Integral Equations, Calculus of Variations And Their Applications
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Lecture-06
Fredholm Integral Equation with Separable Kernels: Theory

Hello Friends, today in this lecture we'll going to discuss Fredholm integral equation with the separable or degenerate kernels, so this is the topic of the today's lecture, so just recall some basic things that an equation of the form alpha x y of x equal to f of x plus lambda a to b, k (x, t) y (t) dt

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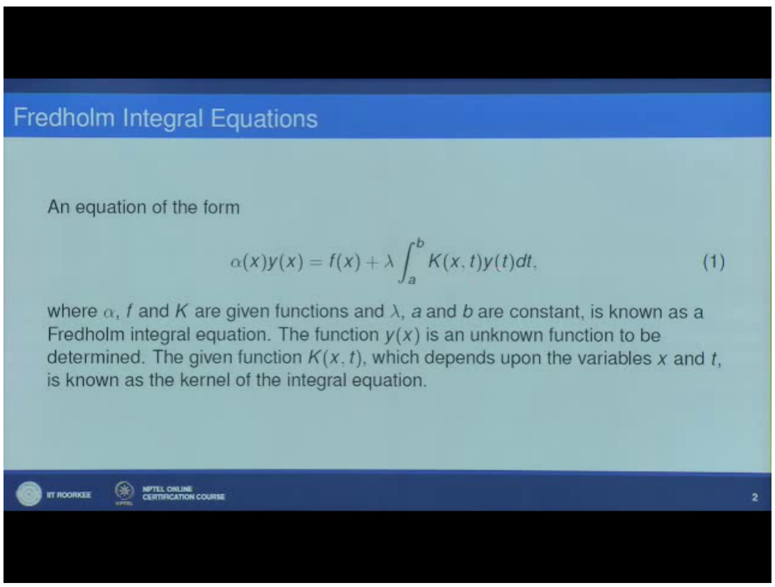

This equation unknown function y x is also coming in under the sign of integration and outside, this kind of integration is known as integral equation.

And if the limit is constant we say that it is a Fredholm integral equation, so here we are assuming that this alpha f and k are some given function and lambda a and b are some constant value and we call this equation one as Fredholm integral equation and this y (x) is unknown function which we are going to find out and the given function k x t which depend upon the variable x and t is known as the kernel of the given integral equation.

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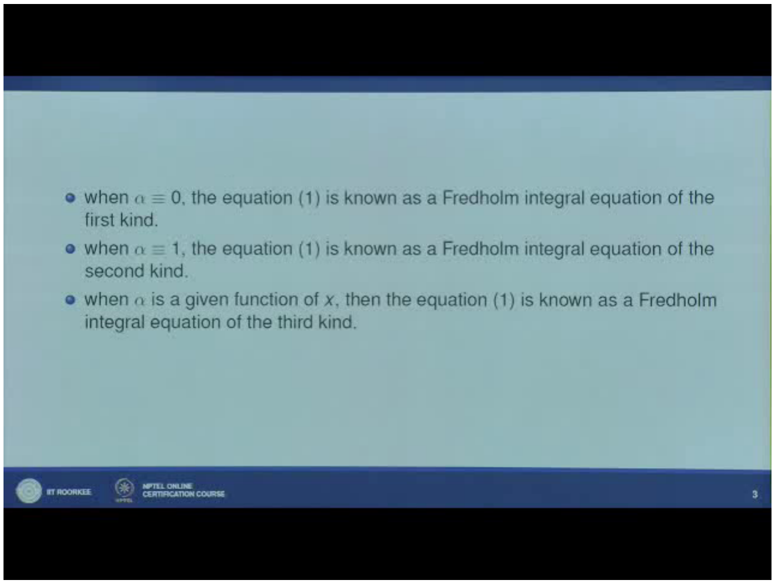

On the basis of this function alpha of x here we can categorize this integral equation into three kind, first is when this alpha x is (01:36 directly) equal to zero, this equation one is known as Fredholm integral equation of the first kind, so it means that here this term is simply vanish and we say that f (x) plus lambda a to b k (x, t) y (t) d t is equal to zero,

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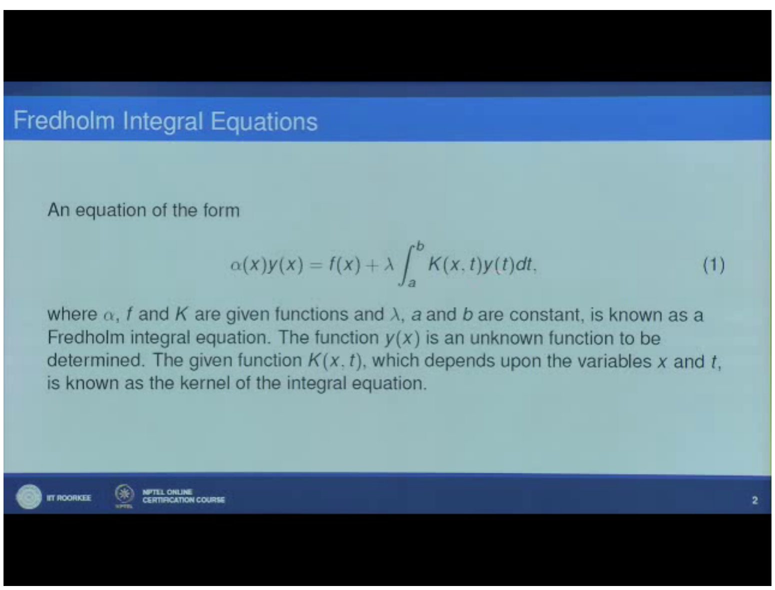

We call this as Fredholm integral equation of first kind, and if this alpha x is an ideally equal to one. The constant value one then this is known as Fredholm integral equation of the second kind and if alpha is not a constant value but is a function of x then this integral equation is known as

Fredholm integral equation of the third kind and generally in practice we discussed only the first and second kind generally in a very less occasion we discussed the Fredholm integral equation of third kind.

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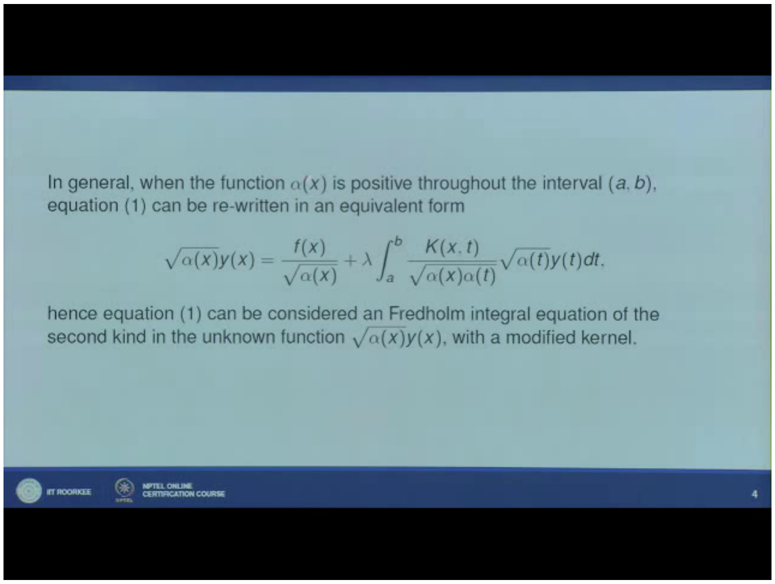

The reason we can say that if we can take alpha x is positive. Or we can say same sign without lots of generating we are assuming that alpha x is positive through the interval a and b then you can take, we can simplify our integral equation which is convertible into Fredholm integral equation of second kind, just look at here.

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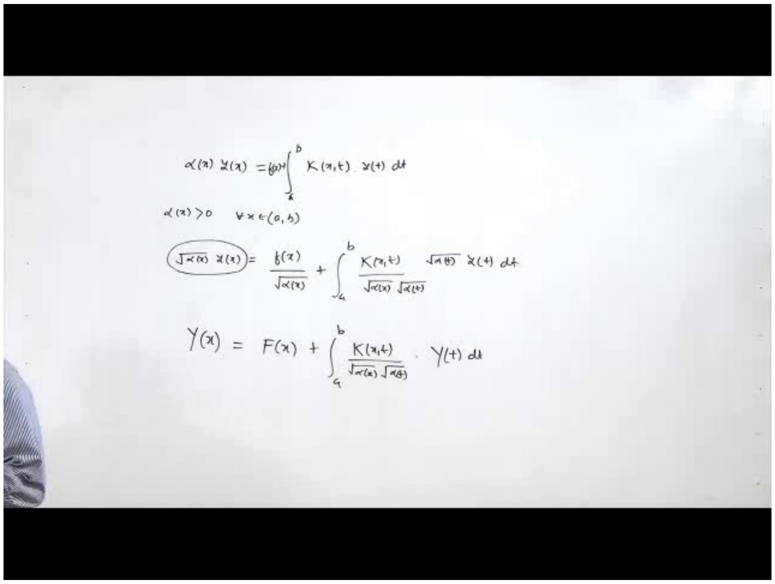

What we try to do here, we have alpha of x, y (x) is equal to f of x here plus a to b is limit k of xt and here we have y of t here.

So what we do here, since alpha x we are assuming that it is all the time positive for all x between this interval a to b then we can write it here as alpha x and y of x here divide by root of alpha x which we can do f x of root of alpha x here plus a to b and I can take inside because this integration is with respect to t, so under root alpha of x here and here I can divide by alpha of under root alpha t here and under root alpha t we can write it here y of t, d of t.

So if you look at, if we assume this as new variable y of x then it is given as y of x equal to f of x plus a to b, k of x of t divided by root of alpha x, root of alpha t here and this is y of t, d of t, so it means that if we assume that your alpha x is having the same sign or we can say without loss will generate that it is positive throughout the interval then Fredholm integral equation of third kind maybe converted into Fredholm integral equation of the second kind.

So this is the reason why we discuss Fredholm integral equation of the second kind in mode of the discussion, so now let us consider right now a very simple case of kernel kexec, we take kernel kexec in this form that we take k (x, t) here as separable or degenerate kernel.

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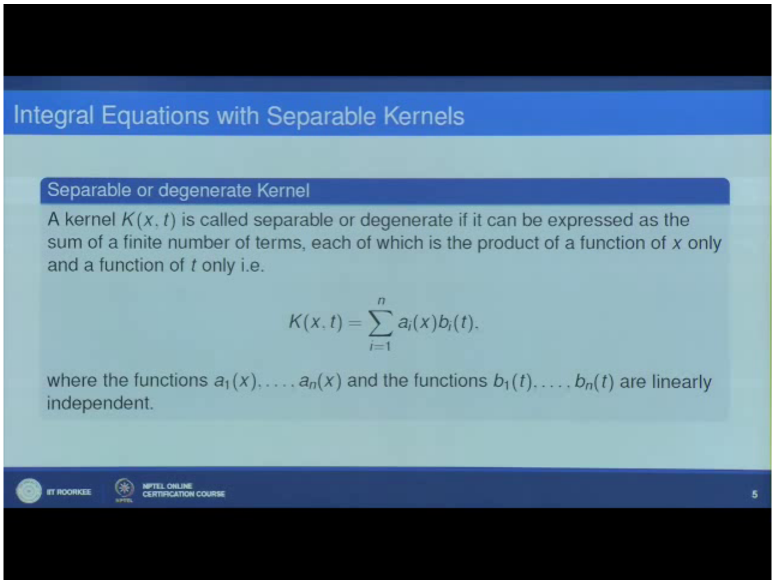

So here we are considering this Fredholm integral equation of second kind when the kernel is given in separable or degenerate form, so what is separable and degenerate form.

So we can simply say that a kernel k x t is called separable or degenerate if it can be expressed as summation of finally many terms which is written in terms of function of x into function of t here, so here I am assuming that k x t is given as here a i x is a function of x only and b i t is a function of t only, and I am writing here k x t as summation of these and terms, I equal to one to n, a i x, b i t, and here we can assume that all these a i x are linearly independent.

In a same way we can say that all b i t, b one t to b n t are linearly dependent, if it is not then we can use into lesser sum of the similar kind of event, so if we say that if these are not linearly independent, so it means that one of the, it can be written in terms of other pies, so for example if i assume that a one to a n x are linearly dependent, so it means one of a i a k x can be written as been a combination of other, so we can reduce this sum to lesser, say summation ok.

So that’s why without loss of generality we are assuming that whatever present here is representing the linearly independent function, so here A one to a n’s are linearly independent, similarly b one two b n’s are linearly independent, so when we have this separable kernel then how we can solve Fredholm integral equation of the second kind,

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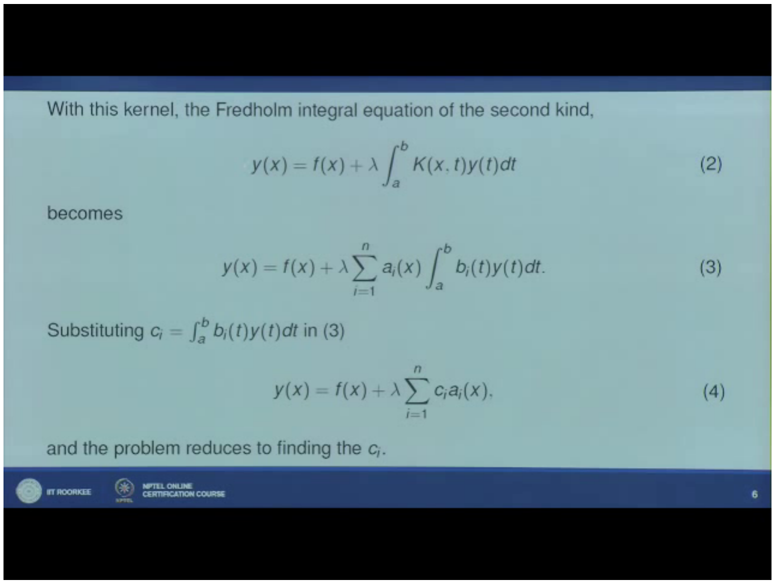

So if you look at with this kernel we have the Fredholm integral equation of the second kind. And in place of k x t now we are writing the summation i e equal to one to n, a i x b i t, now this integration with respect to t, I can take this a i x common and since it is find it’s sum we can interchange the summation and integral sign, so here we right a to b, b i t, y of t d t, now if you look at this quantity is what a to b, b i t, y (t) d t, this quantity is going to be a constant value, so if i denote this constants.

Say c i then we can write, substituting this c i as a to b, b i t, y (t) d t in this thing then our solution y x can be written as f of x plus lambda times, this is c i, c i into t i x, so it means that in this particular case when kernel is given in separable form if solution s given in this form, y of x equal to f of x plus lambda times i equal to one to end, c i e i x, the only thing left here is to find out these constants c one to c n, so how to find out these constants c one to c n.

So we can find out, we can get the solution, so for this what we do, we already know the form of the solution, you take this solution defined in this and put it back into your equation number 3.

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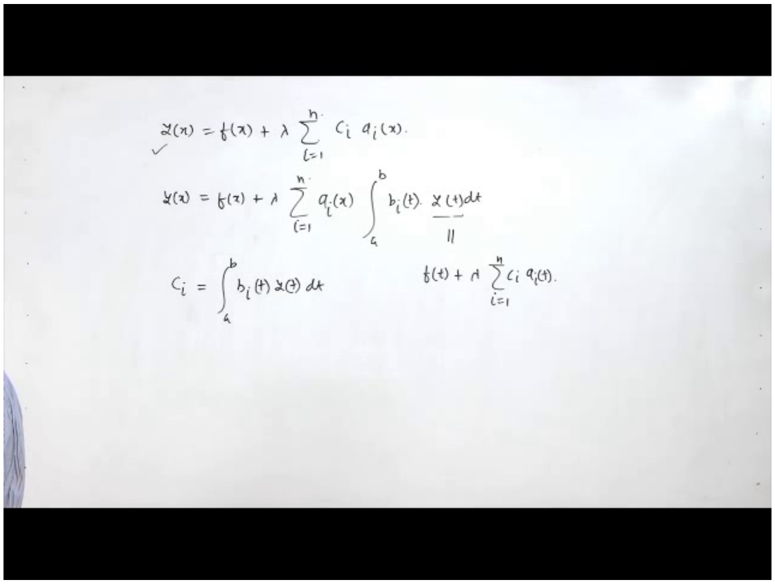

So here we have solution y of x is equal to f of x plus lambda times summation i e equal to one to n and c i, a i x.

Now your equation is what y of x is equal to f of x plus lambda times summation a i x, i e equal to one two n here and this is integral a to b, b i t y (t), d t.

Now y of t you can evaluate from this and here we have c i is equal to a to b, b i t y (t) d t, so what we do here we can write it y(t) d t as f of t plus lambda times summation i e equal to one to n c i or a i t, we can put the value of y t as this and simplify it,

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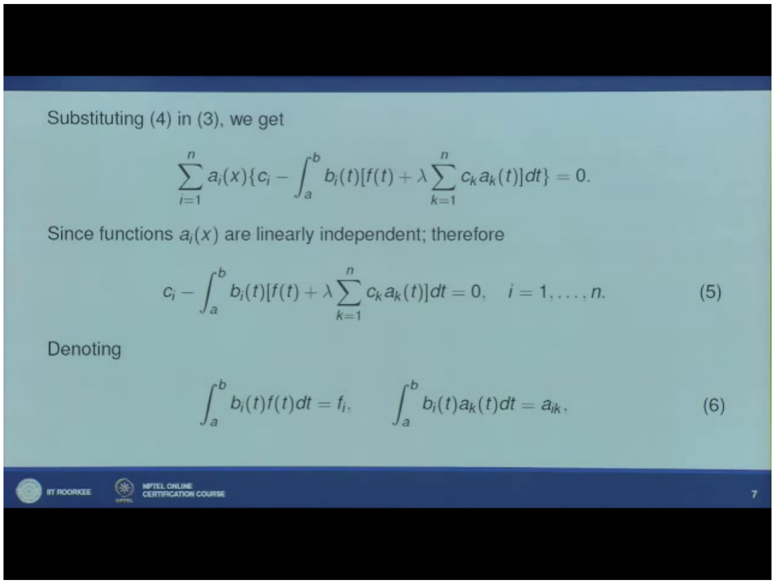

Now here when you simplify it will be reducing to this, summation i equal to one to n, e i x, here if you look at this, we are assuming a different as dummy variable say k so that it should not.

So k is from to n c k, a k t, so if we use this we have this, summation i equal to one to n e i x into this quantity, if you simplify you will get this,

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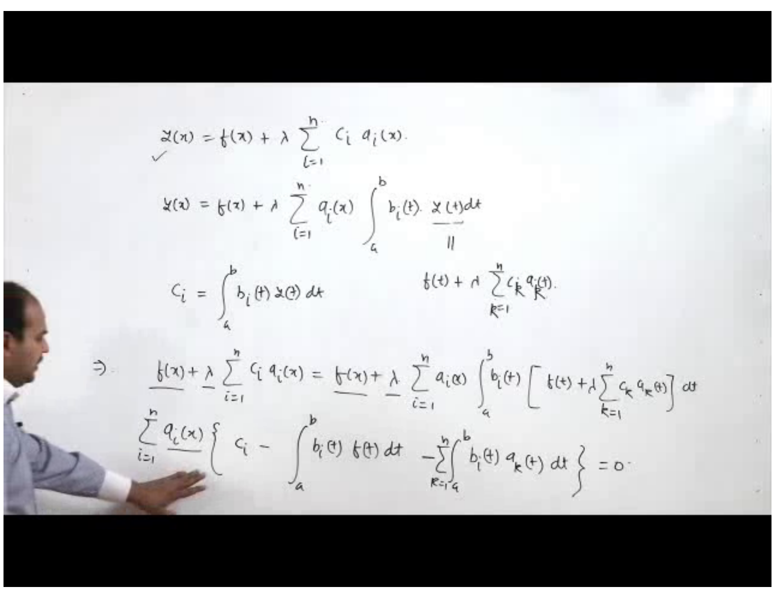

How we can get this let us look at here, I am writing y x here as this f of x plus lambda times summation i e equal to one to n, c i a i x here equal to f of x plus lambda times summation i e equal to one to n, a i x here, a to b, b i t here and in place of y t, I am writing f of t plus lambda times since it is already inside.

So I cannot use the same, so let use k equal to one to n, c k a k t and d of t and when you simplify this fx, this will be cancelled out here, your lambda will also simply gone and I can write it here that summation a i x, i e equal to one to n, if you look at the coefficient of this, coefficient of this is c i minus, here i will get what, I will get this quantity, so let me right it here a to b, b i t f of t here plus f of t dt here plus, I am taking here so minus a to b, b i t and summation here.

I will write it here, summation k e equal to 1 to n, b i t and a k t, d of t equal to zero, anything we have missed, a i x, it’s ok b i t, a k t, a k. So we have this thing and I can write it like this, so this is given here, now this is written here, now here we are using the fact that this a i x are linearly independent, so if they are linearly independent means the coefficient of a i x will be nothing but zero, so here this equation c i these are the coefficient here.

So this will equal to c i a to b, b i t, f t plus lambda times summation c k a k t, d t is equal to zero here, and if we simplify this further by saying that a to b, b i t f of (t) dt is denoted as f of i and a to b b i t, a k (t) d t is equal to a i k then we can write down the simpler form of equation number five as this, so here I can write it c i minus lambda times k equal to one to n, a i k c k equal to f of i.

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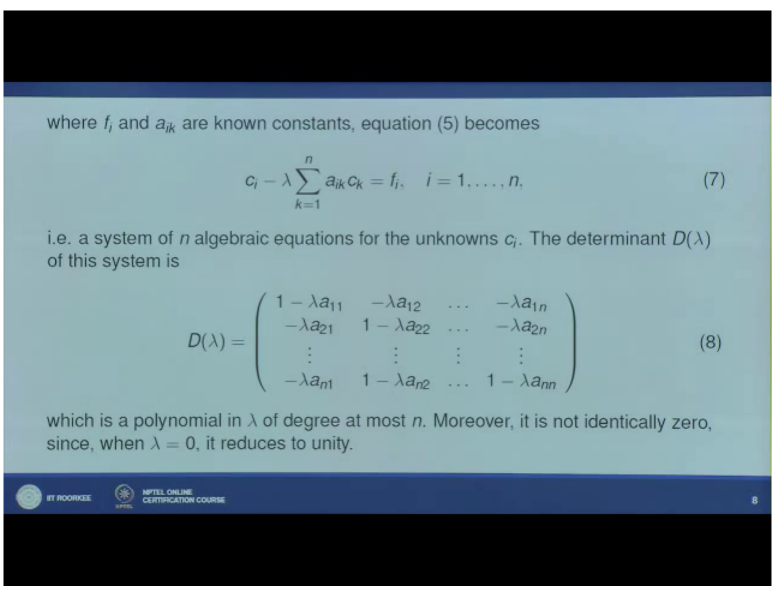

So what I am trying to do here since, just I am summarizing here.

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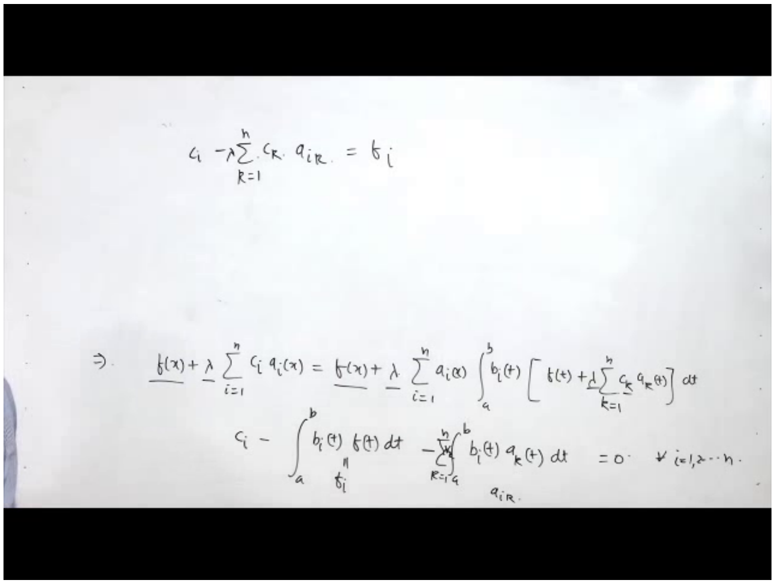

Since a i x are all linearly independent, so this will b e simply zero, so this will be zero means, I am just writing here, equal to zero for all i equal to one, two up to n, is it ok and this I am writing as b i t, this I am writing as f i, so let me erase this, I am writing this as f of i and this I am writing b i t a k t as a i k, so let me write it here, this is I am writing here a i k, so I can simplify this, so here I can write it here c i minus summation some lambda is missing here.

I think lambda is there, some lambda is there, so let me write it here lambda c k, lambda c k is missing here, this is lambda c k here, so let me write it c i lambda c k and we have a i k here and k is from one to h here is equal to f of x, so that is what is written as equation number seven here is it ok, so now this is a simple algebraic equation which we can solve by your knowledge of linear algebra, but here I just want to put one another way to find out this c i.

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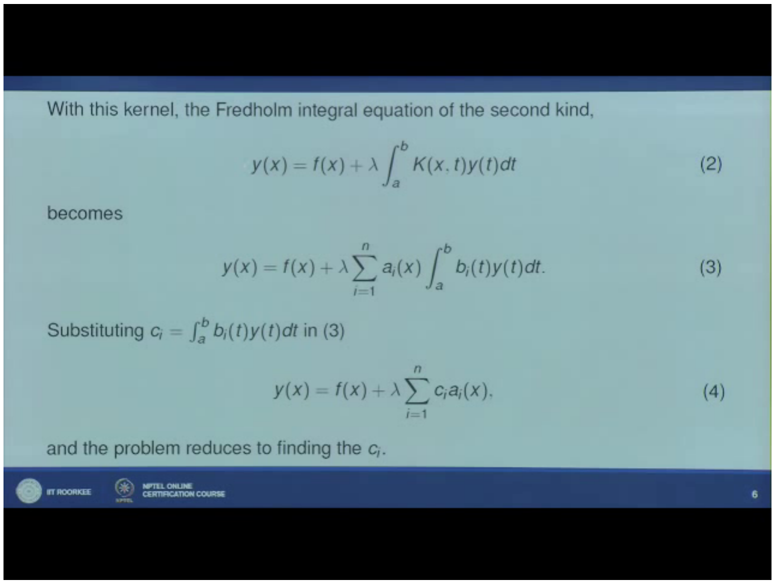

If you look at what we did here, once we have a solution here we have put it back to three, but rather than putting it back into three we can simply right it here, look at this the coefficient c i is given by c i equal to a to b b i t y t, so if you use y t which is given by this, I can write it y t equal to f of t plus lambda times i equal to one to n, c i a i t if you put it back you will get what let me write it here.

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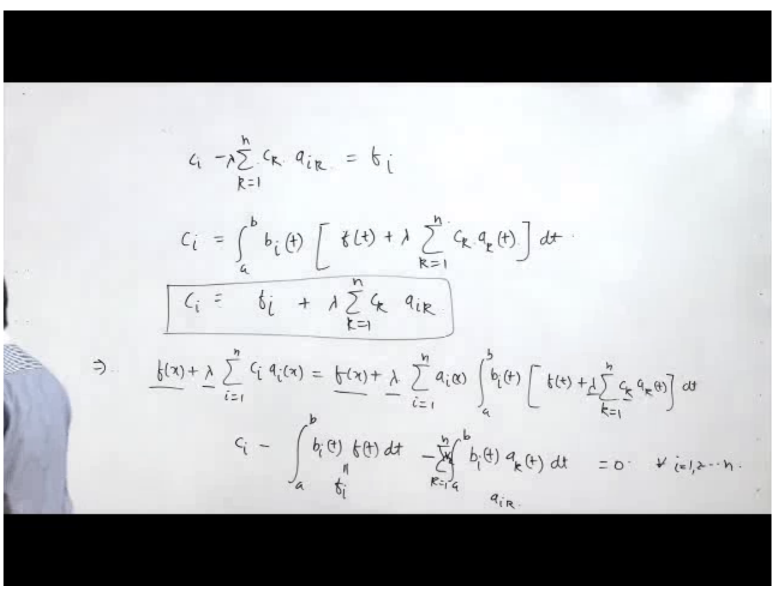

These are your c i and I am writing a to b b i t and in place of f t I am writing what, sorry y t I am writing f of t plus lambda times summation k equal to one two and I am writing k equal to one to n, c k a k t and this is d t and if you simplify you will get what, if you take it this here a to b b i t f of t as f of y so this is f i plus lambda times summation c k and b i t a k t we are denoting as a i k, so this is a i k, k is from one to n, so here we get the same equation.

So c i equal to f i plus lambda times k equal to one to n c k a I k which is same as what we have already obtained here, so this is an alternative way for obtained the equation for this c i’s,

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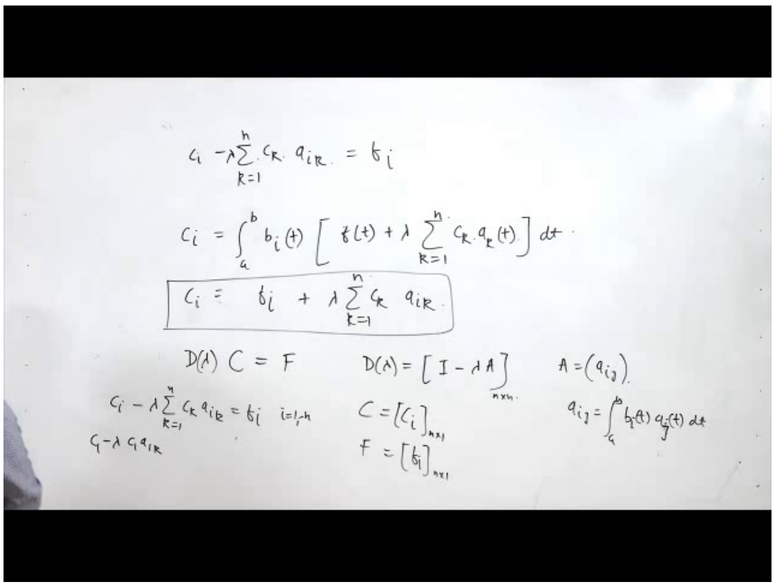

once we have c i whether you obtain through this equation number three or values from the coefficients, so here what we have seen here we can simplify the system of linear equation into this part d lambda, c equal to f.

So here your d lambda is i minus lambda a where a matrix is given by a i j and c is your c i and f is nothing but small f i, basically we are writing in this particular format so we are writing here say i minus, so here we are writing here c i minus lambda summation k equal to one to n c k a i k f of i for every i equal to one to n, so if I write it for one so it is c one minus lambda, I will write it here c one a i c one k c one one.

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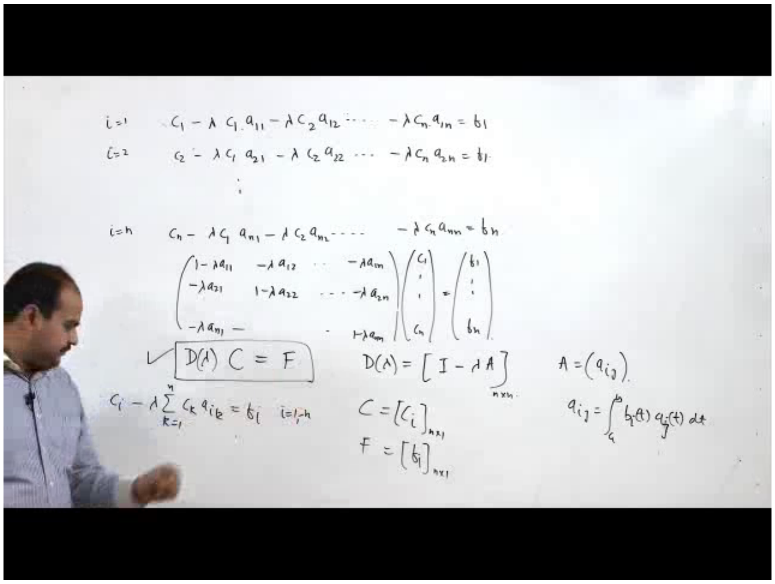

Let me write it here. So here I will write it for so i equal to one we have c one minus lambda c one a one one minus lambda c two a one two and so on and this minus lambda c n a one n equal to f one and for i e equal to two it is c two minus lambda c one a two one minus lambda c two a two two and so on minus lambda c n a to n equal to f one and in so on if we write it i e equal to n it is c n minus lambda c one a n one minus lambda c two a n two and so on.

Minus lambda c n a n n equal to f of n, so if you erase this you will get here first we have one minus lambda a 11 and here we can write it minus lambda a one two and minus lambda a one n and if you look at the second one it is c one is here so minus lambda a to one will be there and if you look at this term it is one minus lambda a two two and so on minus lambda a to n and in this way we can write it here.

If you look at the last one it is what minus lambda a n one and it is what, it is one minus lambda a n n and this is c one to c n and it is f one to f n, which we are denoting as d lambda c equal to f, now here cases arises when we discuss the coefficient matrix d lambda and the forcing term this f, now let us discuss cases here, so this is the equation 9 I am representing here.

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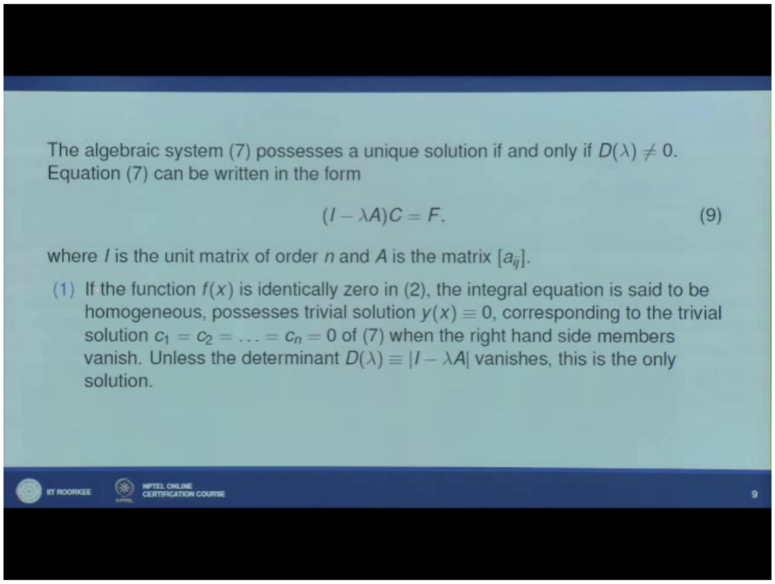

Now this f which is nothing but this a to, basically f is your f i t. Now if the function f x is ideally equal to zero in this two, so what is two here, two is this, so if I assume a homogenous Fredholm integral equation then in this case this f i, if you look at equation number 6, a to b b i t, f of (t) dt equal to f of i and if f of t is equal to zero, all your f i is going to be zero, so in this case if this equation is reduced to d lambda c equal to zero this is happening when we are assuming that f of x is ideally equal to zero.

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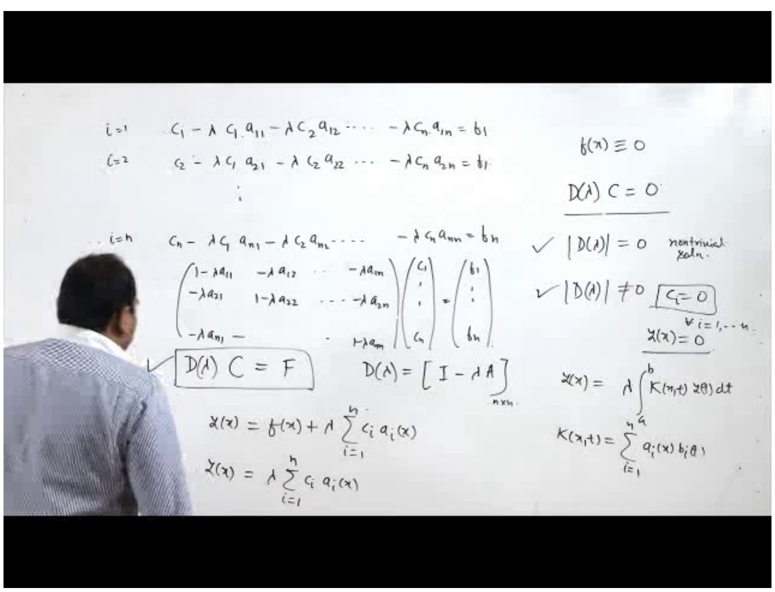

So when we take this f of x is equal to zero or we can say that we are taking the homogenous Fredholm integral equation then you can solve our constant using this, now here we have two cases arise that if determent of b lambda is zero and determent of d lambda is non zero, so in this particular case when we have determent of d lambda is equal to zero then we have a non trivial solution and in this case we can say that we have a non trivial solution.

And not only one solution, we have infinite number of solution here, so in this particular case when determent of d lambda is equal to zero, here we have infinite number of solutions and if determent of d lambda is non zero then we have a unique solution that is c equal to zero, so it means that all c i's equal to zero for i equal to one to n here, and in this case if solution y of x is given by one, the solution s always define as s.

If you remember the solution is given by this formula y of x equal to f of x plus lambda times summation c i a i x, i equal to y to n, so in particular this case when determent of d lambda is non zero all those c i is equal to zero, we have already assumed that fx equal to zero, so in this particular case your y of x is nothing but zero solution, so this is a zero solution for what, this is zero solution for homogenous Fredholm integral equation.

So this is the solution for what, this is solution for this particular problem, so here we have y of x equal to f of x is simply zero here, a to b k of x t and y of (t) dt lambda here and here k (x, t) as a degenerate or a separable kernel, i equal to one to n, a i x and b i t, so here when determent of d lambda is non zero we have a trivial solution for this homogenous Fredholm integral equation, now for this particular case when determent of d lambda is zero we have a non trivial solution.

It means that they are some c i's which are non zero and with the help of this, your solution you can write it here as, let me write it solution is basically what solution here, here f(x) is zero so we can write it y(x) plus equal to lambda times summation i equal to one to n and we have some c i zero, so it means that in this case when determent of d lambda is equal to zero, the solution s given by this and we have infinite many solutions in this case ok.

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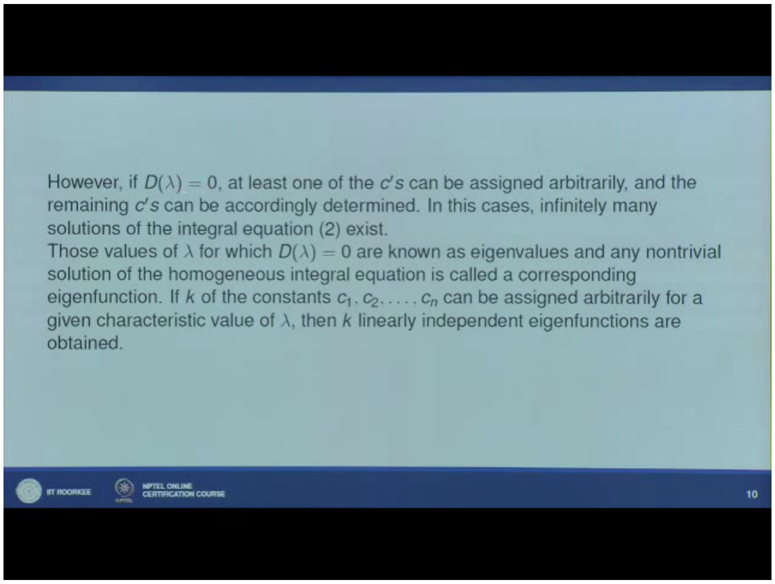

So now let us move to next case, next case is that this is what we have discussed here that if d lambda is equal to zero means determent of d lambda is equal to zero then at least one of the c's can be assigned arbitrary and the remaining c's can be accordingly determined or in this case we have infinite number of solutions.

And those values for this d lambda is equal to zero, d lambda equal to zero means the determent of coefficient matrix is zero.Those values of lambda are known as eigen values of this system, so we call this as lambda for which we have a solution here, we say that these lambdas are eigen values of homogenous integral equation and the corresponding non trivial solution, we call this as corresponding eigen functions, so those values of lambda for which d lambda is zero called as eigen values and the corresponding non trivial solutions are known as eigen functions.

It may happen that a eigen value may have more than one independent non trivial solutions, so in this case we may have corresponding to one eigen values may have more than one eigen functions available to us.

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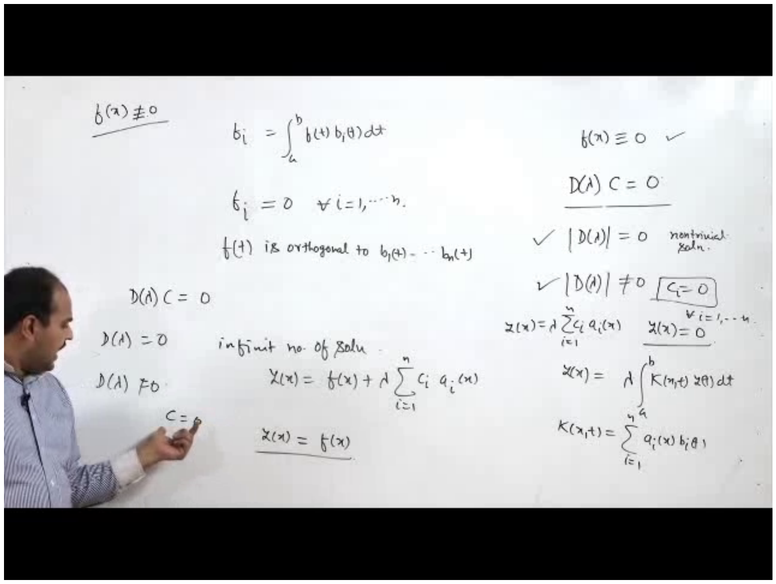

Now let us consider the second case, the second case when we don't have this function f of x is not ideally equal to zero, when we don't have f x not ideally equal to zero but in this case we may have two cases.

One case is that if you look at f i is basically what, F i is basically a to b, b i t f t b i (t) dt, now it may happen that though your function f of x is not ideally equal to zero but this ft is orthogonal to this b i t it may happen that this ft is orthogonal to this b i t, in this case also your f i is simply vanishing for all i equal to one to n, so this may happen when f t is orthogonal to your b one t to b n t, in this case also your problem reduce to this d lambda c equal to zero right.

So again, as we are discussing in as a previous case, here also we have two case arise where d lambda is zero and d lambda non zero right, and in this case your f x is non zero but still because of this property orthogonal d we have again the homogenous equation and if we have d lambda equal to zero we have more than one solution available, so we can say infinite number of solutions here, and solutions are given as y of x equal to f of x plus.

It is given by this particular formula this thing f (x) plus lambda times summation i equal to 1 to n c i a i x right, so d lambda equal to zero means they are more than one solution for c i's so it means the infinite number of solutions of c i available, so we have infinite number of solutions available here and when we have d lambda is not equal to zero, in this case we have only unique solution that is of c equal to zero right.

And in this case the solution will be y of x is nothing but f of x right, so if you compare these two cases when f x is equal to zero and when f x is non zero but orthogonally condition is there then here in this particular case your solution is y x is equal to lambda times summation c i a i x i equal to one to n, so in this particular case the solution is not involving any function f of x, but here since f x is non zero, solution is given in this form.

So here we have infinite number of solutions and in this case, when d lambda is non zero you have y x equal to f (x) here and here we have only y (x) equal to zero, now consider one more case when f x is non zero and this property is not true for all i equal to one to n, it means that it may happen that sum of f i is zero but not all f I s are zero.

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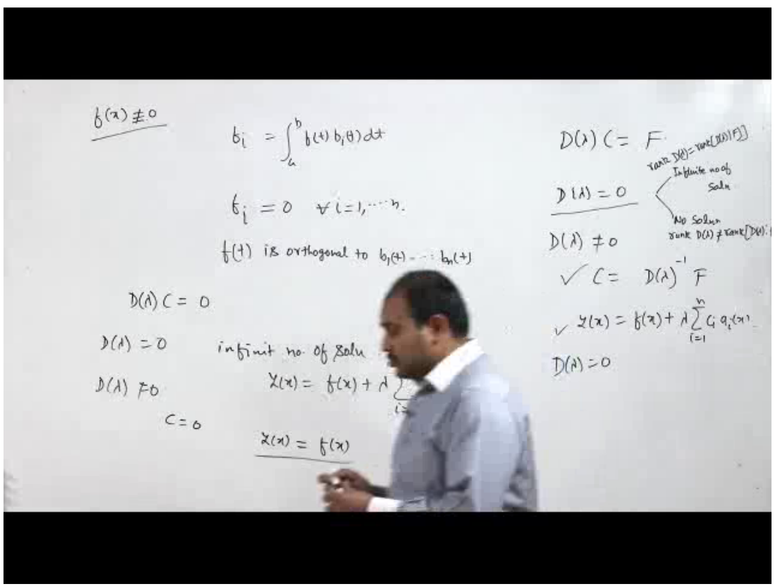

In this case it is d lambda c equal to your f but here your f is not a zero factor now. Now again we have two condition, d lambda is equal to zero and d lambda non equal to zero, now when d lambda is not equal to zero then here we have the unique solution that is c equal to d lambda inverse f, so we have a unique solution and solution is given by again y of x is equal to f (x) plus lambda times summation i equal to 1 to n, c i a i x so c i you can find out using this and you can put it here and you have a solution.

Now when we have d lambda equal to zero then here we have coefficient matrix which is a similar coefficient matrix and here f is a non zero factor, it may happen that we don't have any solution at all, here your knowledge of solving linear algebraic equation is very very important, if you know how to solve a x equal to b you can solve this thing, so it may happen that when d lambda is equal to zero it may happen that there is no solution at all for this.

That you check from the rank condition here, so if rank of d lambda and rank of d lambda f are same then we have a solution and in that case we have infinite number of solution and if rank of d lambda and rank of d lambda f is not same then we have a no solution, so here we have two conditions, no solution and infinite number of solution, that depend on rank condition that here no solution means rank of d lambda is same as rank of.

I hope this you have already seen d lambda f and if it is not same then no solution, if it is same then we have infinite number of solution, so here we have condition rank of d lambda is equal to rank of d lambda f, so in this particular case we have infinite number of solution, here we have no solution, so now we have discussed all the cases, I hope we have considered all the cases ok, so in next lecture we will discuss the example based on this theory.

Is it ok, thank you for being with us.