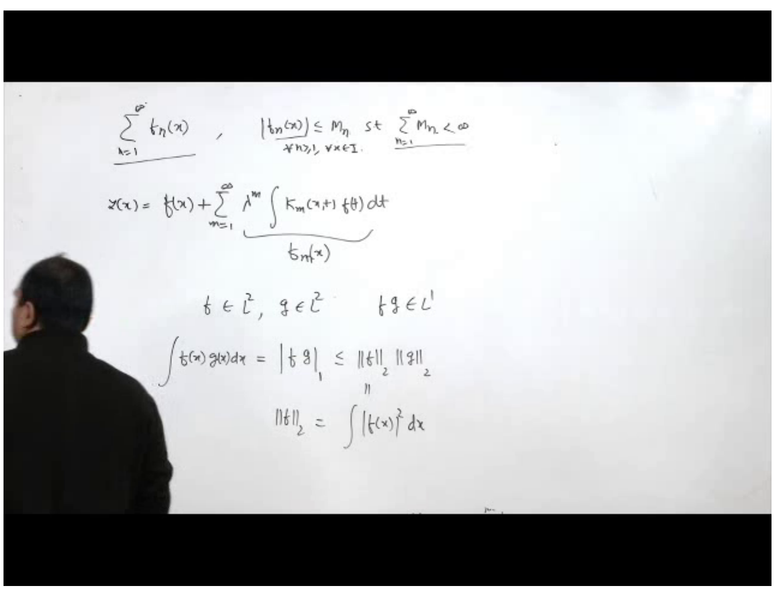

So what this Cauchy Schwarz inequality says that if we have that f from L two function and g is L two function then this f of g is in L one function and modulus of f of g is less than or equal to nom of f and nom of g two, so we are using this fact and here this nom is denoted by this f of x is

L two nom basically d of x, so here this is denoted, so this one nom is basically, one nom is given by this f of, f f x d of x and you can get this, so this is your Cauchy Schwarz inequality and we are using this,

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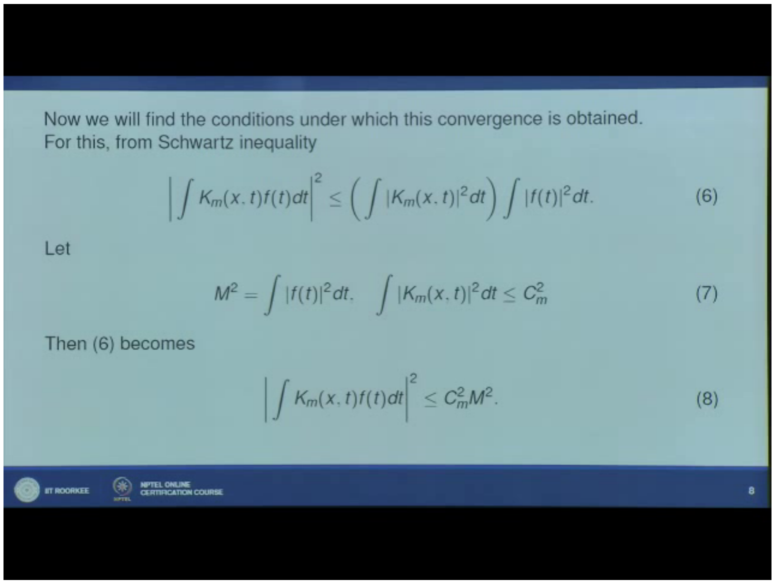

So here if you look at your k m x t is L two function, f t is L two function, so we can apply our Cauchy Schwarz inequality and it says that modulus of this quantity is less than or equal to L two function, L two nom of k m x t and L two nom of f t, so L two nom of k m x t is defined like this, so here integration k m x t square dt and this thing, so here since f t is given function.

So we can denote that this quantity is a known function we call it capital M square and we say that this is k m x t square dt, again this is known but this is depending on m, so we call it that this is the bound for this is denoted by c m, so we can say that integration of square dt is less than three m square, we want to find out this c m square, and we want to relate this c m square with c one square which is the bound of k x t, because k x t is known to us.

So then using this notation I can simplify this as less than or equal to this term is bounded by c m square and this term is bounded by capital M square, so this is bounded by this but I want that this should be written in terms of k x t, bound of k x t, so here we again use Cauchy Schwarz inequality for k m x t, so k m x t is L two function.

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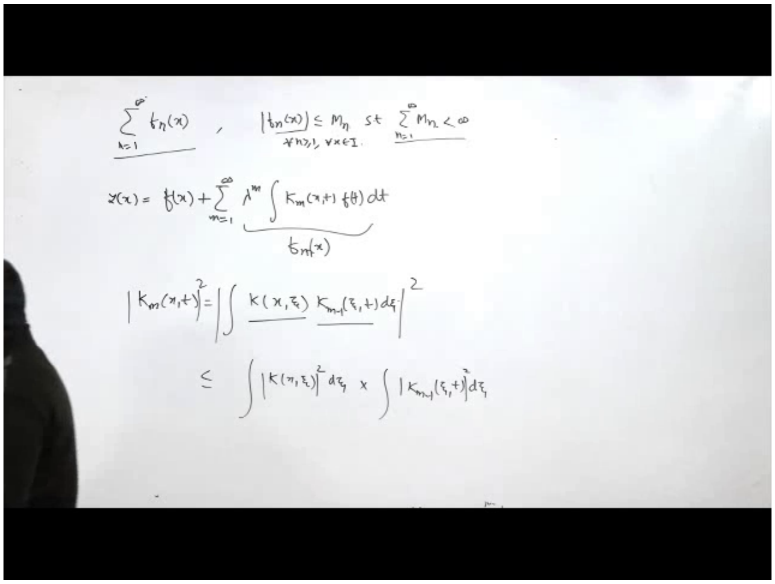

So here let me write it here, so here you use the formula of k m x t, so k m x t is basically what. It is basically k x zi and k m minus one zi(t) dt here, now I want to find out the square of this which is given by this, now here this is L two function, this is L two function,

So we again apply our Cauchy Schwarz inequality and we can write it like k x zi, this is zi here because this is integration with respect to d zi, so d zi square of this and we are talking about the square so into k m minus one zi t, d zi square of this, so we are using this.

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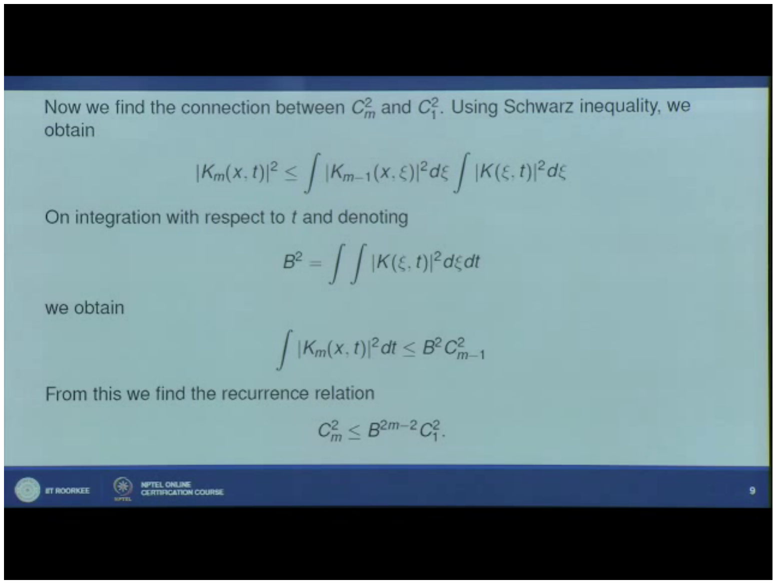

So here we have this, we are using the Cauchy Schwarz inequality for k m x t, so I am writing the expression for k m x t in terms of k m minus one x zi and k zi t and then we are using the Cauchy Schwarz inequality

And if we further integrate this, because what I want, I want the bound on this thing integration of k m x t square dt, so what we do we integrate it with respect to t, so when you integrate with respect to t you will get integration of this with respect to t.

And integration of this with respect to t, so we call this as this k zi t square d zi, this is one more integration we denote this as b square of this, so when you integrate with respect to this one more time then and denoting that, we will have a double integral here with respect to t also, so have b square double integral k zi t square dt, d zi dt and call this as d square, and then we can write this as integration of k m x t square dt is less than.

We calling this as b square and this as c square m minus one, so it means that here this is denoted at c m square, so c m square is less than or equal to b square, c square m minus one and then we repeat this process again and again, and we can write it that c m square is less than b to power two m minus two c one square, so I hope this is well understood because it is c m square, is less than or equal to b square c square m minus one and then again you repeat the same thing.

Then you will get say b square, I am writing this is bounded by b square, c square m minus two and keep allowing, so we will get, every time we are getting b square term and it is reduced by one, so it is will get this kind of relation here, c m square less than or equal to b to power two m minus two c one square, by the way here c m is what, c m we are assuming that we can take this c m square as equality here ok, so this k m x t square dt is equal to c m square.

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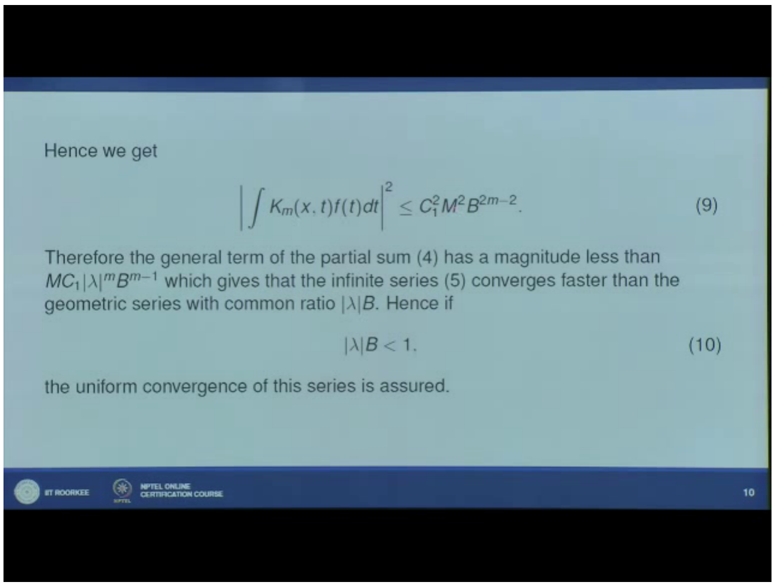

So that this is quite clear now ok, so once we have this the c m square we can put it back and we can say that this integral k m x t f(t) dt whole square is less than c one square m square b to power two m minus two so using this expression here we can say that this term here,

This thing the bound of this will be what, lambda to power m bound of this, so it is lambda to power m and square root of, because it is a square here, so it is what that we write it here,

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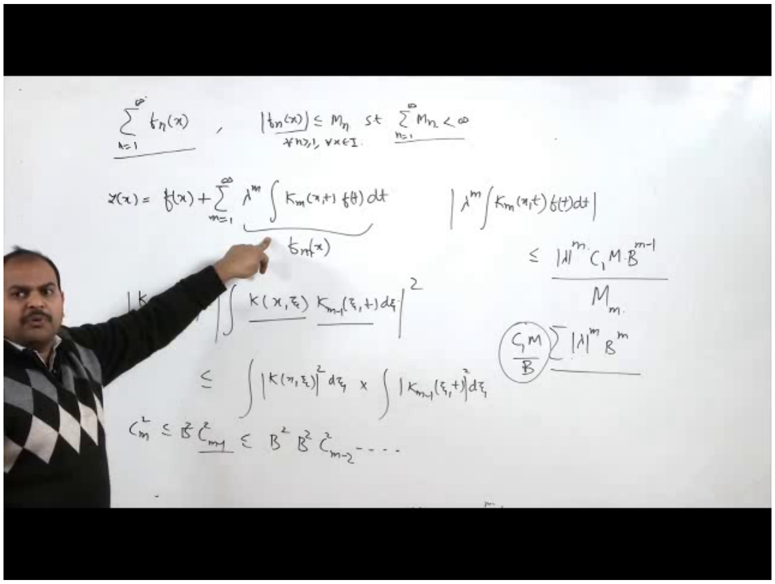

So here this is what, so lambda to power m integration of k m x t f of (t) dt is going to be less than what modulus of this is going to be less than modulus of lambda to power m, square of this is bounded by c one square m square b to power two m minus two, so this will be bounded by c one m and b to power m minus one right, so we will get this right, now so it means that this is your capital M depending small m, now here we say that the summation mm will be what.

This c one m is basically independent of small m, so I can take it out c one m and you can write it here modulus lambda to power m and b to power m minus one, I can take lambda b also here and write it here, I can write it here divided by b and you can write it here this c b m, now these are all caution so it will not create any fact on say convergence here, then if you look at this can be considered as a geometric series with common ratio lambda b.

So if we say that if lambda b modulus of lambda b is less than one then this mm this series will be convergent, so here this series will be convergent provided that modulus of lambda b is less than one, so if it is conversion than this series which is known as this, this series will be convergent absolutely and uniformly provided that modulus of lambda b is less than one, so here I can say that this series will be convergent if modulus of lambda b is less than one.

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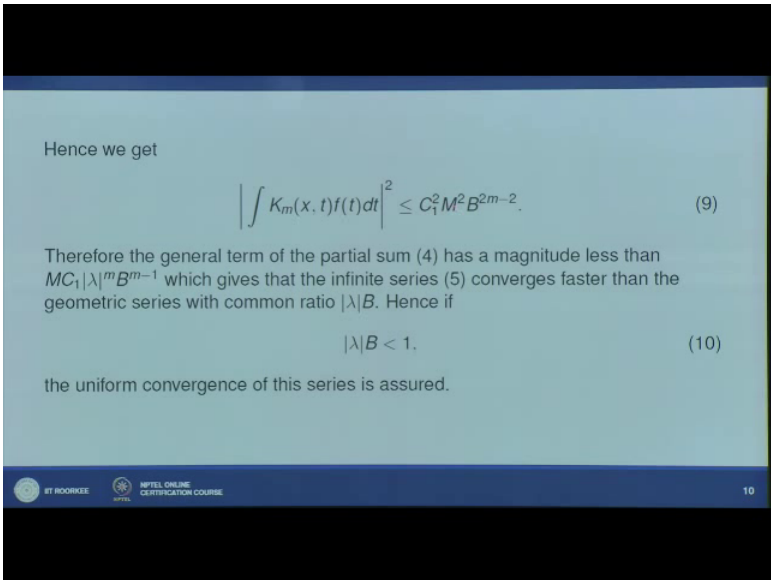

And by the help of (29:13 research and test) we can say that the convergence is uniform ok, now we can prove the uniqueness part here, so for uniqueness part this quite easy, let us suppose that we have limit exist and we want to show that that limit is unique.

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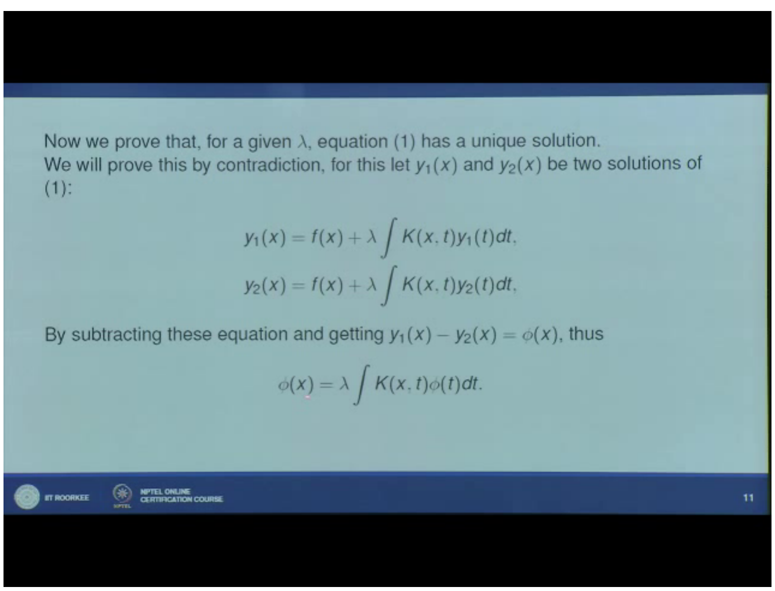

So for that we say that they are suppose two limit possible, so we call it y one and y two, and y one y two will satisfy the Fredholm integral equation here and we try to show that this y one and y two are annually equal.

For that you separate these two and denote this y one x minus y two x as equal to five of x then it will five x will satisfy this equation, this is homogenous integral equation, so here again we use the Schwarz inequality, how we can use because y one is L two function and y two is also an L two function, so difference is also be on L two function and if you look at here, modulus of five of x square will be what, this is modulus of k x t y (t) dt integration whole square.

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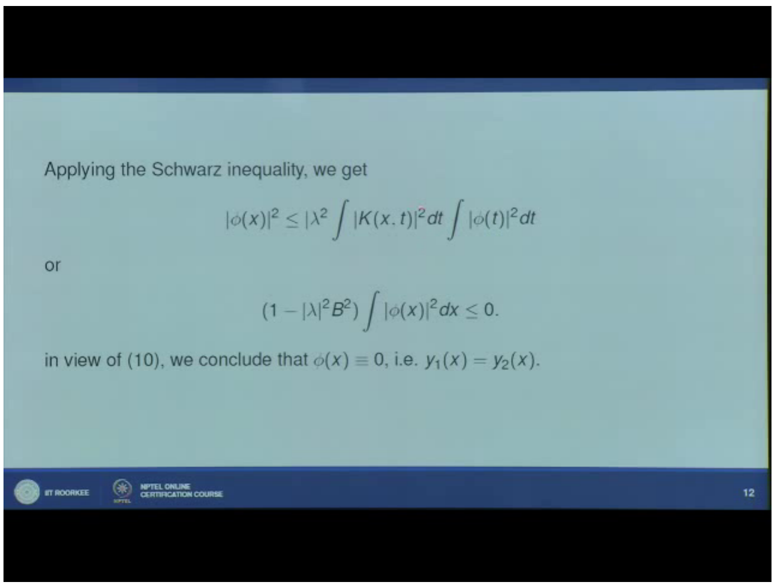

So this is L two function we can apply our Cauchy Schwarz inequality and we can write it like this and here this if further integrate with respect to x here and when you integrate with respect to x this is what

A integration five x square d of x less than or equal to lambda square now this quantity is what, one more integration with respect to x and this quantity we have denoted as d, so we can write it like this.

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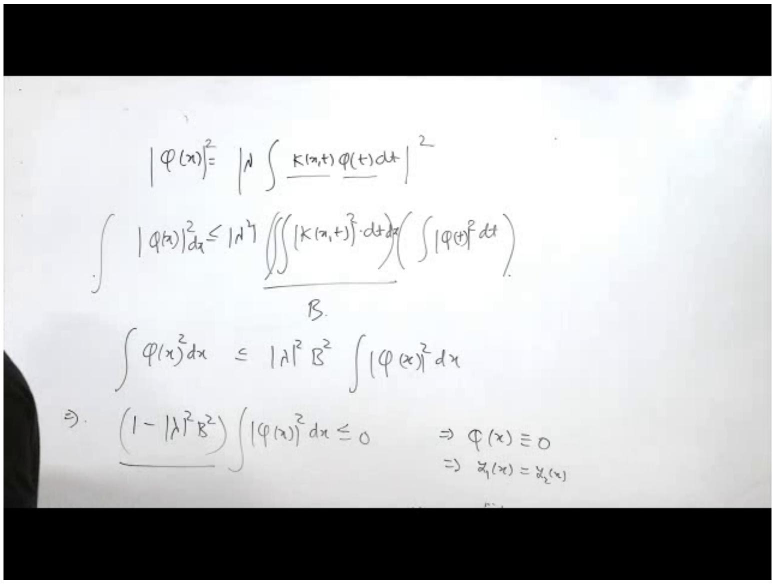

So here we have let me write it quickly y of x is equal to lambda you have k of x t and y of (t) dt right, so let us take the square of this so it is square of this and this is L two this is L two, so we can write it lambda square taken out and this is k of x t and square dt, I am using Cauchy Schwarz inequality and it is y of t square d of t here right, and then you integrate one more time with respect to say x, so integrate with respect to x.

So this is integration with respect to x here and this is which is independent of x, so it will be taken out so this is what this we denote as b here, so we say that integration of y of x whole square d of x less than or equal to modulus of lambda square and b square and this into this quantity here we have integration of y of x square d of x and then you can take it out and what you will get this is one minus modulus of lambda square b square.

And integration of y of x whole square d of x is equal to less than or equal to zero, now if you look at this modulus of lambda b is taken as less than what, so this quantity is positive, this quantity is positive less than or equal to one, this is possible only when this integral is zero and this integral is possible equal to zero because limit we have taken as positive whether it is a to x or a to b, limit is going to be positive.

So your integrant has to be ideally equal to zero function, so ideally equal to zero function means y x is ideally equal to zero which says that y one x is equal to y two x, so it means that if modulus of lambda b is less than one then this series converges absolutely and uniformly, and limit is going to be the solution of integral equation and this solution is going to be unique solution, thanks for listening me and we will meet again in a next lecture, thank you.