Integral Equations, Calculus of Variations and their Applications

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Lecture 07

Fredholm Integral Equation with Separable Kernels: Examples

Hello Friends! Welcome to this lecture, in previous class we have already seen the theory to solve Fredholm Integral Equation with Separable Kernel and we have also discussed the concept of Eigen values and Eigen functions. And now in this lecture we discuss some example based on that theory given earlier. So let us go to the first example here.

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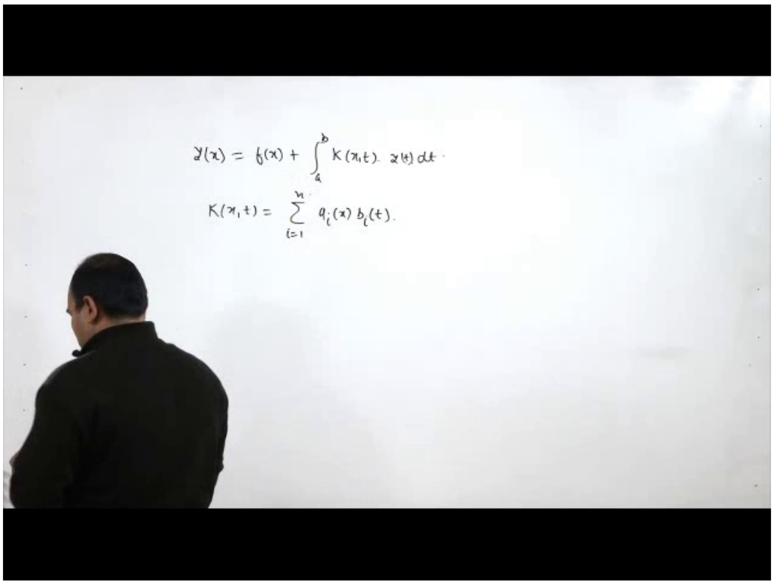

Here I just want to note here that in previous class we have discussed y(x) equal to f(x) plus a to b and k(x, t) and f(t) dt sorry here it is f(t) it is y(t) here, y(t) dt here. And here we have assumed that this kernel k(x, t) is separable means we have assumed that it can be written as in this separable form are equal to 1 to n. And we have discussed how to solve this.

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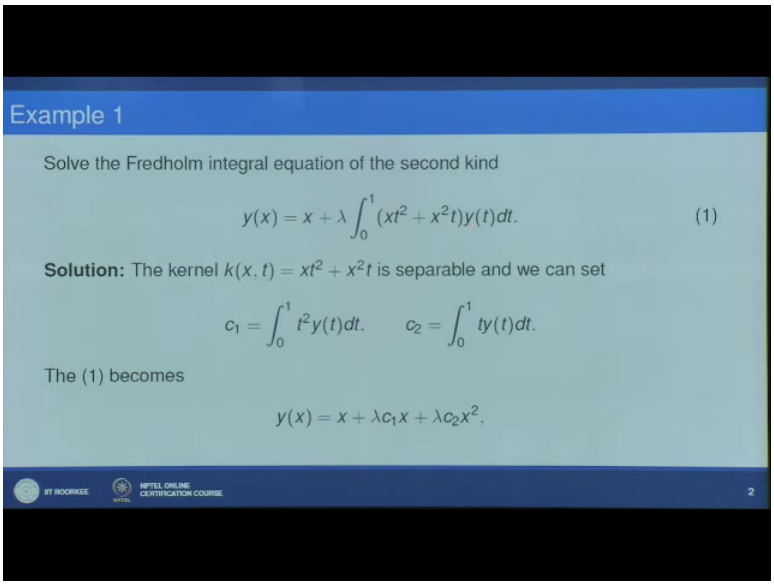

So let us take the first example given here and here the example is that solve the Fredholm Integral Equation of the second kind y(x) equal to x plus lambda 0 to 1 xt square plus x square t y(t) dt. So here if you look at kernel is this x t square plus x square t.

So it is here x t square is written as the function of x and function of t square plus this also can be written as function of x and function of t. So this is of the separable kind. So here the kernel k(x, t) is a separable and now we can use our theory which we have already discussed to solve this.

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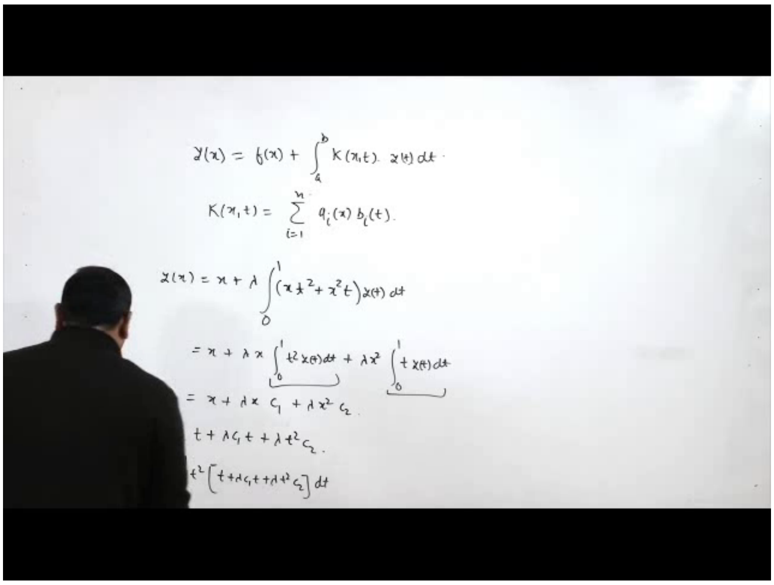

So here what we trying to do here, just I am going to write it here so in this example y(x) is equal to x plus lambda 0 to 1 here x t square plus x square t and y(t) dt I can simplify this as x plus lambda. Now x say here variable is t so I can take x out so it is 0 to 1 t square y(t) dt plus I can take x square here so lambda x square 0 to 1 y(t) dt.

So if you look at this quantity is going to be a constant quantity. So I am going to write it x plus lambda x if I denote this as say c 1 then it is c 1 plus lambda x square again the same this quantity is also constant so I will write it like C 2. So here if I denote this c 1 as 0 to 1 and t square y(t) dt and C 2 as 0 to 1 y(t)dt. And I can write equation 1 as y(x) equal to x plus lambda c 1 x plus lambda c 2 x square.

So it means that solution is given like this only thing we have to discuss here how to find out the value c 1 and c 2 the constant c 1 and c 2. For this what we can do here we can look at the formula here C 1 is equal to 0 to 1 t square y(t) dt. Now we already know what is the form of y(t) I can write down I can use this y(x) equal to this to put it here. So when you put let me write it here.

So here this is your y(x) so I can write y(t) as t plus lambda c 1 t plus lambda x square this x is t square so this is lambda t square c 2. So using this I can write it here C 1 as 0 to 1 t square, now in place of y(t) I am writing t plus lambda c 1 t plus lambda t square c 2 and dt and we can solve this. So similarly we can do for C 2 also.

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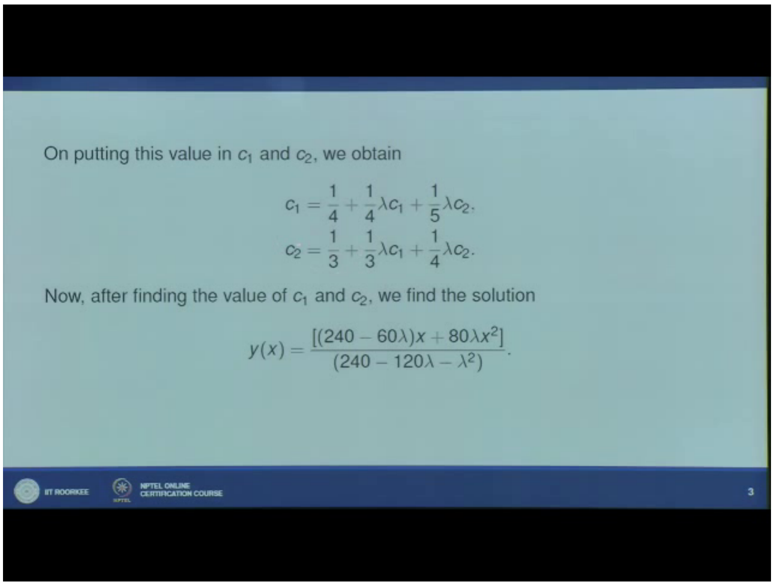

So if we use this then I can write c 1 as 1 by 4 plus 1 by 4 lambda c 1 plus 1 by 5 lambda c 2. Similarly we can have expression for c 2. Now if you look at this is a simple algebraic equation in c 1 and c 2 say non homogenous equation because 1 by 4 and 1 by 3 is also there. So this we can solve by you use one theory of linear equation. So by solving this we can get c 1 and c 2. And if you plug this again the value of c 1 and c 2 here in this expression then your y(x) can be given as this particular form. You can easily check that your c 1 and c 2 can be given in this particular formula. So here our solution is given by y(x) equal to this.

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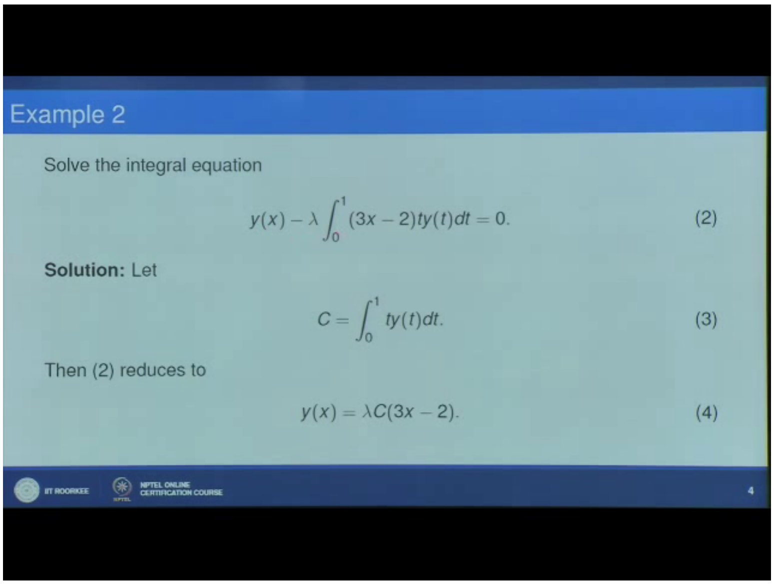

Now let us go to the second example so here second example is I solve this homogeneous integral equation Fredholm integral equation y(x) minus lambda 0 to 1 3x minus 2 t y(t) dt equal to 0. Here again it is given in separable form it is 3x minus 2 is a function of x here and then it is t. So here k(x, t) is given by. So ok,

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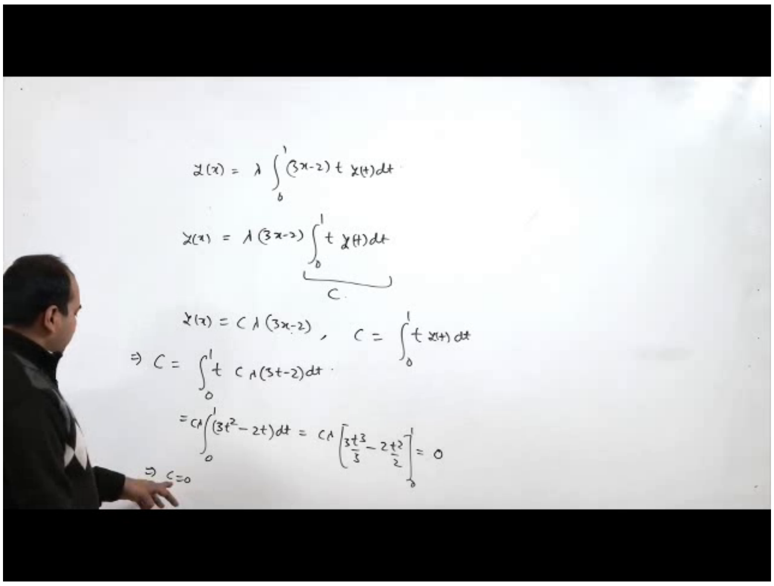

So here I am writing here y(x) if you take that sides then it is lambda 0 to 1 3x minus 2 of t y(t) dt (())(06:09).So now here if you write it then it is lambda you can take 3 x minus 2 out because variable the integration is t so 3x minus 2 0 to 1 t of y(t) dt and that is given as y(x), ok.

Now this quantity is going to be a constant call it c then we have y(x) is equal to c lambda 3x minus 2. So now our form of the solution is already known to us that y(x) is equal to c lambda 3x minus 2.

The only thing we need to find out is the constant c here. So here we have denoted c as 0 to 1 t of y(t) d (t). So here if you write down then c is going to be 0 to 1 t now y(t) is going to be what c lambda in place of x we are writing t here 3t minus 2 d(t). So if you look at this is what 0 to 1 c lambda you can take it out then it is what 3t square minus 2t and dt.

And if you solve you will get c lambda c lambda and this is what 3t square t cube by 3 minus 3 is there minus 2 t square by 2 and if you use a integral boundary limit, boundary 0 to 1 then it is coming out to be 1 minus 1 that is 0. So this is coming out to be 0. So here we have c equal to 0. So it means that if you use c equal to 0 here then our solution is going to be y(x) is equal to 0. It means that here the solution of this homogeneous problem is nothing but trivial solution.

So if I look at this problem as Eigen value Eigen function problem then this homogeneous Fredholm integral equation is not having any Eigen value because we do not have any non trivial solution corresponding to any lambda here. So and it is not very surprising because if you look at the kernel here, kernel is 3x minus 2 and t. So kernel is not symmetric and we can say that a Fredholm integral equation with non symmetric kernel may not have Eigen value and Eigen function.

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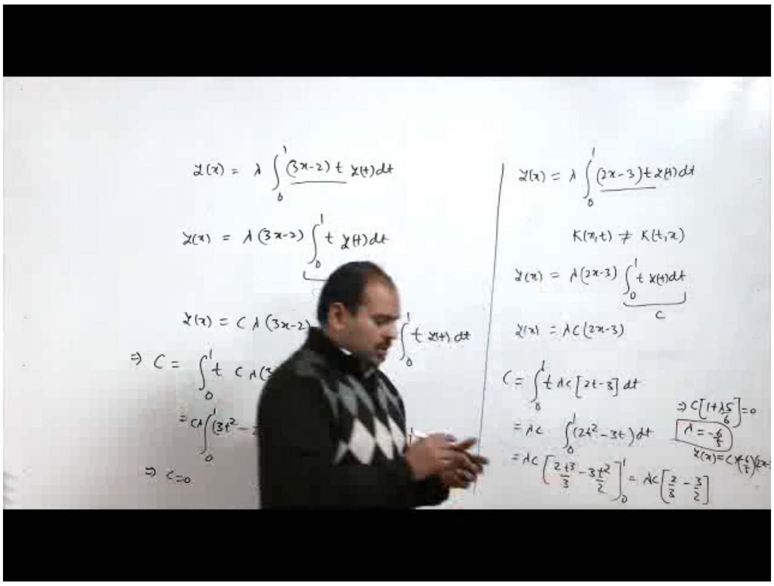

So but if you look at the same problem this problem in a different manner, so if I look at this problem as y(x) is equal to lambda 0 to 1 in place of 3x minus 2 I can write it 2x minus 3 t y(t) dt. So here if you look at your kernel is again in separable kind and kernel is also not symmetric. So I can say that here symmetric means k(x, t) is not equal to k(t, x). I am talking about a real kernels so I am writing k(x, t) is not equal to k(t,x).

So it is a problem with non symmetric kernel but if you try to solve the same problem I am using the same technique I can write it y(x) is equal to lambda I can take 2x minus 3 out and in the same way 0 to 1 t y(t) dt I can write this as c here then it is what lambda c 2x minus 3. So that is your y(x) and if we calculate your constant c then c is basically what 0 to 1 t and this is what lambda c and it is 2(t) minus 3 dt.

So this I can write it here lamda c I can take it out and this is what 0 to 1 2t square minus 3t dt and if you simplify this is lambda c and this is what 2t cube by 3 minus 3t square by 2 and here you get 0 to 1. So if you simplify you will get lambda c and this is what 2 by 3 minus 3 by 2 and you can simplify and you can get value for this I can say that c if you take it 1minus lambda and this expression is what 4 minus 9 so it is 5 by 6 minus 5 by 6 so it is going to be plus 5 by 6, so it is equal to 0.

So here if I take c equal to 0 then we are getting a trivial solution but if I take lambda 1 plus 5 lambda by 6 as 0 or you can say that lambda is going to be minus 6 by 5 then in this case I can chose any value of C or I can say that in this case when lambda is equal to minus 6 by 5 then our solution y(x) is given by c lambda , lambda is minus 6 by 5 so I can write it minus 6 by 5 here, this value to x minus 3.

So in this particular case we too have non symmetric kernel but in this particular case we have Eigen values because corresponding solution is non zero solution. So here we can sarize these two problems as that a non symmetric Fredholm integral equation is non symmetric kernel may or may not have Eigen value, ok. So so here we have solved these two problems with the help of separable kernel, ok. So this is what we have discussed now.

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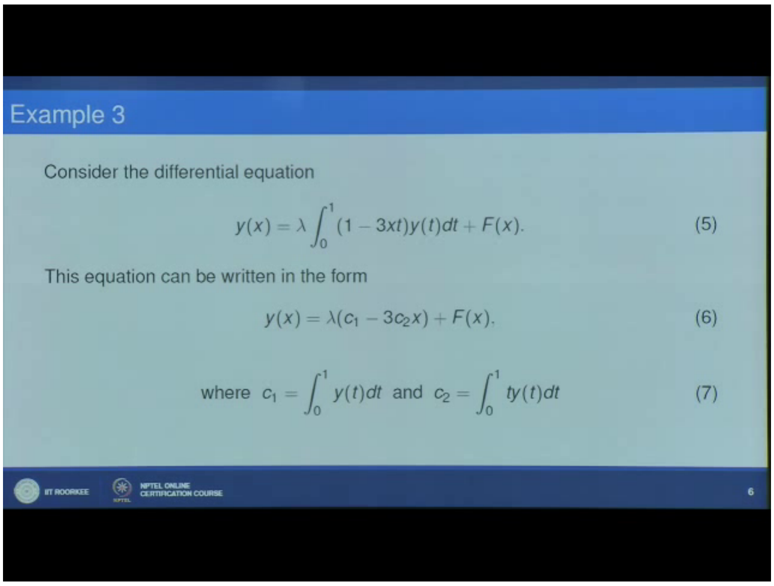

Now let us go to example 3 here, so here we want to solve this Fredholm integral equation of second kind so y(x) is given as lambda 0 to 1 (1 minus 3xt) y(t) dt plus f(x). If you look at here kernel is symmetric, ok. So k(x, t) is basically 1 minus 3xt) so k(x, t) is same as k (t, x). Now you also want to solve this particular problem.

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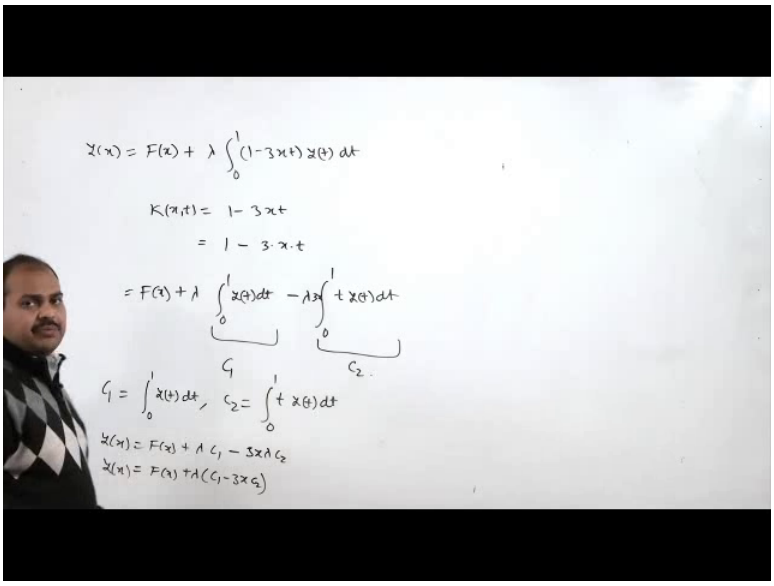

Let me write it here ok, so problem is this y(x) I am writing f(x) here plus lambda times 0 to 1 (1 minus 3x t) and y(t) dt here. So here your k(x, t) is written as (1 minus 3xt) which I can write it 1 minus 3 times x times t. So I can write it that it is a function of x it is function of t so it is separable kind and if you simplify this further then it is f(x) plus lambda 0 to 1 y(t) dt plus you can write it here as minus lambda 0 to 1 3x you can take it out.

So 3x also you can take it out here and it is what t y(t) dt. So again as we have pointed out this is going to be a constant function call it c 1 and this is 0 to 1 t y(t) dt is your c 2. So c 1 is going to be 0 to 1 y(t) dt and c 2 is going to be 0 to 1 tof y(t) dt. So with this our solution y(x) can be written as f(x) plus lambda c 1 minus 3x lambda c 2. Or I can simplify this further f(x) plus lambda you can take it out c (1 minus 3x) c 2. So only thing we need to find out these constants c 1 and c 2.

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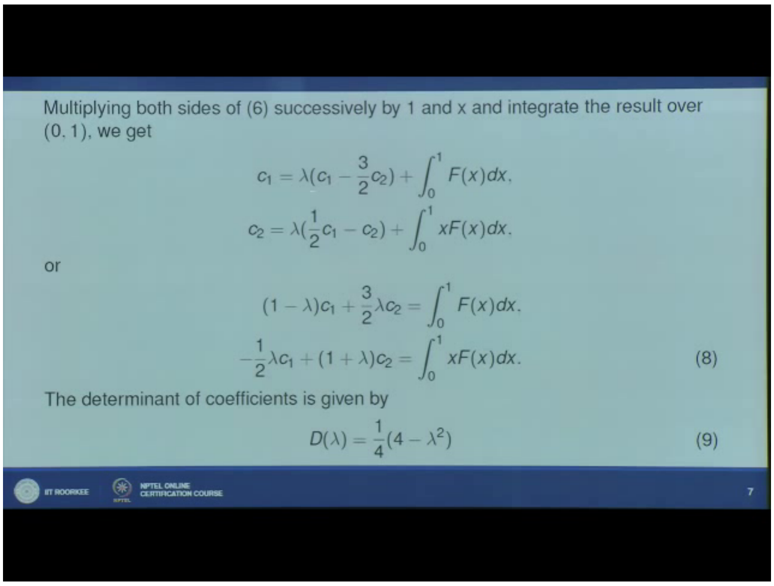

So to find out this what we try to do here, you can use this thing. So what we try to do here look at the (question) equation number 6 here we multiply by 1 and integrate between 0 to 1. So if you look at the equation number 6 multiply by 1 with no change and integrate between 0 to 1. So this side will give you c 1 and this i you will get something which we want to integrate here. So that will give you 1 equation in terms of C 1.

And similarly if you multiply this with respect to x here and integrate between 0 to 1 then we can get c 2 here, so multiplying both side of 6 successfully by 1 and x and (integral) indicate the result over 0 to 1 we can get these two equations. So first equation is this c 1 equal to lambda c 1 minus 3 by 2 c 2 plus 0 to 1 f(x) dx and c 2 equal to lambda c 1 by 2 minus c 2 plus 0 to 1 x f(x) dx.

And if you simplify this then you will have an algebraic equation in terms of c 1 and c 2 so if you look at this is coefficient matrix I can represent as a d lambda so I am just going to write it equation number 8 here.

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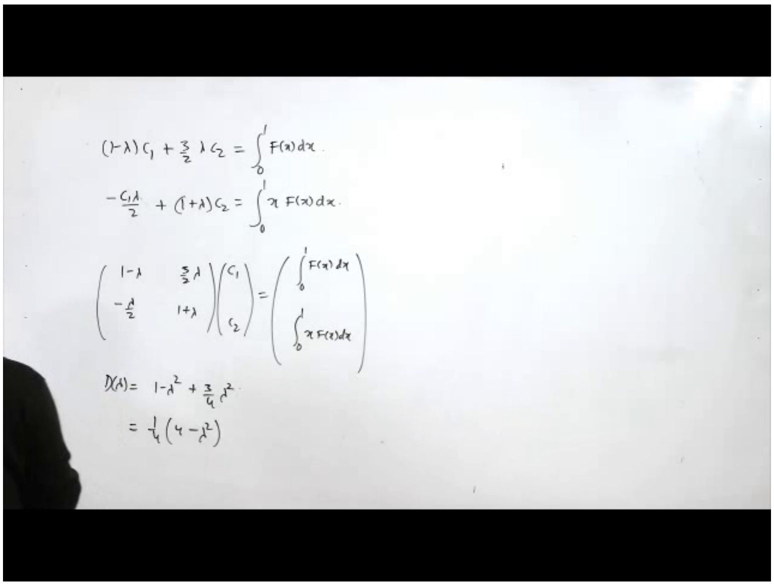

So if you look at 1 minus lambda c 1 plus 3 by 2 lambda c 2 is equal to 0 to 1 your f(x) d(x) plus here it is second equation is minus c 1 lambda by 2 plus 1 plus lambda c 2 equal to 0 to 1 x f(x) dx.

So here your lambda coefficient matrix is given by 1 minus lambda and 3 by 2 lambda minus lambda by 2 and 1 plus lambda here. So this is a coefficient matrix here and c 1 and c 2 and it is given by this thing. Here it is 0 to 1 f(x) d(x) and 0 to 1 x f(x) d(x).

Now to solve this look at the determinant of this coefficient matrix so d lambda is basically 1 minus lambda square plus 3 by 4 lambda (squa) 3 5 3 by 4 lambda square. So I can write it here that this is going to be 1 by 4 4 minus lambda square is that ok.

Now so here solution on this algebraic equation will depend on the quantity lambda that d lambda is 0 and d lambda is not equal to 0. So here we consider the case that if lambda is not equal to plus minus 2 means this d lambda is non zero. In that case we have a unique solution and we can get that unique solution by solving this algebraic equation.

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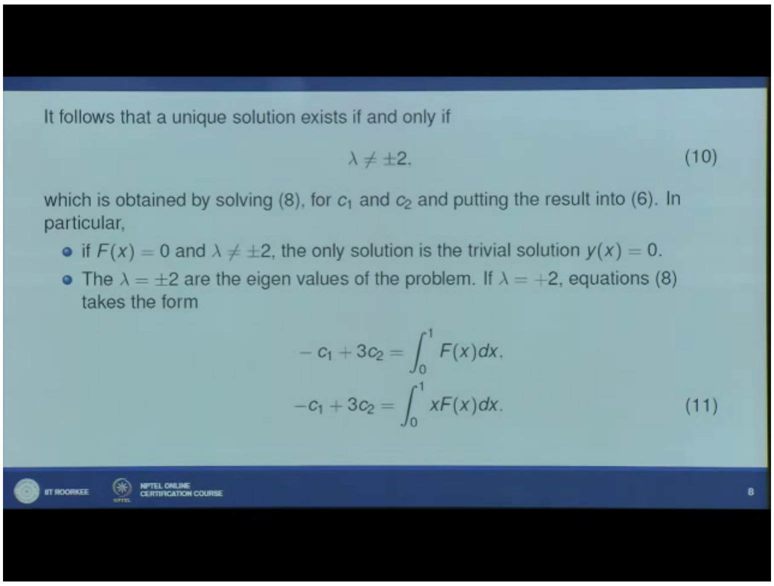

So you can get c 1 and c 2 by just inverting this non singular matrix and you can get values of c 1 and c 2.Once we have c 1 and c 2 we can write down our solution y(x) in terms of c 1 and c 2. So now here there are certain cases depending on the f(x) and the value is lambda here. If I assume that f(x) equal to 0 and lambda is not equal to plus minus 2.

So when we assume that lambda is not equal to plus minus 2 means coefficient matrix is non singular matrix. So in that case our f(x) equal to 0 means we have a homogeneous equation.

So homogenous Fredholm integral equation is given with the thing that coefficient matrix has is a non singular matrix. So in this particular case the value here if you look at the value here c 1 and c 2 is going to be what. So here this is f(x) is 0 so this is going to be 0 0 and this is a non singular matrix so you can get only a trivial solution that c 1 equal to 0 and c 2 equal to 0.

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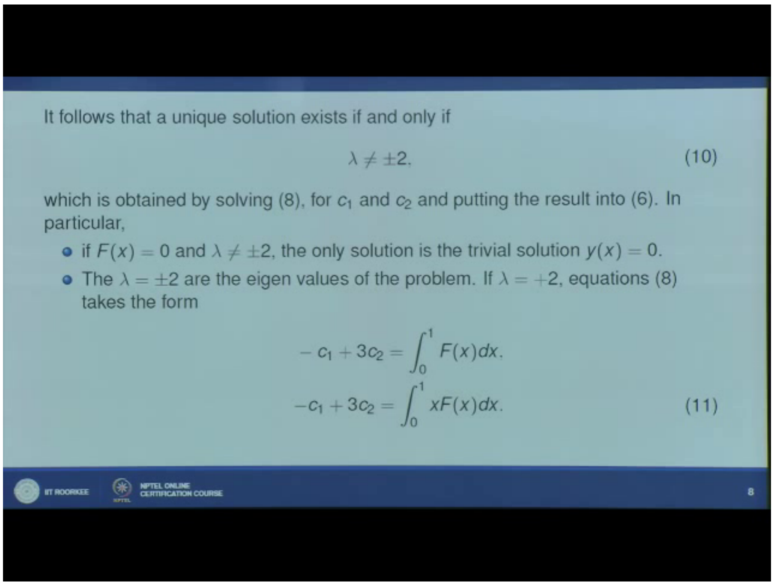

So I can say that when f(x) is equal to 0 I can sarize this result as this, if f(x) is not f(x) equal to 0 and lambda is not equal to plus minus 2 the only solution is the trivial solution y(x) equal to 0. But if I assume that lambda is plus minus 2 and then the coefficient matrix is going to be a singular matrix. So I can find out the Eigen values of the our problem.

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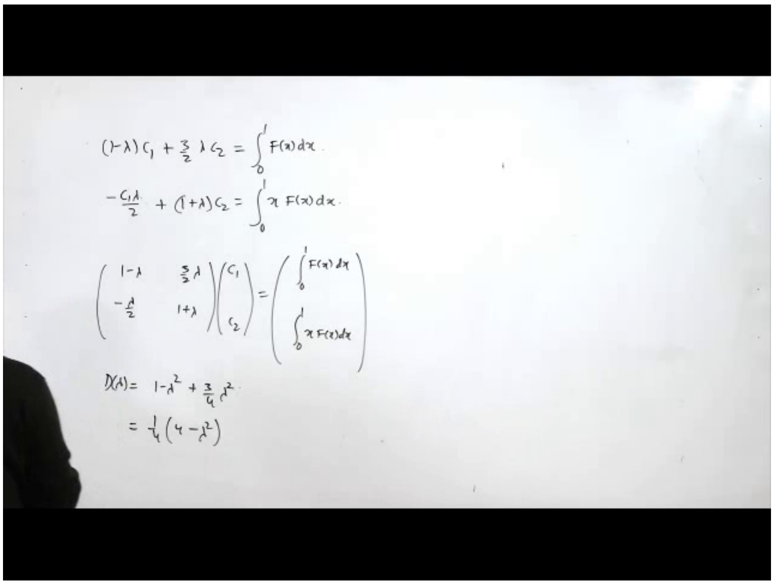

So here we can say that d lambda equal to 0 gives you the values for Eigen values. So here lambda equal to plus minus 2 gives you Eigen values of the problem.

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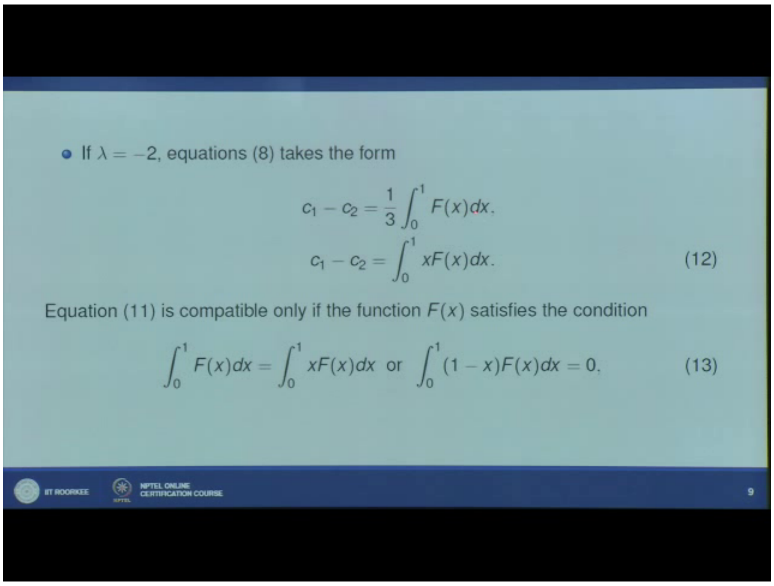

Now we look at the particular case when lambda is equal to plus 2. So when lambda is equal to plus 2 then your equation this equation, equation number 8 here you put lambda equal to 2 then this will reduce to equation number 11 which says that minus c 1 plus 3c 2 equal to 0 to 1 f(x)dx.

And if you look at the second equation it is also minus c 1 plus 3 c 2 equal to 0 to 1 f(x) dx. Now these two equation will be valid only when the value of 0 to 1 f(x)dx is same as 0 to 1 x f(x) dx. Or I can say that, that this equation number 11 is compatible if 0 to 1 f(x) dx is equal to 0 to 1 x f(x) dx. So we will discuss this case.

Now let us move to case lambda equal to minus 2. In this case when lambda equal to minus 2 equation 8 means this equation, equation number 8 is reduced to this form c 1 minus c 2 equal to 1 by 3 0 to 1 f(x) dx and c 1 minus c 2 equal to 0 to 1 x f(x) dx. So again here we have a c 1 minus c 2 equal to this and c 1 minus c 2 equal to this. This is possible only when the right hand side of these two equations match.

So I can say that this equation number 11 which is corresponding to lambda equal to plus 2 and equation number 12 which is corresponding to lambda equal to minus 2 will be compatible if the right hand side of this equation number 11 and equation 12 should match. So I can write that equation 11 is compatible only if the function f(x) satisfy the condition this. So here equation number 11 is given here, so this will be compatible if these two are equal.

Or I can say that 0 to 1 f(x) dx equal to x f(x)d x or you can say 0 to 1 (1 minus x) f(x) dx equal to 0. So here if f(x) satisfies this condition then only equation number 11 makes sense.

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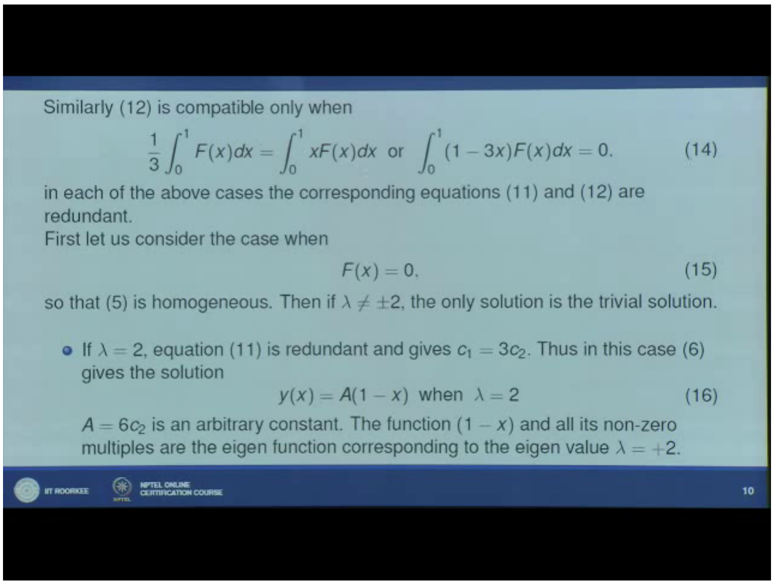

Similarly when lambda equal to minus 2 is compatible only when these two things are equal or I can say that 0 to 1 (1 minus 3x) f(x) dx is equal to 0. So it means that when lambda is equal to 2 then equation will give a makes sense. If 0 to 1 (1 minus x) f(x) dx is equal to 0.

Similarly when lambda equal to minus 2. You can calculate c 1 and c 2 only when we have 0 to 1 1 minus 3x) f(x) dx is 0. So let us sarize everything. We can say that if I consider f(x) equal to 0 r lambda is plus minus not equal to plus minus 2. We have seen that only solution is trivial solution. But if you take lambda equal to 2 then equation 11 is redundant.

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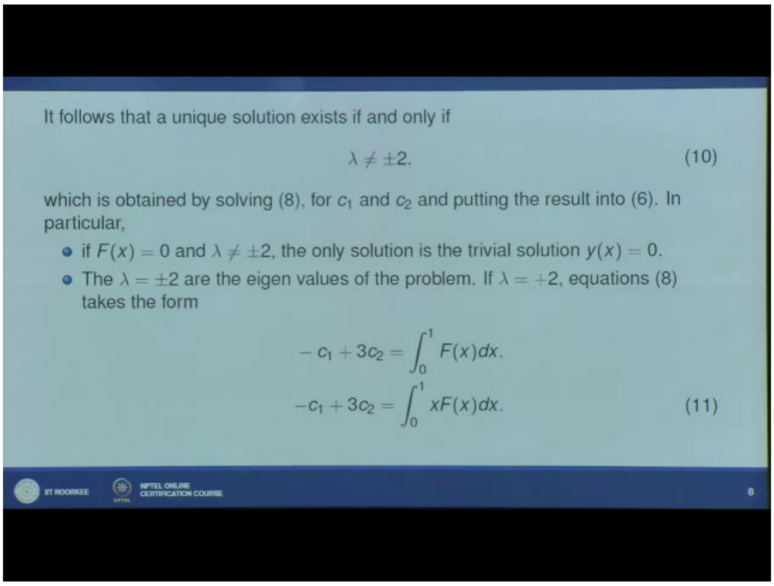

If you look at equation number 11 these two will give the same things redundant it will make sense only when we have 0 to 1 (1 minus x) f(x) dx is 0. Or I can write it here if lambda equal to 2 equation 11 is redundant and gives c 1 equal to 3 c 2, is it ok. And in this case we have solution like y(x) equal to a 1 minus x when lambda equal to let me write it here, ok.

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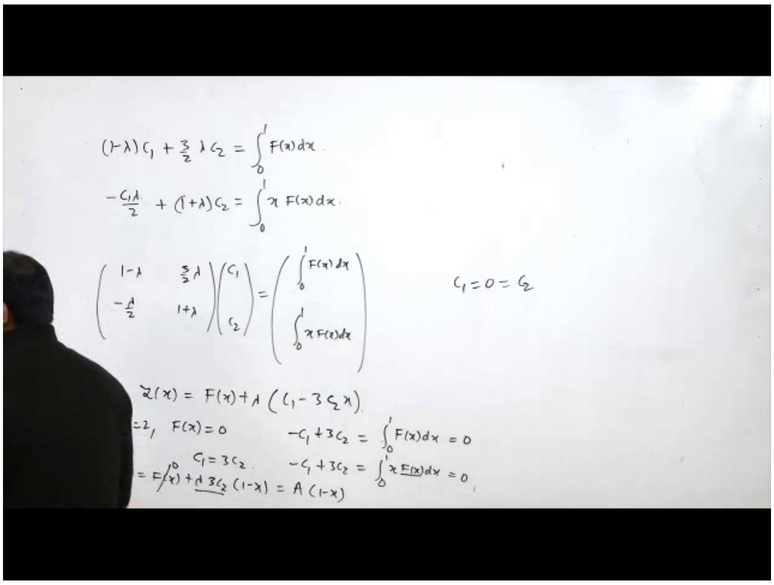

So here your solution is what solution is y(x) is equal to f(x) plus lambda times let me look at here solution is what we have written here we have a solution lambda times c 1 minus 3 c 2 x. 3c 2 this is c 2, ok. So here when lambda equal to 2 and f(x) is equal to 0 so in this case lambda equal to 2 equation number 11 is basically what, equation number 11 is minus c 1 plus 3 c 2 is equal to 0 to 1 f(x) dx.

Now here f(x) is 0 so it is basically 0, if you look at the second equation minus c 1 plus 3 c 2 equal to 0 to 1 s f(x) dx. So again this is 0 f(x) is 0 so it is equal to 0. So this basically these two equations is the same equation so I can write it c 1 equal to 3 c 2.

So when we take c 1 equal to 3 c 2 a then we have y(x) equal to f(x) plus lambda I am writing c 1 as 3 c 2 so i can take 3 c 2 common and I can write it here 1 minus x.So if I represent f(x) is already 0 so this is 0. So i can write it this as a times 1 minus x where a is some constant here constant is 3 lambda c 2.

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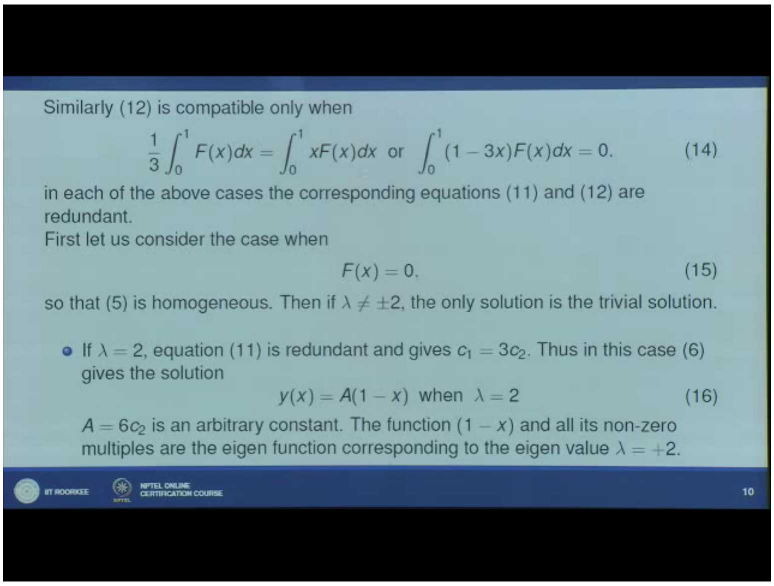

So I can write our solution here that if lambda equal to 2 and f(x) equal to 0 then in this case our solution y(x) is written as a into (1 minus x) where a is some arbitrary constant here it is given as lambda c 2 oh sorry 6 c 2. The function 1 minus x is called Eigen function corresponding to lambda equal to 2.

So here lambda equal to 2 is one of the Eigen value and the corresponding Eigen function is given as (1 minus x) and any non zero multiple of 1 minus x

Similarly look at the (quotient) look at the solution for lambda equal to minus 2 and f(x) equal to 0. So when lambda equal to minus 2 in that case.

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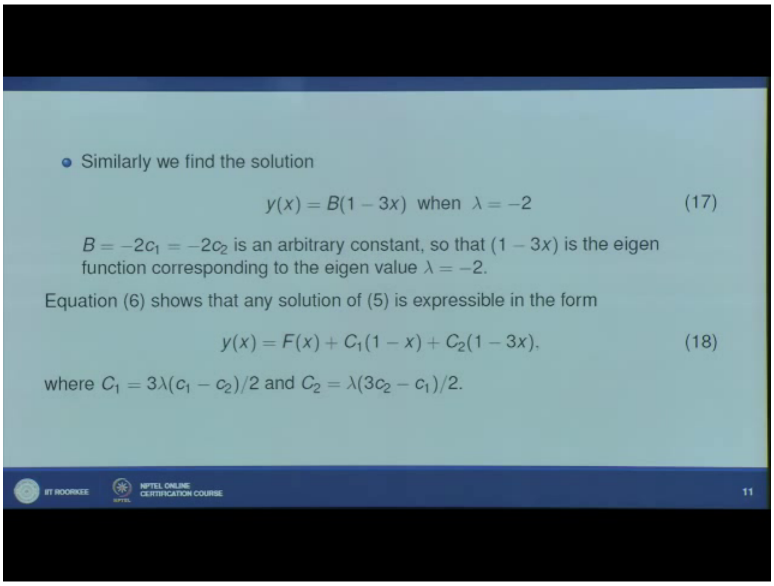

If you look at the equation number 12 here then we will get c 1 equal to c 2 because f(x) is 0 so c 1 is equal to c 2 in this case the solution is given here. So corresponding to lambda equal to minus 2 and f(x) is equal to 0 we have c 1 equal to c 2 and your solution is reduced to y(x) is equal to f(x) is simply 0 here f(x) is simply 0 here and c 1 is equal to c 2.

So c 1 I can take it out so it is lambda equal to minus 2. So minus 2 c 1is out c 1 and this is 1 minus 3(x). So I can write it here y(x) equal to some b 1 minus 3(x) where b is simply minus 2 c 1 or minus 2 c because here we have c 1 equal to c 2.So in this case where lambda is equal to minus 2 and f(x) is simply 0 then y(x) is given as b into 1 minus 3x.

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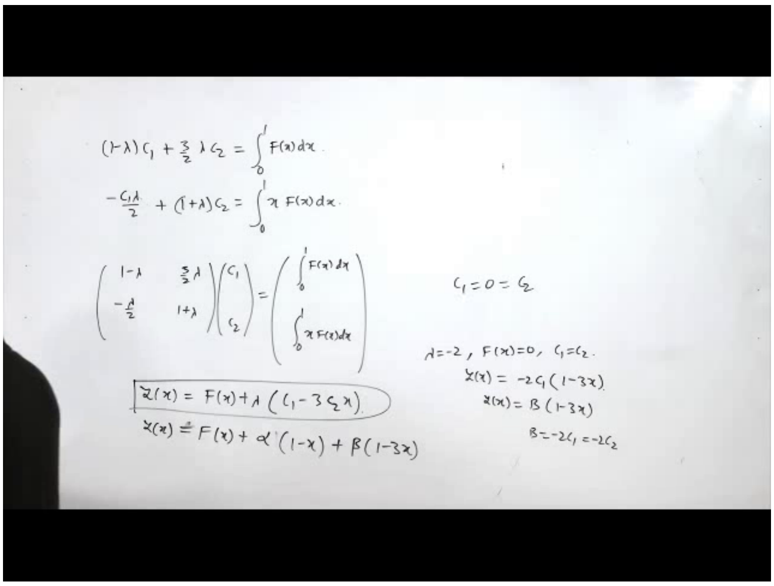

So I can say that lambda equal to minus 2 is an Eigen value and the corresponding Eigen function is given by 1 minus 3x and a non zero multiple of 1 minus 3(x). Now if you look at with this information I can say that our solution y(x) which is given here as f(x) plus lambda times c 1 minus 3(c 2) x. I can simply rewrite in terms of Eigen function that is (1 minus x) and (1 minus 3x).

So I can simply write this as write it here I can write it here as y(x) plus f of sorry equal to f(x) plus lambda times. Now I am going to write it here I am going to write alpha times say 1 minus x lambda time (1 minus x) plus you do not need this alpha here, alpha times (1 minus x) and beta times 1 minus 3(x) and you can easily find out the value alpha and beta from comparing the comparison of these two things.

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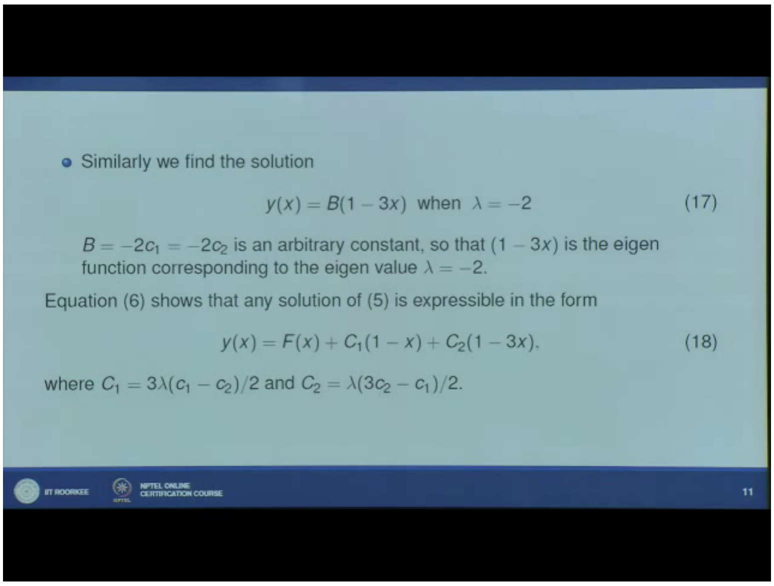

So here if you look at we can rewrite this solution y(x) as this, equation number 18 which says that y(x) can be written as f(x) plus c 1 times 1 minus x capital C times minus x plus capital C 2 times 1 minus 3x where you can find out capital C 1 and C 2 just like comparing the previous form of the solution and this form of the solution here capital C 1 is given as my 3 lambda C 1 minus C 2 by 2 and capital C 2 is given as lambda times 3 C 2 minus C 1 divided by 2.

So it means that your solution can be written as function f(x) which is non homogeneous terms plus linear combinations of Eigen function corresponding to Eigen values here. So solution is given by this.

Now we can sarize this any solution of 5 can be expressed as the sum of f(x) and some linear combination of the Eigen function. Now look at the non homogenous case it means when f(x) is not equal to 0 then we know that if lambda is not equal to plus minus 2 then unique solution exists. But what happens when lambda plus 2 or minus 2.

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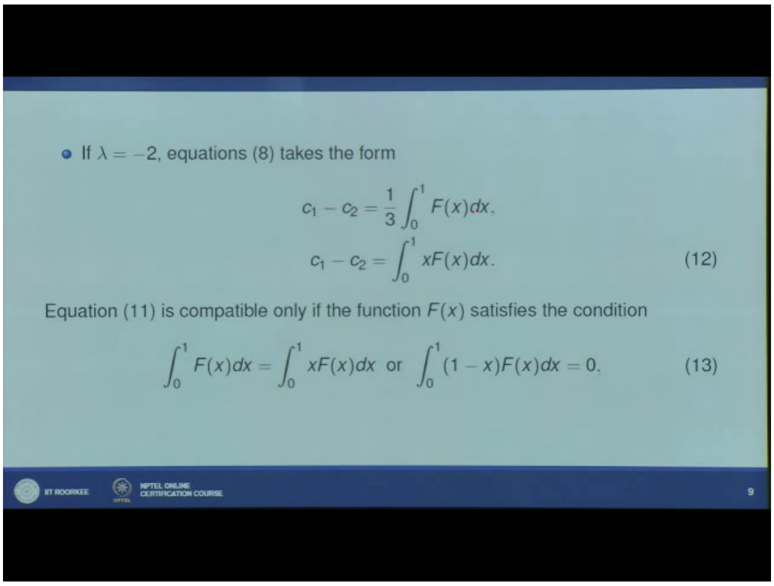

So if I look at lambda equal to 2 then in that case look at here this thing this is corresponding to lambda equal to 2. So lambda equal to 2 and f (x) is non zero then this equation 0 to 1 f(x) dx equal to 0 to 1 x f(x) dx which we have obtained from here. So this will make sense only when 0 to 1 (1 minus x) f(x) dx is 0.So it means that this function f(x) is orthogonal to 1 minus x in the interval 0 to 1.

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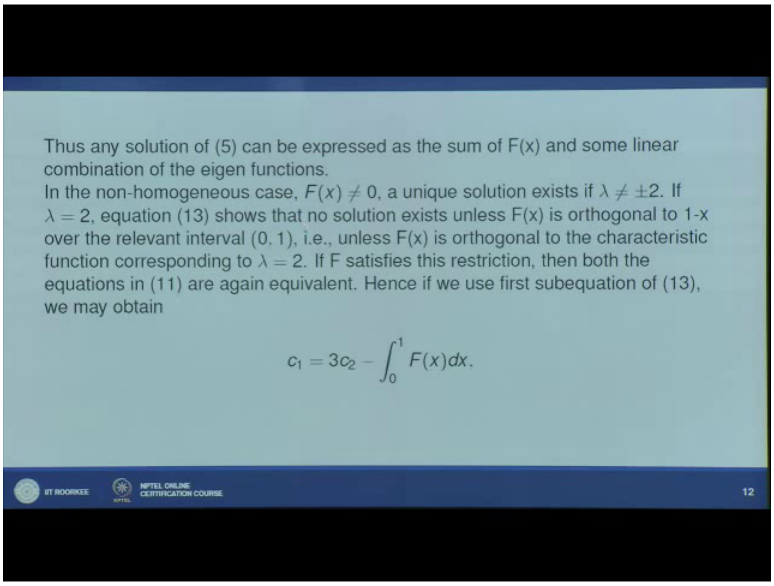

So I can write it here like this that if lambda equal to 2 this equation, equation for finding small c 1 and c 2 will give you solution when f(x) is orthogonal to (1 minus x) over the interval (0 to 1). Or I can say that f(x) is orthogonal to charesteristic function what is 1 minus x it is the characteristic function or Eigen function corresponding to lambda equal to 2. So we can say that for lambda equal to 2 we can find out c 1 and c 2 only when that f(x) is orthogonal to the Eigen function corresponding to lambda equal to 2.

Or I can write it that if satisfy this instruction then in that case you can use any one of the formula of eleven. So here if you look at in eleven if we have 0 to 1 f(x) dx is same as 0 to 1 s f(x) dx, so I can use any of these two integral I can write it c 1 equal to 3c 2 minus 0 to 1 f(x) dx, or I can write it this thing, ok.

In that case your let me here then I can write it the value of c 1 as in terms of c 2 as c 1 equal to 3 c 2 minus 0 to 1 f(x) dx. I am using the first sub equation of eleven. You can use second also no problem, right?

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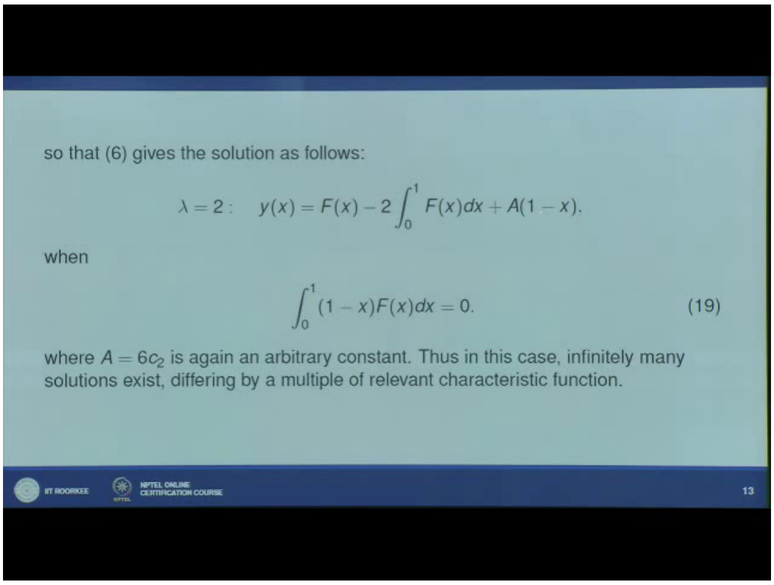

In that case lambda equal to 2 y(x) is given as f(x) minus 2 times 0 to 1 f(x) dx plus a times 1 minus x.Now here f(x) is a function which satisfy this property 0 to 1, 1 minus f(x) 1 minus x f(x) dx equal to 0.

So it means that this solution exist only when f(x) you have a function f(x) which is orthogonal to Eigen function corresponding to lambda equal to 2 is that ok. And here we already know that here we have infinite many solution because a is completely arbitrary here, a is 6 c 2 and c 2 is completely arbitrary. So here we can have infinite many solution.

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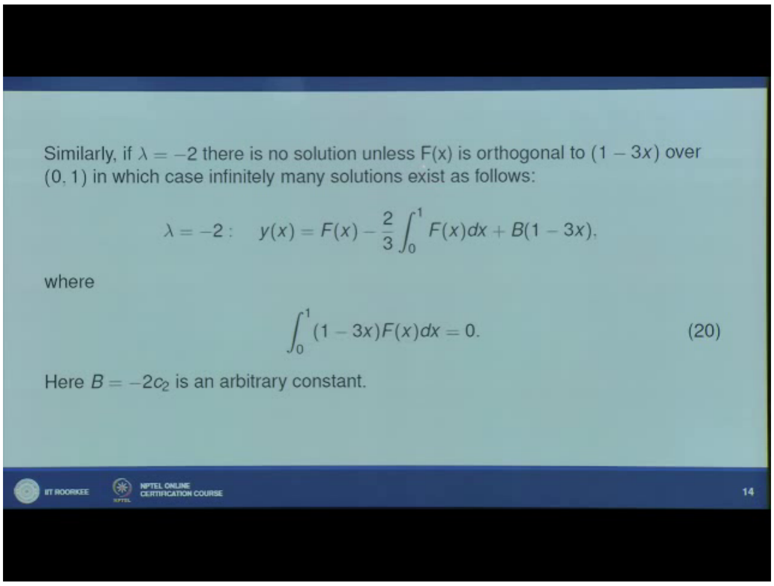

Now the same similar case we can consider for lambda equal to minus 2 so when lambda equal to minus 2 if you go back to your equation number this or we look at here. So if lambda equal to minus 2 so this will make sense if these two are equal, these two are equal means you are 0 to 1 1 minus 3x f(x) dx is equal to 0.

So it means that f(x) is orthogonal to 1 minus 3x, now what is this 1 minus 3x if you look at we have already calculated that 1minus 3x is an Eigen function corresponding lambda equal to minus 2 it means that if we use this information I can say that for lambda equal to minus 2 there is no solution unless f(x) is orthogonal to this. And what is this function it is an Eigen function corresponding to lambda equal to , lambda equal to minus 2.

So I can say that in case lambda equal to minus 2 y(x) is given as f(x) minus 2 times 2 by 3 0 to 1 f(x) dx plus b times 1 minus 3x. Again here f(x) is satisfying this condition orthogonal condition and b is minus 2c 2 or minus 2 c 1 whatever and c 2 is an arbitrary thing so this is going to be infinitely many solutions here.

So it means that when we have a coefficient matrix which is a singular matrix then we had to look at the Eigen values and Eigen function and if function capital F(x) satisfy certain orthogonality condition then only we can have it non trivial solution. And in this particular case we have infinite many solutions here .

If you look at some example of capital F(x) for which this condition holds an we have a solution like this. So if you look at I can take f(x) as say 1 minus x or you can say like this that here if I take f(x) equal to 1 minus x which is an Eigen function corresponding to lambda equal to 2. Then 20 will hold and we have a non trivial solution here, is that ok.

So in todays lecture we have seen that how to find out an solution of Fredholm integral equation of second kind when the kernel is separable kernel, Thank you for listening and we will meet again for next lecture, Thank you!