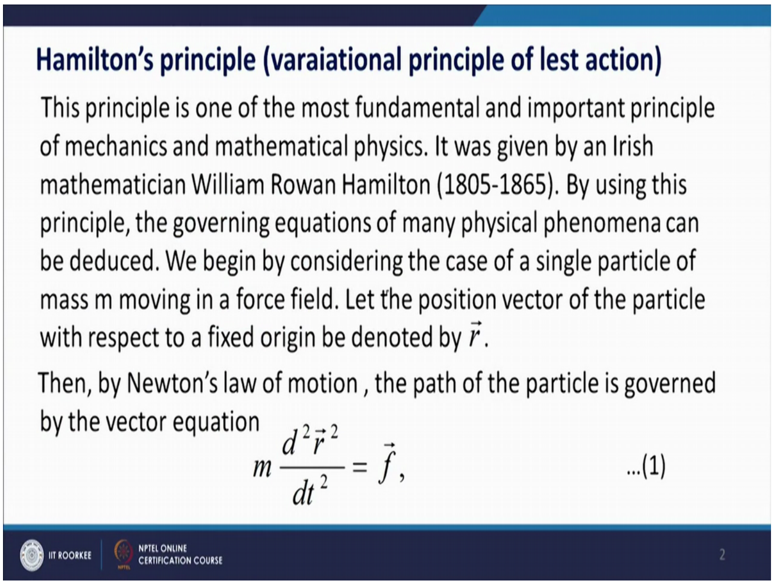
Integral Equations, Calculus of Variations and their Applications
Professor Dr. P. N. Agrawal
Department of Mathematics
Indian Institute of Technology Roorkee
Lecture 60
Hamiltons principle: Variational principle of least action

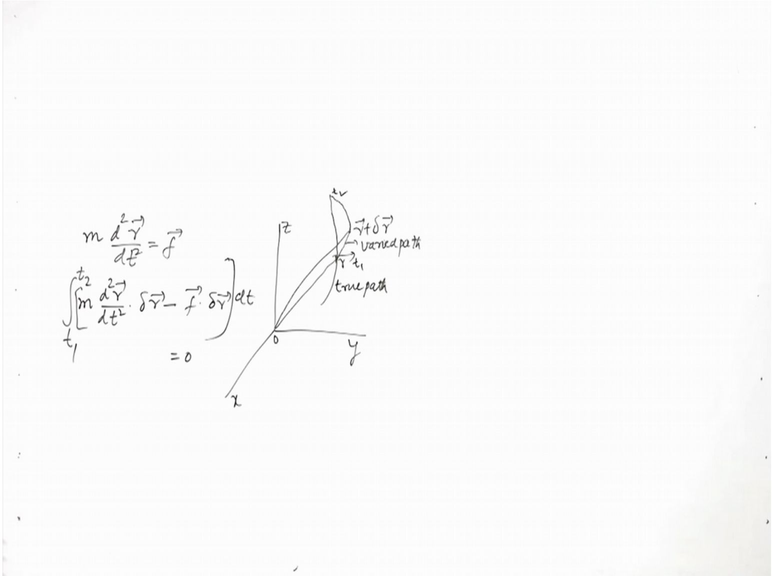
Hello friends I welcome you to my lecture on applications of calculus of variations where we will first consider the Hamiltons principle which is also called the variational principle of least action.

(Refer Slide Time: 0:34)



This principle is one of the most fundamental and important principle of mechanics and mathematical physics. It was given by an Irish mathematician named William Rowan Hamilton during the period 1805 to 1865. By using this principle, the governing equations of many physical phenomena can be reduced. Let us begin by considering the case of a single particle of mass m which is moving in a force field. And let us consider that the position vector of the particle with respect to a fixed origin be r.

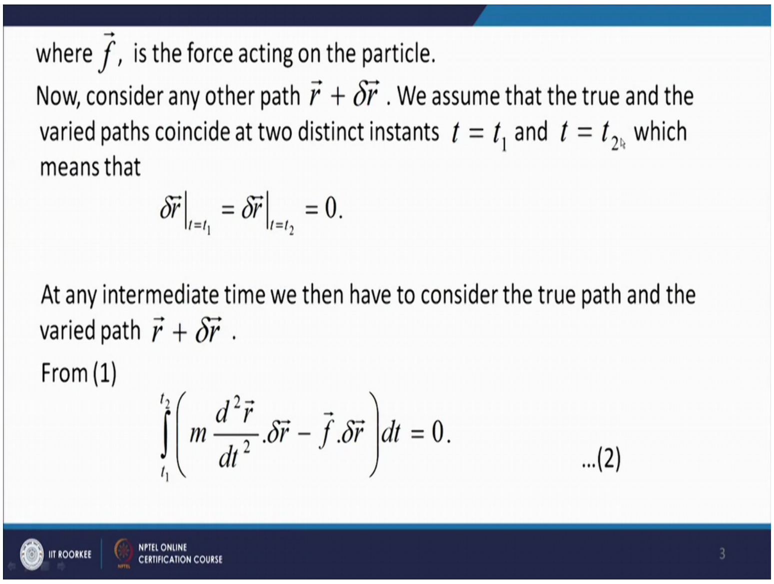
(Refer Slide Time: 1:10)

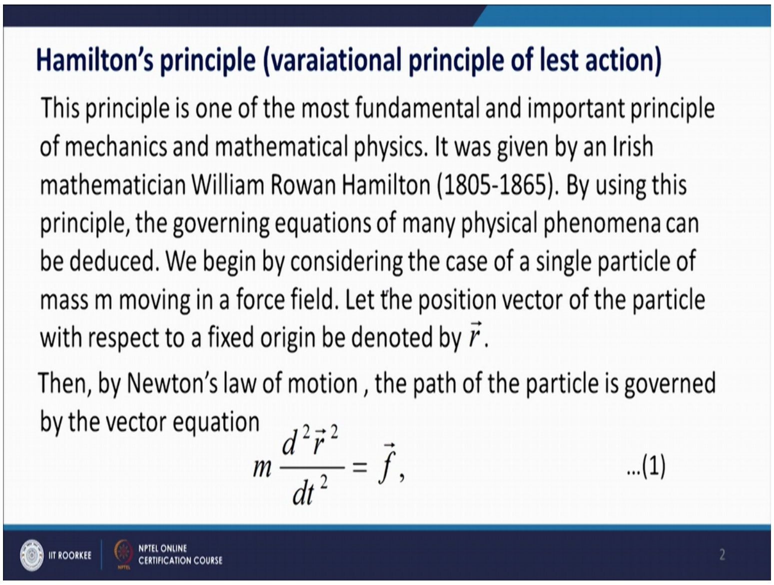


So let us say so let us say this is the portion vector of the particle, this is this is moving this is the path of the particle, this is the portion of the particle at any point at any time t and r be the position vector of the particle with respect to a fixed origin this is your x, y and z axis. So by Newtons law of motion the path of the particle is then governed by the vector equation m d square r by dt square equal to f, where f is the force acting on the particle.

So now let us consider any other path r bar plus delta r bar, let us consider any other path r bar plus delta r bar, let us consider any other path r bar plus delta r bar, we assume that the for true and the varied paths. So this one is the true path this is true path and this is varied path. So let us consider any other path r bar plus delta r bar, we assume that the true and the varied path coincide at two distinct instants t 1 and t 2. So let us say this path and this path they coincide at some instant t 1 and t 2, this is t 2 instant and this is t 1 instant.

(Refer Slide Time: 3:02)

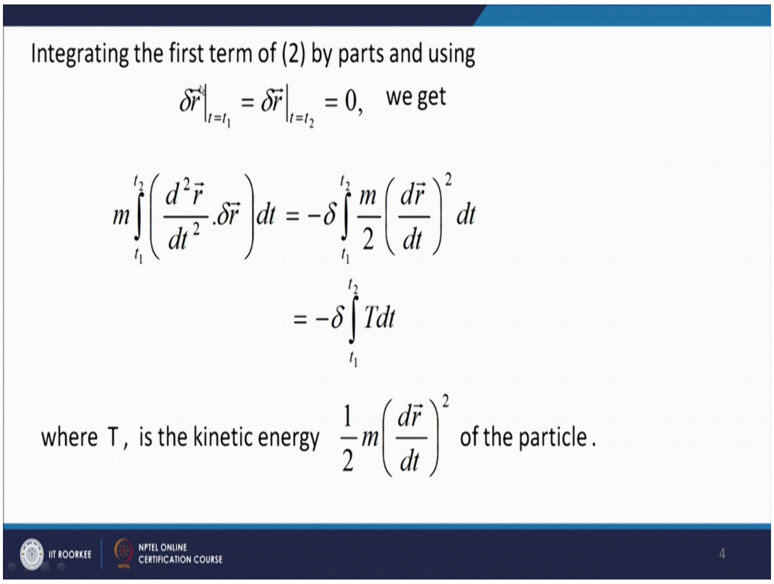


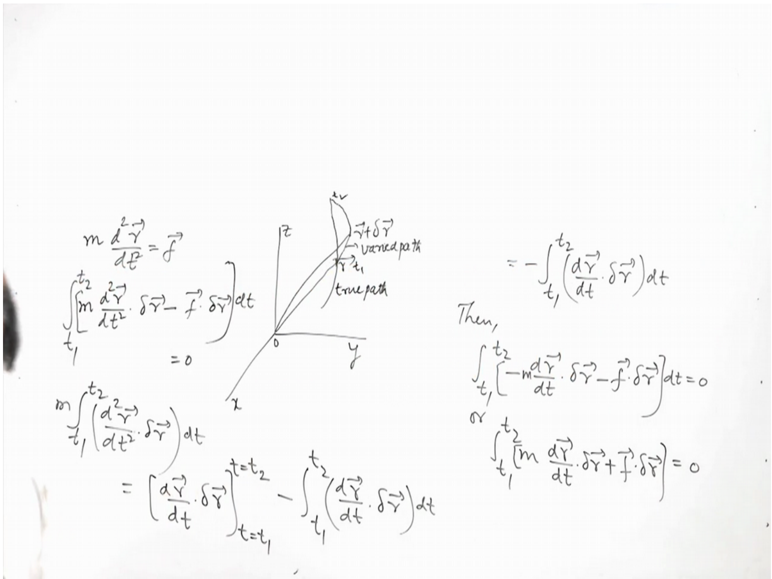


So the actual path and the varied path let us assume that they coincide at two distinct instants t equal to t 1 and t equal to t 2 which will means that delta r is equal to 0 at t equal to t 1 and delta r is equal to 0 at t equal to t 2. At any intermediate time we have to consider the true path and the varied path r bar plus delta r bar. Now from the equation 1, from the equation 1, m into d square r by dt square equal to f we can write this m times d square r by dt square dot dr is equal to f dot dr and let us integrate with respect to t with respect to dt, so we have let us integrate with respect to dt so we shall have this equal to 0.

So we take the dot product on both side with dr vector, delta r vector and then integrate with respect to t integrate with respect to t from t 1 to t 2, so what we get is this equation integral t 1 to t 2 m d square r by dt square dot delta r minus f dot delta r into dt equal to 0.

(Refer Slide Time: 4:42)

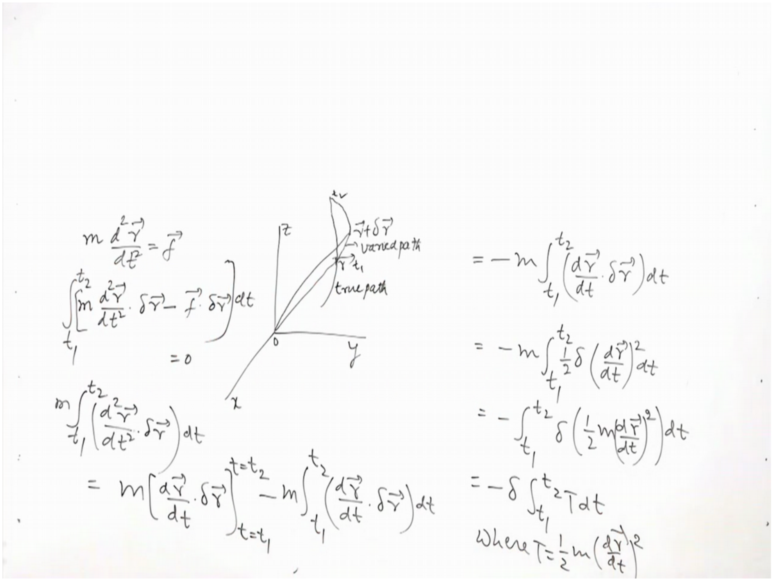




Now and the condition here is that let us apply the condition that delta r at t equal to t 1 and at t equal to t 2 is 0. Let us evaluate the first term, so integral t 1 to t 2, m times let us put m outside d square r by dt square dot delta r let us integrate this with respect to t by parts. So we have dr by dt integral of this d square r by dt square is dr by dt into dot delta r at t equal to t 1 and t equal to t 2 minus integral t 1 to t 2 dr by dt dot delta r into dt.

Now delta r is equal to 0 at t equal to t 2 and delta r is equal to 0 at t equal to t 1, so this will give you minus integral t 1 to t 2 dr by dt dot delta r dt. So let us put this value in this equation, so then we shall, have integral t 1 to t 2 minus dr by dt dot delta r this is m times m times minus f dot delta r dt equal to 0 or we can say integral t 1 to t 2 m times dr by dt delta r plus f dot delta r equal to 0 and this can be then written as so m f is equal to m d square r by dt square, so what we can say this is this is equal to when we integrate it by parts we get this so d square r by dt square (okay okay sorry this is like this not not like this).

(Refer Slide Time: 7:38)



So we get this so this is equal to okay so m times this I have not written m here, so this is minus m times and then I can write it as minus m times t 1 to t 2 this is delta r dr by dt whole square we can write like this or we can write it as minus t 1 to t 2 delta 1 by 2 dr by dt m dr by dt whole square like this. So this is equal to this and or I can write it as minus delta times the T is the kinetic energy, where T is equal to 1 by 2 m dr by dt whole square.

So T is the kinetic energy, so I can write the first expression can be written like this minus delta t 1 to t 2 T dt and let us substitute this value in that in this equation in this equation here then what do we get we get the following.

(Refer Slide Time: 9:26)

Substituting the value of the first term in (2), we get

$$\int_{t_1}^{t_2} \left(\delta T + \vec{f}.\delta\vec{r}\right) dt = 0. \qquad ...(3)$$

This is Hamilton's principle for motion of a single particle. If the force is conservative, we can write it in a more concise form.

Let $\vec{f} = X\hat{i} + Y\hat{j} + Z\hat{k}.$

We know that a force field $\vec{f}$ is conservative if

$$\vec{f}.d\vec{r} = Xdx + Ydy + Zdz$$

is the differential $d\phi$ of a single valued function ϕ. Also, then $\vec{f} = grad\phi.$

Integrating the first term of (2) by parts and using

$$\delta\vec{r}\big|_{t=t_1} = \delta\vec{r}\big|_{t=t_2} = 0, \quad \text{we get}$$

$$m\int_{t_1}^{t_2} \left(\frac{d^2\vec{r}}{dt^2}.\delta\vec{r}\right) dt = -\delta \int_{t_1}^{t_2} \frac{m}{2}\left(\frac{d\vec{r}}{dt}\right)^2 dt$$

$$= -\delta \int_{t_1}^{t_2} T dt$$

where T, is the kinetic energy $\frac{1}{2}m\left(\frac{d\vec{r}}{dt}\right)^2$ of the particle.

where $\vec{f}$, is the force acting on the particle.

Now, consider any other path $\vec{r} + \delta\vec{r}$. We assume that the true and the varied paths coincide at two distinct instants $t = t_1$ and $t = t_2$ which means that

$$\delta\vec{r}\big|_{t=t_1} = \delta\vec{r}\big|_{t=t_2} = 0.$$

At any intermediate time we then have to consider the true path and the varied path $\vec{r} + \delta\vec{r}$.

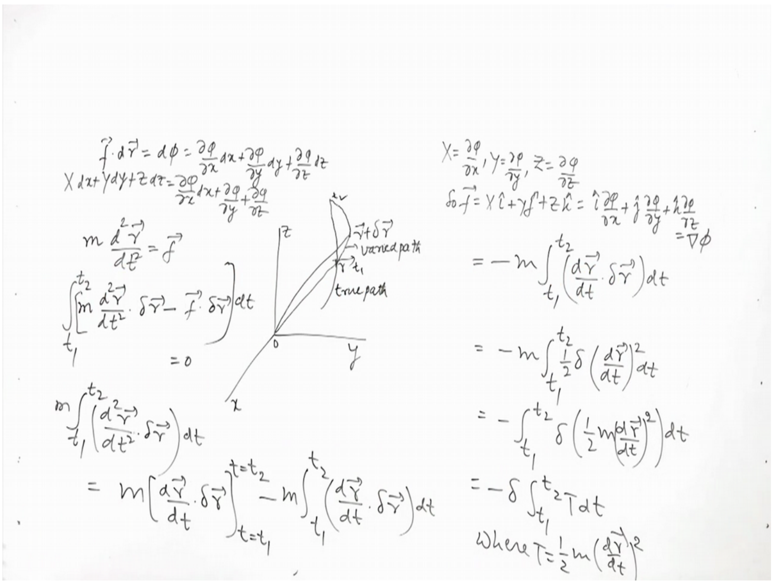
From (1)

$$\int_{t_1}^{t_2} \left(m\frac{d^2\vec{r}}{dt^2}.\delta\vec{r} - \vec{f}.\delta\vec{r}\right) dt = 0. \qquad ...(2)$$

So substituting the value of the first term, the value of the first term this this this is the first term the value of the first term is minus delta integral t 1 to t 2 T dt where T is the kinetic energy in equation 2 in equation 2 means here here we put the value of this first term. So then what we get is this integral t 1 to t 2 delta T plus f dot delta r dt equal to 0. This is called the Hamiltons principle for motion of a single particle.

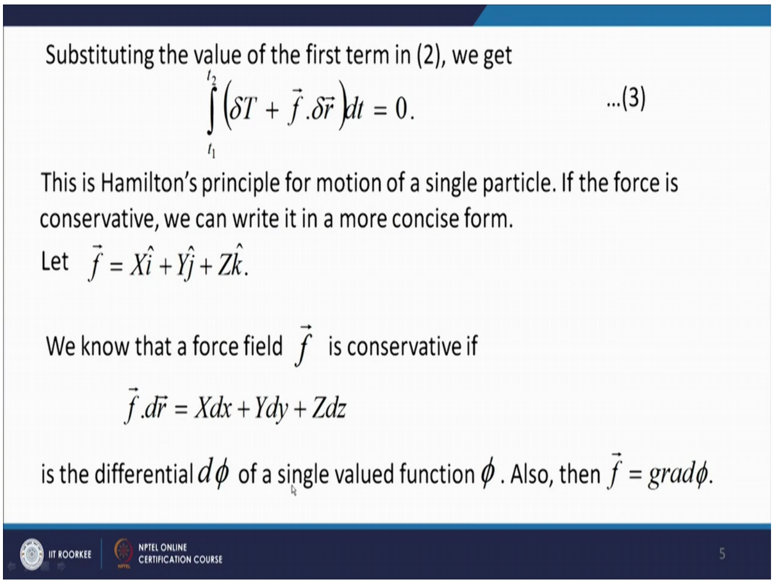
Now if the force is conservative we know that the force is conservative provided can be written as the gradient of a scalar function. So if the force is conservative we can write it in a more concise form that is we can let us say f is equal to Xi plus Yj plus Zk, X, Y, Z are the components of the force in X, Y, Z directions. Then we know that the force filed is conservative if f dot dr, f dot dr is dr is dXi plus dYj plus dZk, so f dot dr will be Xdx plus Ydy plus Zdz this is equal to differential of a single valued function phi.

(Refer Slide Time: 10:40)



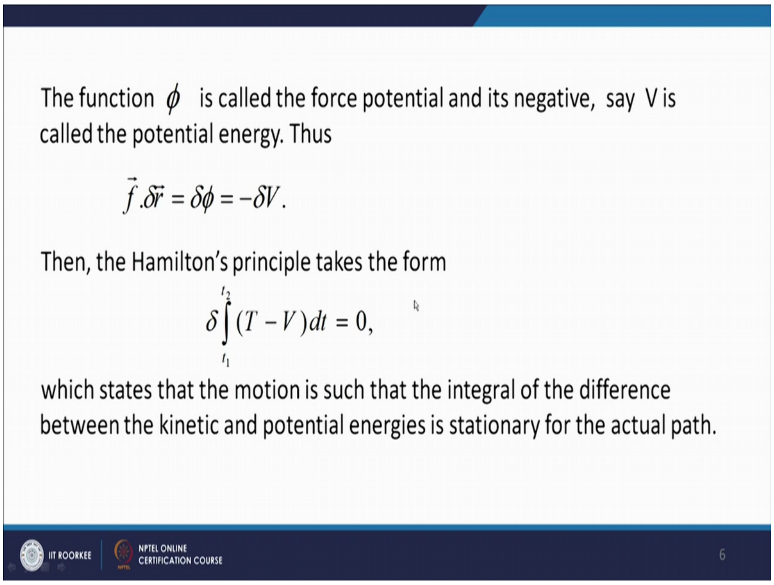
Now if f is f dot dr is equal to d phi, f dot dr is equal to d phi, so d phi is equal to delta phi over delta x dx plus delta phi over delta y dy delta phi over delta z dz, f dot dr is Xdx plus Ydy plus Zdz, so this is equal to delta phi over delta x dx plus delta phi over delta y dy delta phi over delta z dz, which means that X is equal to delta phi over delta x, Y is equal to delta phi over delta y and Z equal to delta phi over delta z and so the (field) force which is Xi plus Yj plus Zk this is equal to i delta phi over delta x plus j delta phi over delta y plus k delta phi over delta z which is nothing but gradient of a (phi) grade phi.

(Refer Slide Time: 12:06)



So we know that the force field f is conservative if f dot dr is the differential d phi of a single valued function phi and then f is equal to grade phi.

(Refer Slide Time: 12:12)



Substituting the value of the first term in (2), we get

$$\int_{t_1}^{t_2} \left(\delta T + \vec{f}.\delta\vec{r}\right)dt = 0. \qquad ...(3)$$

This is Hamilton's principle for motion of a single particle. If the force is conservative, we can write it in a more concise form.

Let $\vec{f} = X\hat{i} + Y\hat{j} + Z\hat{k}$.

We know that a force field $\vec{f}$ is conservative if

$$\vec{f}.d\vec{r} = Xdx + Ydy + Zdz$$

is the differential $d\phi$ of a single valued function ϕ. Also, then $\vec{f} = grad\phi$.

IIT ROORKEE | NPTEL ONLINE CERTIFICATION COURSE 5

So the function phi is called the force potential and its negative, let us say the negative of phi is V, so V is called the potential energy and thus f dot delta r which is equal to delta phi is equal to minus delta V. And then the Hamiltons principle takes the form, so Hamiltons principle which was this integral t 1 to t 2 delta T plus f dot dr dt equal to 0, it takes the form delta t 1 to t 2 T minus V dt equal to 0, which states that the motion is such that the integral of the difference between the kinetic, T is kinetic energy and V is potential energy. So the integral of the difference between kinetic and potential energy is stationary because delta of this integral is 0, so it is stationary for the actual path.

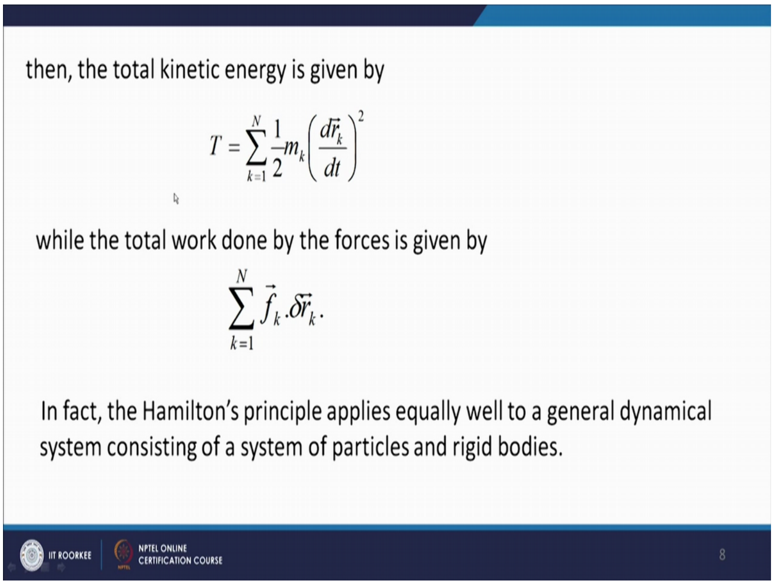
(Refer Slide Time: 13:04)

The difference T-V is called the Lagrangian function. If the time interval is sufficiently short, then it can be proved that the Hamilton's integral

$\int_{t_1}^{t_2} (T - V)dt$ is necessarily a minimum. In this form the Hamilton's principle is called the principle of least action.

The above study can be easily extended to a system of N particles by summation, and to a continuous system by integration. For example, if the position vector of the ith particle of mass m_i be $\vec{r}_i$ and if m_i is subjected to a force $\vec{f}_i$,

IIT ROORKEE | NPTEL ONLINE CERTIFICATION COURSE 7



And the difference T minus V is called the Lagrangian function. If the time interval is sufficiently short then it can be proved that the Hamiltons integral which is integral t 1 to t 2 T minus V dv is necessarily a minimum. In this form the Hamiltons principle is called the principle of least action.

Now this study can be extended to a system of N particles in the case of discrete system we will take summation and the case of continuous we will take integration. So for example, if the position vector of the ith particle of mass m i be r i vector and if m i is the subjected to the force f i bar then we shall have the total kinetic energy will be T equal to sigma k equal to 1 to N 1 by 2 m k dr k by dt whole square.

While the total work done by the forces will be sigma k equal to 1 to N f k dot delta r k. So in fact the Hamiltons principle applies equally well to a general dynamical system which consist of a system of particles and rigid bodies. So this is one example of of the calculus of variations. In our next lecture we shall take up some more examples on calculus of variations, thank you very much for your attention.