Integral Equations, Calculus of Variations and their Applications
By Dr. D.N. Pandey
Department of Mathematics
Indian Institute of Technology Roorkee
Lecture 08
Solutions of Integral Equations by Successive Substitutions

Hello Friends! Welcome to this (Fredholm) Integral Equaltions, Calculus of Variations and its Applications lecture. And in today's lecture we are going to discuss the solution of Fredholm integral equation with the help of iterative methods or we can say that method of successive approximation.

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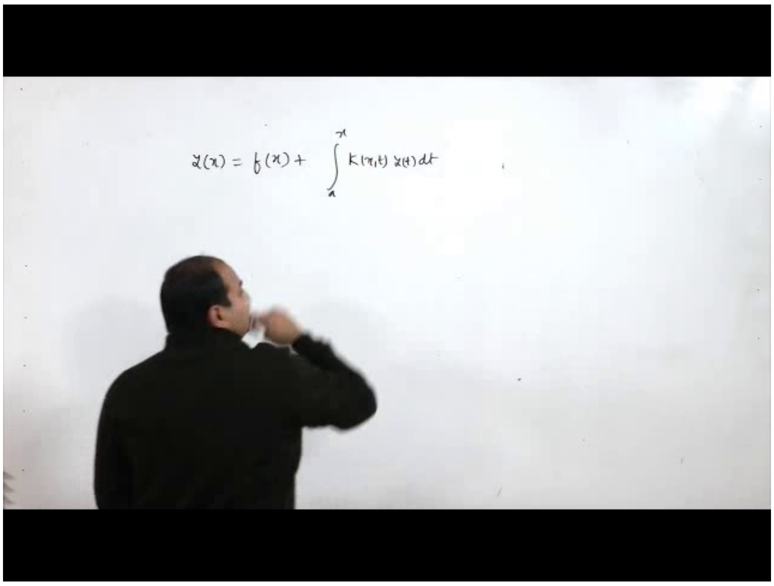

So if you look at here Fredholm integral equation of second kind I can write it here y(x) equal to f(x) plus initial we have k(x, t) and y(t) dt here. I am not writing any limit here because if I write a to b then it is your Fredholm integral equation. But if I write the upper limit as function this x then it is known as Volterra integral equation of second kind.

But whatever we are going to discuss today is equally applicable for both kind of equation means for Fredholm integral equation as well as (Volterra differential equa) Volterra integral equation. So here I am I am going to discuss this with the Fredholm integral equation and at the end we are going to discuss for Volterra integral equation also.

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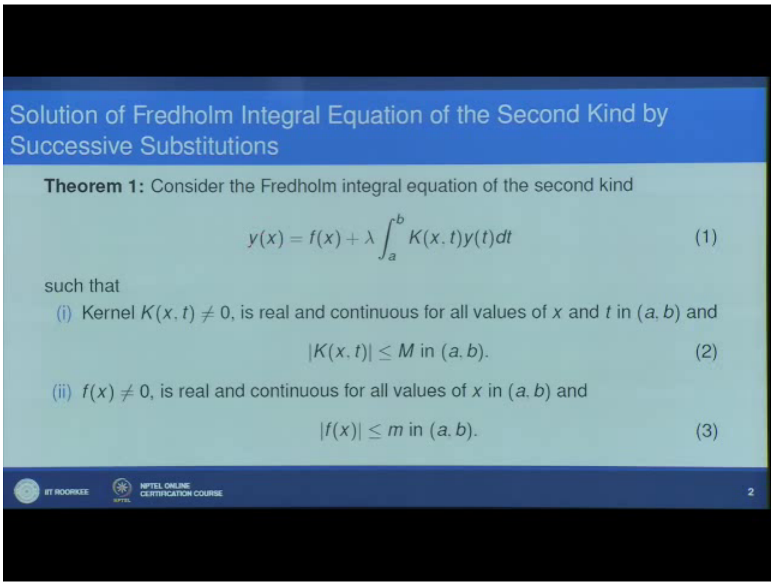

So let us look at the theorem here. So first theorem to solve this is this. So here we are assuming that we have a Fredholm integral equation of second kind is given as y(x) equal to f(x) plus lambda times a to b k(x, t) y(t) dt. I am putting lambda just to consider the Eigen values and Eigen function (())(02:08) So we have assumed we are assuming that this kernel (x, t) is a non zero and is real and continuous for all values of x and t in this interval a ,b.

And also we are assuming that this k(x, t) is bounded by this capital M for all values of x and t in this interval a, b. Also that this small function f(x) is again non zero real and continuous for all values of x in a, b. And here f(x) is also bounded by say this is small m. And for every x in this interval (a, b).

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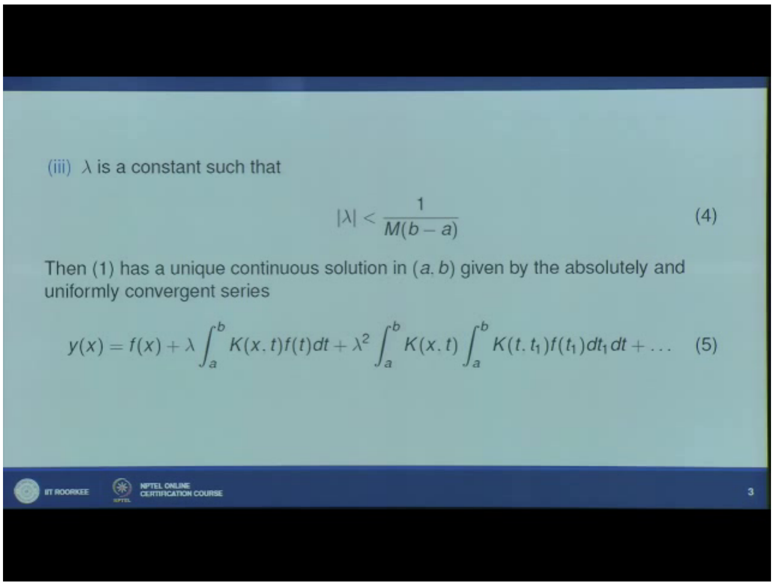

And here we are assuming that this lambda which is a constant here is satisfying this condition that mod of lambda is bounded by this quantity 1 upon M (b minus a) where M is the bound of k (x, t) (())(03:05). So in this condition we can we will show that our Fredholm integral equation 1 is a having a unique solution in the interval (a, b). And it is driven by this series infinite series which is absolutely and uniformly convergence series.

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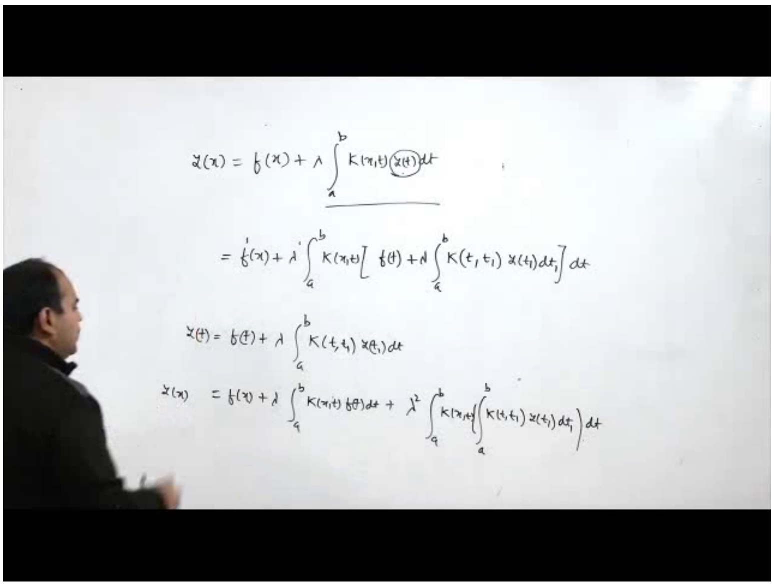

So if you look at here it is a series given in terms of lambda. So if you look at this is how to prove it. Let us try to prove it here. So this proof is based on a very simple idea, the idea is this that if you look at this is the equation here. So since I am discussing right now Fredholm, so I am just using the upper limit as b here.

So if you look at y(x) is given as this thing. Now problem is that this is a y (t) which is unknown function so we cannot see that this is a solution here. But if I assume that if we know what is y(t) here then we can find out the solution. So how to get how to do this kind of integration if I can do this integration we are done. But this is quite difficult here because y(t) is still unknown to us.

So what we try to do here we can write it here as f(x) plus this lambda is missing here lambda a to b here and k(x, t) and here in place of y(t) let us use this formula again. So we know that y(t) y(x) is a function which satisfy this Fredholm integral equation. So I can use this equation for minus y (t) also.

So it means that I am writing here y(t) can be could be what is a y(t) here I can write it here f(t) here plus lambda times a to b, now here I have to write it k(t) if I am using in place of x I am using t then I have to use some other variable here. So let us use t 1 here y(t 1) here d(t 1) right? This is d (t).

So what we have done here we have say find out the value of y(t) with the help of this integral equation itself. So here how we have done here we have written here y(t) we have just changing the value x here y(t) so it is f(t) plus lambda times a to b, I am using x y(t) replacing x y(t) so this variable t integration variable I am writing as t 1 and y(t 1) and d(t), ok.

So that means I can write it like this and if you simplify this further you will get what? This is what y(x) is given as f(x) plus lambda times a to b I am just multiplying this so it is what k(x, t) f(t) dt plus if you look at this second term, second term is going to lambda square a to b k(x, t) integral a to b k (t) (t 1) and y(t 1) dt 1 d(t), ok.

So now this process we can keep on doing, so it means that here also if we look at if use the same first equation for y(t 1) you can find out what is the value of y(t 1) put it here and we can go to the next step. So this is we can keep on doing. So if you look at it is written here.

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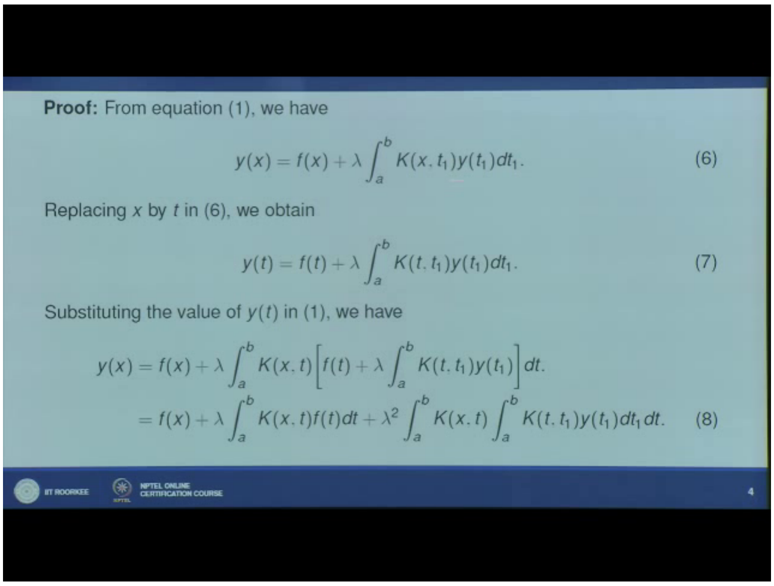

So here we know that Fredholm integral equation of is given as this then if we want to find out the value of y(t 1) here then it is y(t) like this then if you put it here we have y equal to f(x) plus lambda times a to b k(x, t) f(t) dt plus lambda square a to b k(x, t) a to b k(t, t 1) y(t 1) d(t 1) dt, which we have shown just now, ok.

So it means that y(x) is given by this equation number 8. Now if I move the thing again it means that if we want to find out the value y(t 1) here. So if I want to replace this y(t 1) by the value given here then it is basically what? Let me write it y(t 1) here first. So y(t 1) is basically f(t 1) here plus lambda times a to b I am writing here k (t 1)

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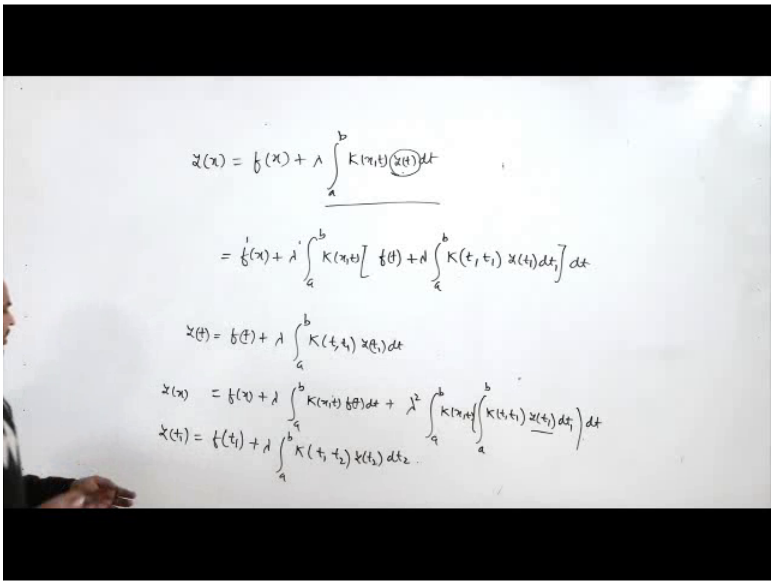

Let us change this integrant a integral variable by say t 2 and y(t 2) and d(t 2).So what we try to do here we use the Fredholm integral equation its already given. And use find out the value of y(t 1) here in terms of y(t2) and put it here.

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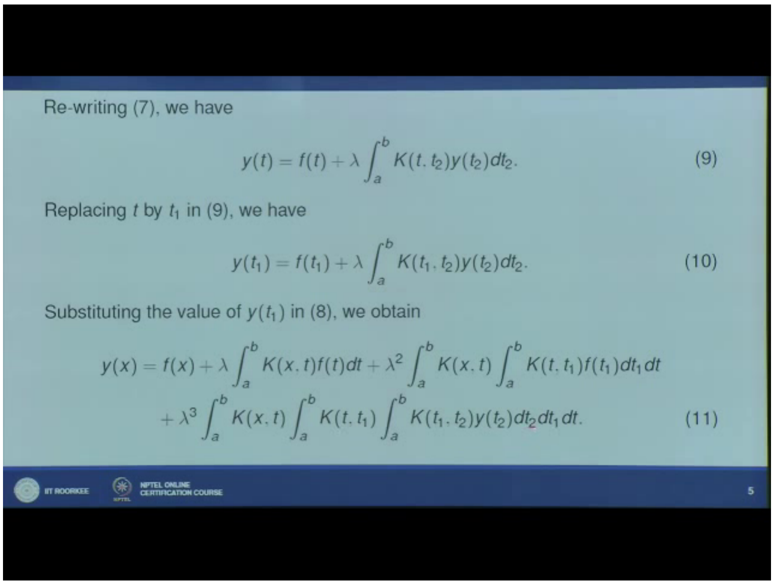

Then if you put it here then we have another term given another term we can add and we can write it like y(x) equal to f(x) plus lambda a to b k(x, t) f(t) dt plus lamda square a to b k(x, t) a to b k(t, t 1) f(t 1) dt 1 dt which we have obtained in a previous iteration also plus there is one more term added here lambda q a to b k(x, t) a to b k(t, t 1) a to b k(t 1, t 2) y(t 2) dt 2 dt 1 and dt. This is total integration basically.

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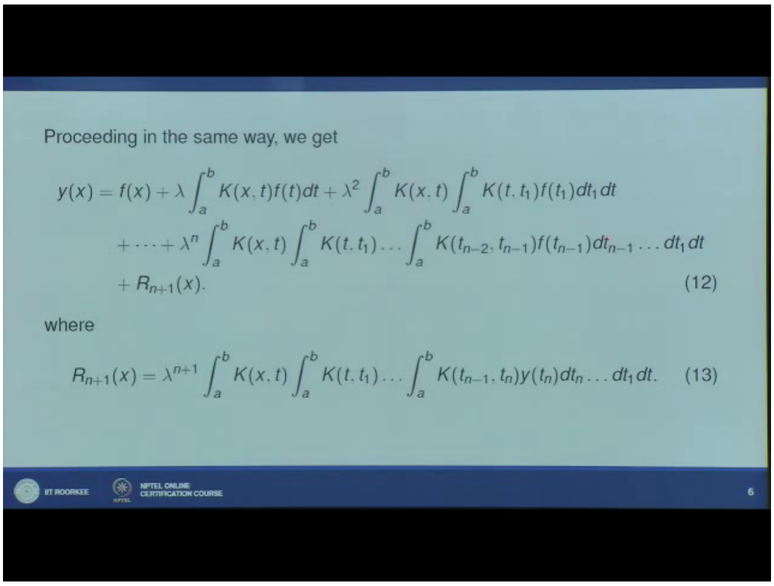

So if you proceed in this way we can write our solution in this series y(x) equal to f(x) plus lambda times a to b k(x, t) f(t) dt plus lambda square this term and so on. So this series will having this remainder term r n plus 1 x which is given by this. So what we can do here if we keep on doing this then we can get a infinite series in terms of lambda for this y(x).

So we can say that the solution of Fredholm integral equation is given by this y(x) equal to f(x) plus these many terms provided that this series is convergent and this r n plus 1 x is tending to 0.

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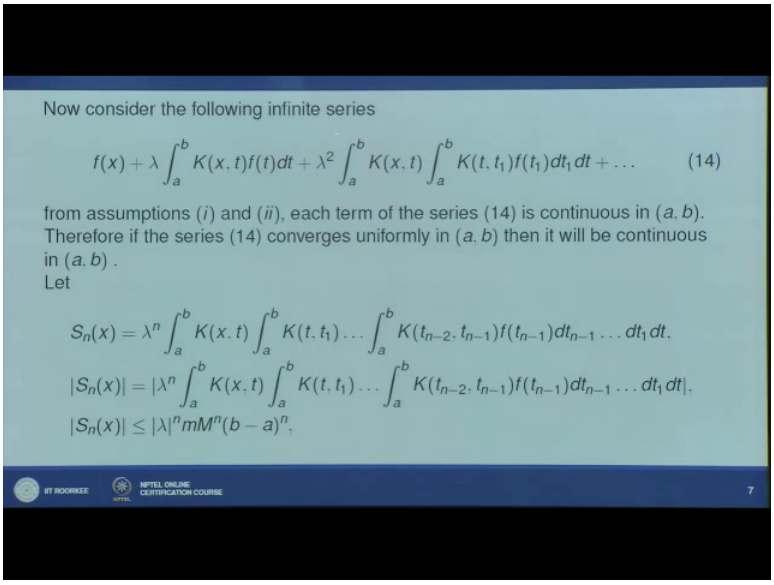

So for that let us look at this following infinite series f(x) plus lambda a to b k(x, t) f(t) dt and so on. We want to show that this series convergence series. For that we will utilize the assumption we have assumed earlier.

What we have assumed earlier is that k this kernel k(x, t) is continuous function and bounded in this interval a to b for values of x and t in interval a to b and this f(x) is also continuous and bounded in this interval a to b. So for that let us consider the other term of this infinite series call it s n x . An s n x is given by this.

Now here what we try to do here we just try to find out bound of this s n x. So if we look at here modulus of s n x is given by this. Now here if you look at this term, this k (t n minus 2) (t n minus 1) and f(tn minus 1). So this f small f (t n minus 1) is bounded by small m. So I can take that upper bound here. And if you look at this, this is bounded by k (x, t) is bounded by capital M here.

So I can write that this small m is taken out and capital M for this term and every term here. If you look at here each t, t 1, t 2 , x everything is lying in this interval a to b. So it means that I can take I can utilize the upper bound as k(x, t) and which is given by capital M. So here we have n times of integration, so we have n to power n outside and upper bound of this that is small m here lambda to the power n is already there.

And then we have constant integral a to b d t n minus 1 and full integration basically. So if you do this integration we will get this b minus a to power n here. So I can say that this n th term is bounded by lambda to power n, m small m capital M to power n (b minus a) to power n.

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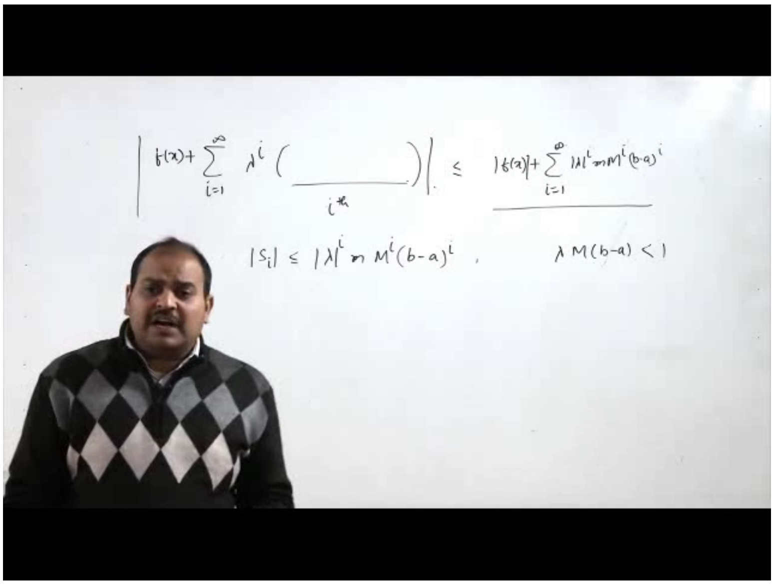

So it means that I can write it here like this I can say that your series f(x) plus I can write it here summation your lambda to power i here from 1 to infinity and here we have expression, we have i th term here and it is i fold integration is given here.

And we say that this is s i and this is bounded by just now we have calculated this is that modulus of lambda to power i and small m capital M to power i (b minus a) to power i. So I can say that if you take the absolute value here then this is going to be bounded by modulus of f(x) plus summation i equal to 1 to infinity and here I am writing here modulus of lambda to power i small m capital M i (b minus a) to power i.

So i can say that this is geometric series with common ration lambda capital M b minus a. So basically this is a geometric series and this geometric series will convey us provided this quality is going to be less than 1. So it means that this series is convergent or every term i th term of this is less than or equal to i th term of this, so by this law I mean test we can say that this series is not only convergent it is uniformly convergent and the benefit of uniformly convergence is that we have if you look at the each term each term is continuous here.

So if you look at here f(x) is continuous this is continuous so every term is continuous function, so we can say that this is an infinite series of continuous function. So your limit is also going to be a continuous relation. So it means that what ever be the limit that is going to be a continuous function of x, ok

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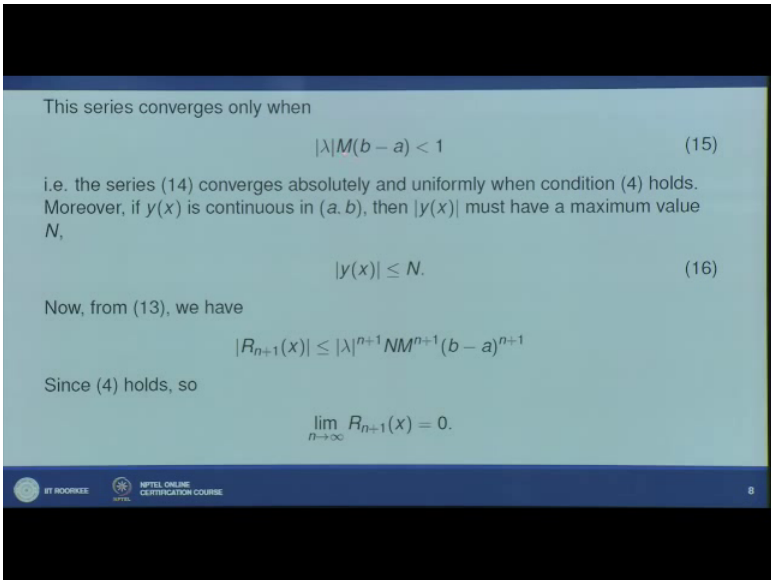

So here this series will converge only when modulus of lambda M(b minus a) is less than 1. And this convergence is absolute convergence and uniformly convergence. And in this case we can assign that value as y(x) here and y(x) is continuous because of this uniform convergence. And if I assume that it has a maximum value if we call this as maximum values as capital M then we can say that the reminder term which is given by equation number 13 this is term y(t) a.

So this is capital M taken out and this is n k(x, t) is coming n plus n times , n plus 1 times so we have bound of that r n plus 1 is given as modulus of lambda n plus 1 n times capital M to power n plus 1 (b minus a) to power n plus 1. So here we already know that this modulus of lambda capital M (b minus a) is less than 1. So as n tending to infinity is also going to be 0.

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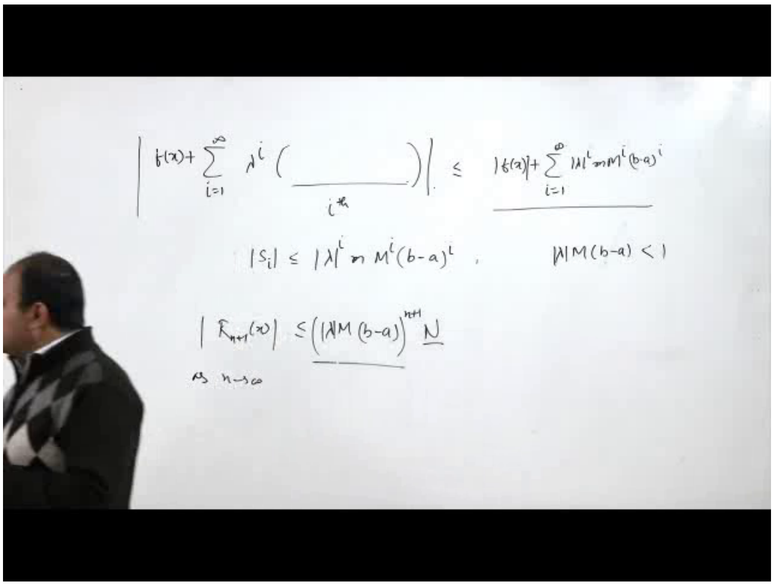

Let me write it here, we have r n plus 1 modulus of this is bounded by lambda capital M (b minus a) here modulus lambda here to power n plus 1 and this is n which is a bound of y(t). And as n tending to infinity, so this term is tending to 0. Because this is already less than 1. So we can say that I hope here it is modulus of lambda, ok.

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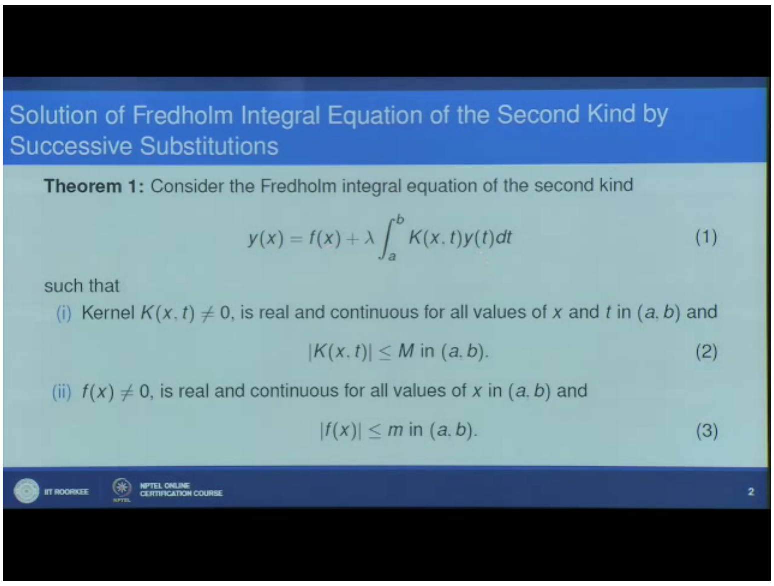

So here we can say that this series is convergent and your solution y(x) is given by this infinite series provided that lambda is a constant which we have taken here is satisfying the condition let me look at here. So it means that this Fredholm integral equation will have a series solution which is uniformly convergent provided that first condition second condition modulus of lambda is less than 1 upon M (b minus a) this condition (())(16:53).

So here we have a Fredholm integral equation. If we look at here the same thing we can do for Volterra integral equation. The only difference between Volterra integral equation and Fredholm integral equation is that the upper limit a is replaced by x variable x here.

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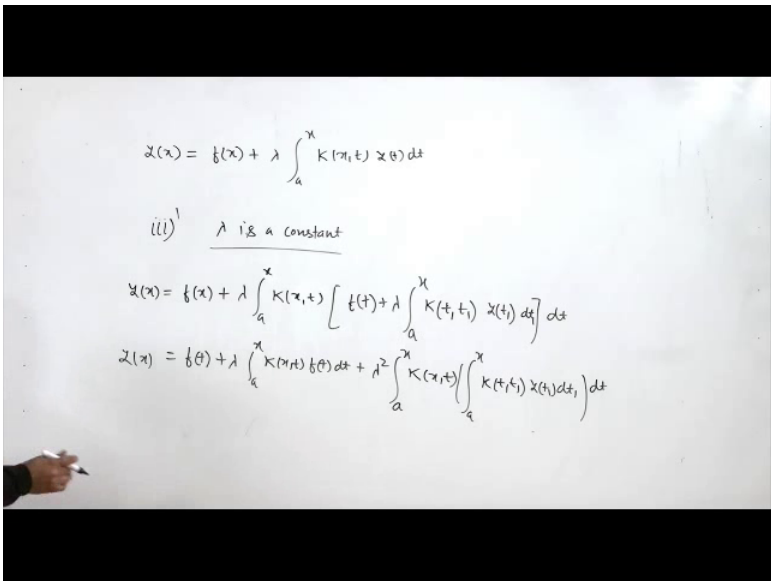

So let us look at the same thing for Volterra integral equation. So y(x) is given here f(x) plus lambda times a to x k(x, t) (f) y(t) here y(t) dt here. So here we are assuming the same condition what condition we are assuming here we are assuming that kernel is real and continuous for all values of x and t in a, b. I am having bound capital M which is the upper bound of this modulus of k(x, t).

And this small f(x) is also real and continuous and satisfying this bound. The third condition which we have assumed for the case of Fredholm integral equation is not required for Volterra integral equation. Here we can simply say that lambda is a constant so I can replace this third condition y third dash which says that lambda is a constant.

So here no need of any condition for this lambda. The reason being that if you look at what we have done here we have started with the first substitution writing this f(x) plus lambda times a to b here if you write it here k(x, t) k(x, t) k(x, t) here I am writing the value y(t) here which is given by this only. So f(t) plus lambda times a to x here and if you write it here I am writing t, t 1, y(t 1) and d(t 1) and dt here.

So if you look at this is same as the previous thing and we can write it here f(x) plus lambda times a to b k(x, t) f(t) dt plus lambda square times ohh sorry here it is not b it is x so this is x plus lambda times a to x k(x,t) and integral a to x k(t, t 1) y(t 1) dt 1 and dt here. So if you look at here also we have not it is the same procedure which we have adopted for Fredholm integral equation.

And in this way we can again write down the expression for y(t 1) using the same equation here and keep on this procedure. So if you look at we can repeatedly we can go upto say this thing we can write down y(x) equal to f(x) plus lambda a to now here all these base are replaced by variable x.

So you can say that for in case of Volterra integral equation your solution y(x) is given as f(x) plus lambda a to x k(x, t) f(t) dt plus lambda square a to x k(x, t) a to x k(t, t 1) f(t 1) d(t 1) dt and so on.

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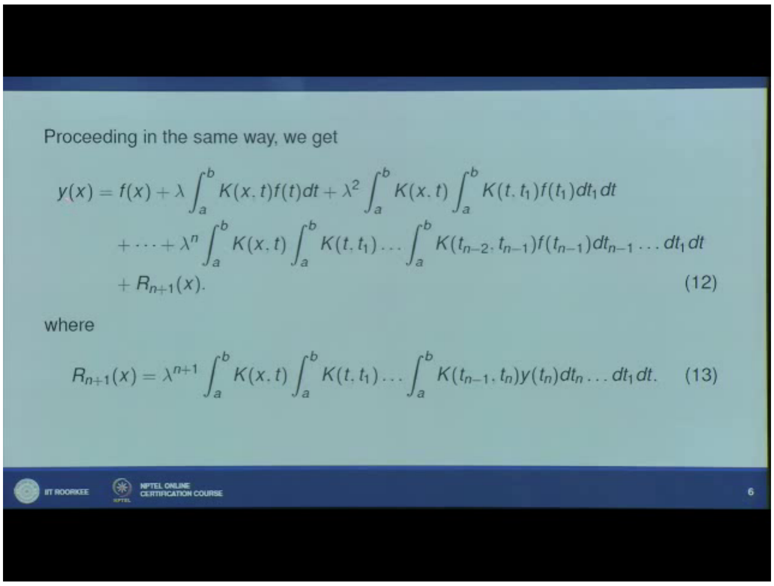

Similarly we have r n plus 1 given by this n plus 1 times full integration lambda n plus 1 upper limit is simply replaced by this x. So this no change other than this b is replaced by the variable x. So we can similarly say that this series in this way we can as n tending to infinity this is going to be an infinite series.

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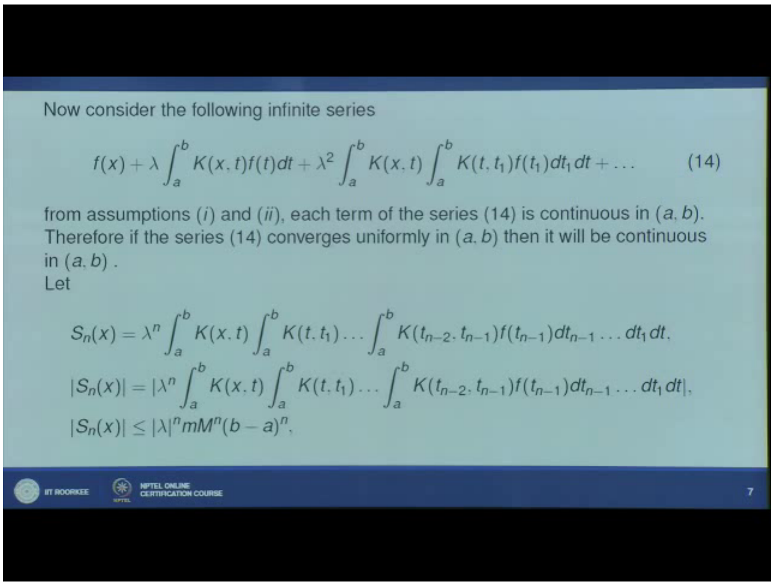

So we say that this is a infinite series and we want to tell that the solution of Volterra integral equation y(x) is given as this infinite series. So what we need to prove here that this infinite series basically converge, so converge so here also look at the n th term of this infinite series so s n x is given by this term, only thing is that upper limit is replaced by x here.

So when we use the condition that k(x, t) is bounded by capital M and your f(x) is bounded by small m we can write down this (capital) small m term outside capital M term outside modulus of lambda to power n is already there. The only things here change is this?

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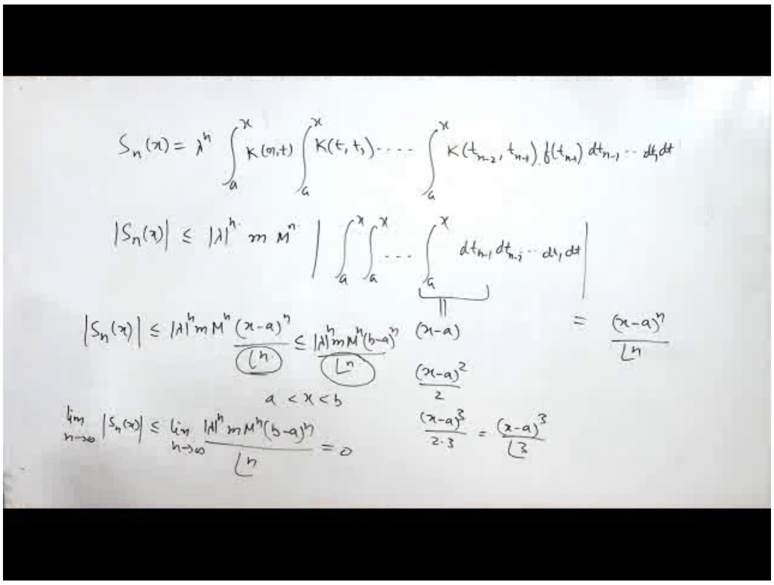

Let me write this expression for S n (x) is going to be lambda to power n a to x here k (x, t) a to x here k (t, t 1) and so on, a to x k (t n minus 2, t n minus 1) f (t n minus 1) d t n d t n minus 1 and upto dt. So it is n th integration.

So modulus of S n x is going to be is less than or equal to modulus of lambda to power n. Now this is bounded by small m I am writing here and k(x, t) is bounded by capital M so it is n times so we have m to power n.

And this is the left here a to x a to x and so on, a to x at d t n minus 1 d t n minus 2 and so on and d t 1 and dt. So this thing is left here. If you look at here if you solve the inner one this is basically what you can write it here as the first one is (x minus a) this one. Now next you if you integrate it you will get what you get (x minus a) whole square upon (factorial) upon 2.

So it means that I can write it here this is (x minus a) and if you integrate further you will get (x minus a) square upon 2 here and if you further integrate you will get x minus a) cube and 2 3 which I can write this as as (x minus a) to power 3 divided by factorial 3. So if you use this notation then this is going to be written as x minus a) to power n divided by factorial n.

So here in this Volterra integral equation if your mod of S n(x) is replaced by bounded by mod of lambda to power n small m capital M and (x minus a) to power n divided by factorial n. And we already know that x is lying between this a to b here. I can take I can include this end point also then I can say that this is further bounded by modulus of lambda to power n small m capital M and b minus a to power n upon factorial n.

If you remember the only difference here is this denominator that is factorial n so in Fredholm integral equation we have only this part but in Volterra integral equation because of the upper limit is variable we have this factor coming out that is factorial n is a very say very fast going to infinity as I am tending to infinity.

So I can say that if I take limit n tending to infinity modulus of S n(x) is less than or equal to limit n tending to infinity I really do not care because this is anything that (())(25:41) and because of this factorial n that is very rapidly going to zero. So we can see that modulus of lambda to power n small m capital M b minus a to power n upon factorial n.

So because of this term this is tending to 0 whatever lambda is so we can say that in case of Volterra integral equation your S n(x) n th term of this infinite series is tending to 0 without any condition on lambda n, right. As n tending to infinity your solution is going to be going to be represented by equation number 14 here. Only thing is b is replaced by x.

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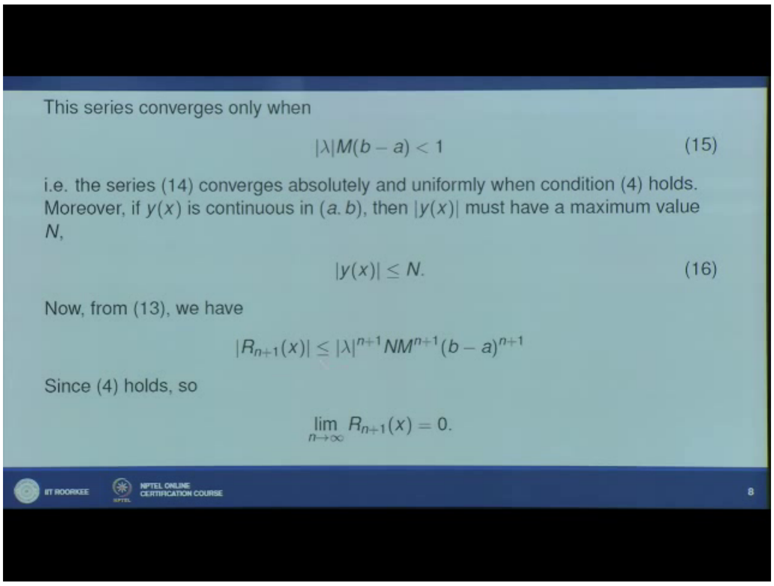

Now if you look at the remainder term, remainder term is again we can use the same theory available here. But here again if we say that solution is given by the previous series we can say that again this convergence is absolute and uniformly convergence. So your solution is y(x) is going to be continuous in this interval (a, b).

And we assume that the solution y(x) is bounded by capital M and in the same way we can find out say bound of this again here we have the same change here. So here also we can have the upper bound of R n plus 1 is this.

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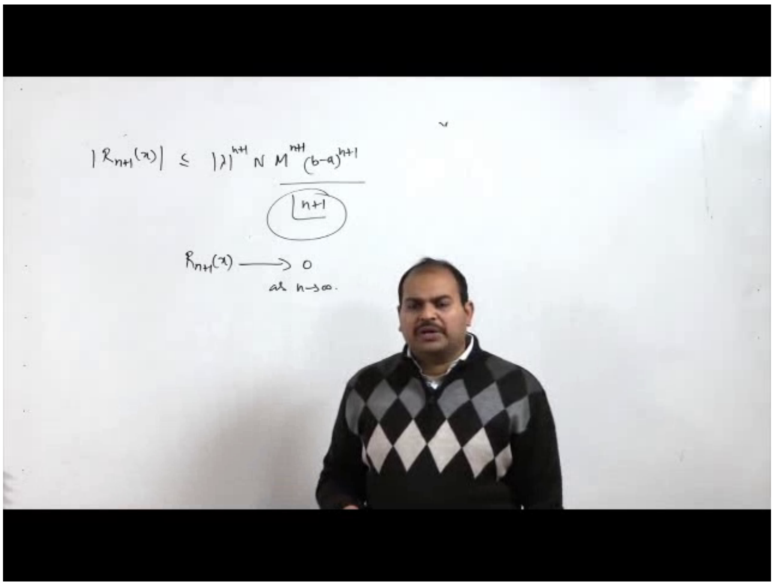

So here we have modulus of R n plus 1 (x) is bounded by modulus of lambda to power n plus 1 capital N here upper bound of y (x) here n to power n plus 1 here (v minus a) to power n plus 1 here and divided by factorial n plus 1.

So again here as n tending to infinity this term tend to infinity very fast and which makes this R n plus 1 (x) tending to 0 as n tending to infinity. So we can say that here also we get a power series in terms of lambda as a solution for y(x) and here we do not have any condition on this lambda. In Fredholm integral equation we have a condition lambda but here if I Volterra integral equation we do not have any condition on y(x) for every lambda that will converge, ok.

So in next lecture we will discuss the approximation method to solve Volterra and Fredholm integral equation. Thank you for listening us thank you.