Integral Equations, Calculus of Variations and Their Applications.

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Lecture-57.

Variational Problems with Moving Boundaries-III.

Hello friends and welcome to the lecture on variational problems with moving boundaries. 1st we will discuss some examples where we will see one endpoint is fixed and the other point moves on a curve and then we will take up a case where the one point is fixed and the other point moves on another curve. So let us see, we start with the example on the finite, find the shortest distance between the point P1, that is 1, 0 and ellipse which is given by 4x square + 9y square equal to 36. We know that they are slim P1, P2 of the curve y equal to yx is given by I yx equal to integral x1 to x2 under root 1+ y dash square dx.

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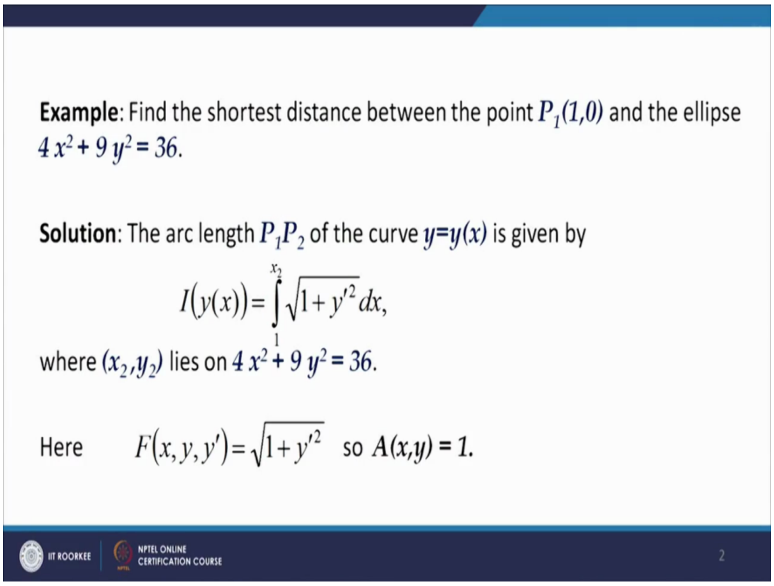

Now here x1, y1 is the point 1, 0, so we will write 1 for x1 and therefore we have integral over x1 to x2 under root 1+ y dash square dx, where x2, y2 will lie on the ellipse 4x square + 9y square equal to 36. Now here this , the function is integral 1 to x2 under root 1+ y dash square dx. So if you compare it with the standard form of the functional, we can see that F x, y, y dash is under root 1+ y dash square. And so Ax, y here is equal to 1. Now from Euler's equation which we write as delta F over delta y - d over dx, delta F over delta y dash equal to 0. We see that the Euler's equation reduces to - d over dx y dash over 1+ y dash square equal to 0.

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From Euler's equation, we have

$$-\frac{d}{dx}\left(\frac{y'}{1+y'^2}\right)=0$$

$\Rightarrow y = c_1 x + c_2.$

Since it passes through (1,0), we have

$$c_2 = -c_1. \quad ...(1)$$

Since the point (x_2, y_2) lies on $4x^2 + 9y^2 = 36$, we have

$$4x_2^2 + 9y_2^2 = 36. \quad ...(2)$$

Also $y = \frac{2}{3}\sqrt{9-x^2} = \phi(x)$ (say). ...(3)

Because here F x, y, y dash is not depending on y, so its partial derivative with respect to y will be 0 and when we differentiate this with respect to y dash, we will get y dash over under root 1+ y dash square. So Euler's equation reduces to - d over dx of y dash over 1+ y dash square equal to 0. And we have seen in our previous lecture that when we solve this equation, what we get is y equal to C1 x + C2 and so which is the equation of a state line. Now since this line passes through the point 1, 0, which is x1, y1, we will have C2 equal to - C1. And since the point x2, y2 lies on 4x square + 9y square equal to 36, we have 4 x2 square +9 y2 square equal to 36.

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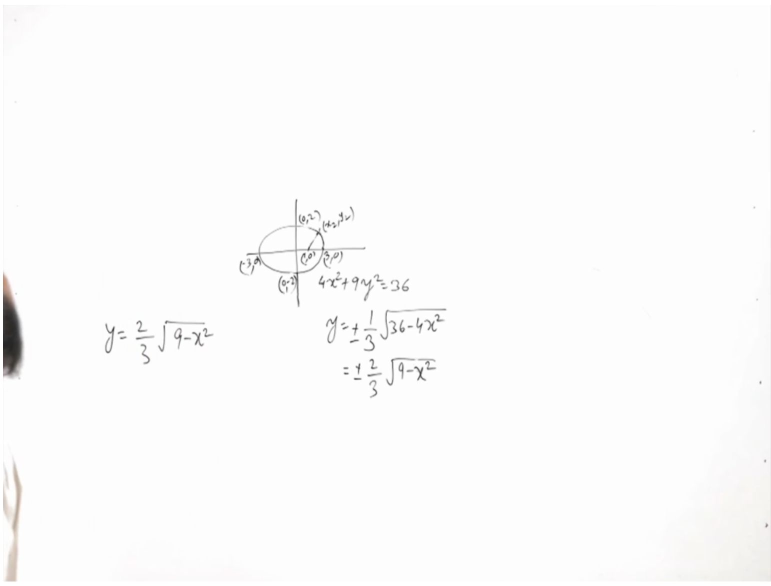

Now if you solve 4x square + 9y square equal to 36, you can write it as a function, y as a function of x and y will be equal to 2 by 3 under root 9 - x square. Now here we see that, we have this situation, this is ellipse 4x square + 9y square equal to 36, so this is 3, 0, here we have -3, 0 and this is 0, 2 and this is 0, -2. And here is the point let say 1, 0 and x2, y2 is say a point here, so shorter distance occurs along a straight line y equal to C1 x + C2 x and 4x square + 9y square, we write as y equal to 36 - 4x square, y is equal to 36 - 4x square divided by under root, divided by 1 by 3, so + -, okay.

And this I can write as + - 2 by 3 under root 9 -, 9 - x square. Now we are taking here y equal to positive, y we are taking, we are taking y here, y equal to 2 by 3 under root 9 - x square. One can take y equal to - of under root, -2 by 3 under root 9 - x square then the other point which is symmetric corresponding to x2, y2 here below this x axis will also be, can be also be taken in x2, y2. So we are taking y equal to 2 by 3 under root 9 - x square and we are calling it as Phi x. Now from the transversality condition because x2, y2 lies on the curve that is the ellipse, it will satisfy the transversality condition F + Phi dash - y dash into f y dash at x equal to x2 equal to 0.

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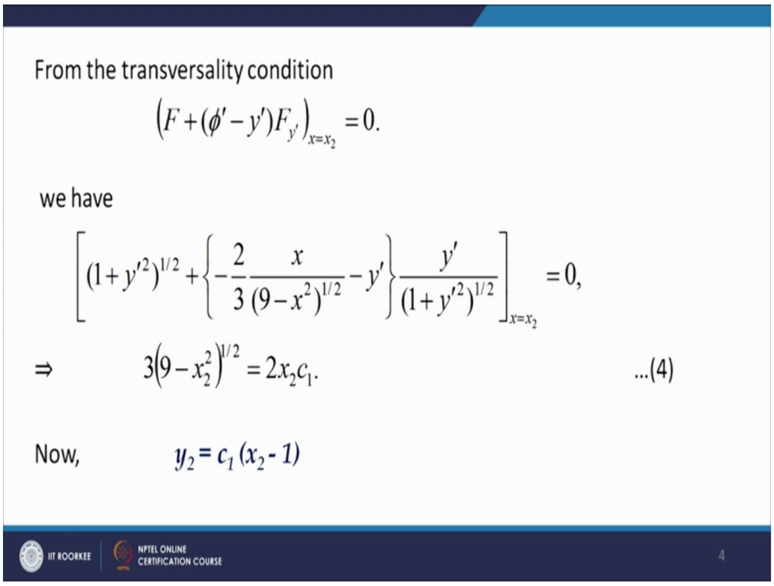

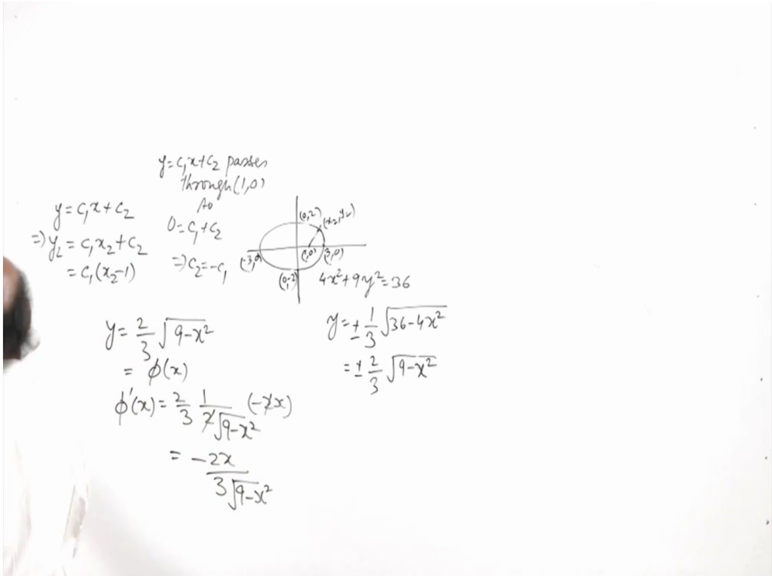

Now here F is equal to1 + y dash square this to the power half and Phi dash, this is y equal to Phi x, so when you differentiate, what you get is Phi dash x is 2 by 3, 1 by 2 under root 9 - x square into - 2x. So this will give you - 2x upon 3 under root 9 - x square. So Phi dash is - 2x upon 3 under root 9 - x square - y dash and then multiplied by Fy dash. Fy dash is y dash divided by under root 1+ y dash square evaluated at x equal to x2 equal to 0. And when we simplify this equation, what we get is 3 types 9 - x2 square raised to the power half is equal to 2 x2 into C1.

And what we have y equal to, y equal to C1 x + C2, okay. So y2 equal to, this implies y2 equal to C1 x2 + C2. Now y equal to C1 x + C2 it passes through, y equal to C1 x + C2 passes through 1, 0, so we will get 0 equal to C1 + C2, which implies that C2 equal to - C1.

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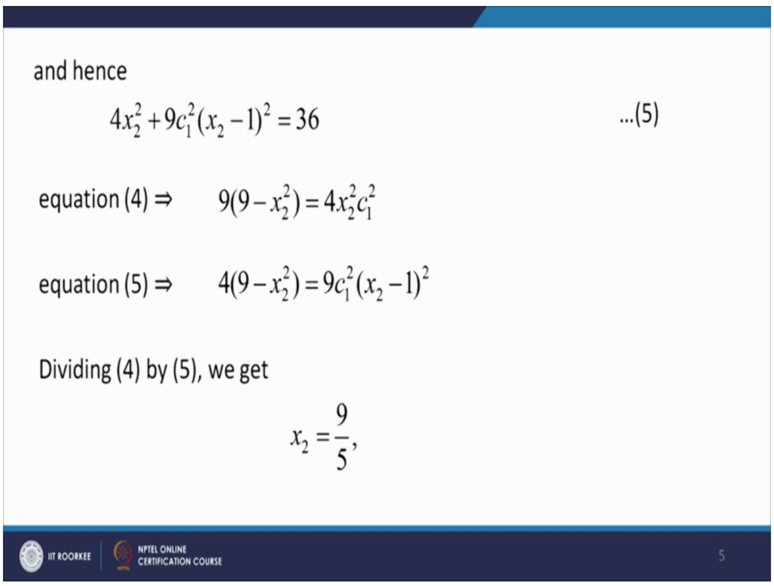

So put here C2 equal to - C1 is, so we get C1 times x2 - 1. So we get y2 equal to C1 into x2 -1 and therefore 4 x2 +9 y2 square equal to 36 reduces to 4 x2 square +9 C1 square into x2 - 1 whole square equal to 36. Now the equation 4, if you look at equation 4, the equation 4 on squaring both sides gives you 9 time 9 - x2 square equal to 4 x2 square into C1 square. And equation 5 gives 4 times 9 - x2 square is equal to 9 C1 square into x2 -1 whole square. So if you divide the equation , divide these 2 equations and simplify, you get x2 equal to 9 by 5.

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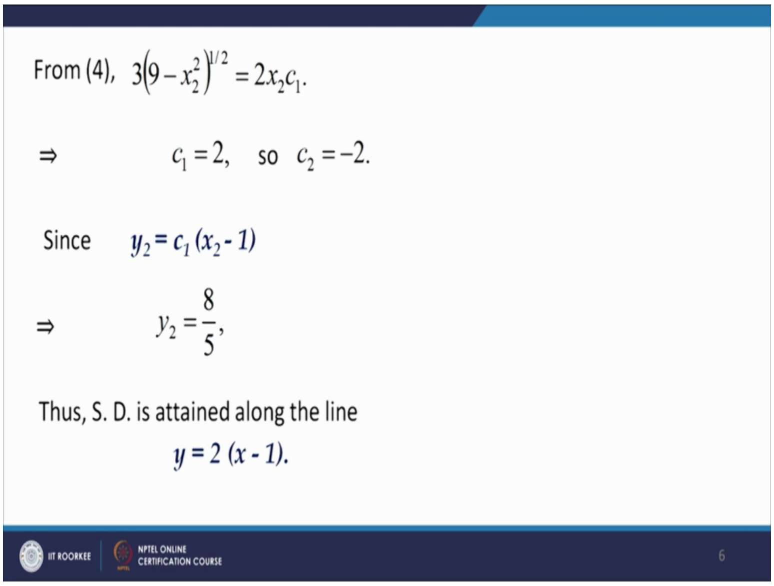

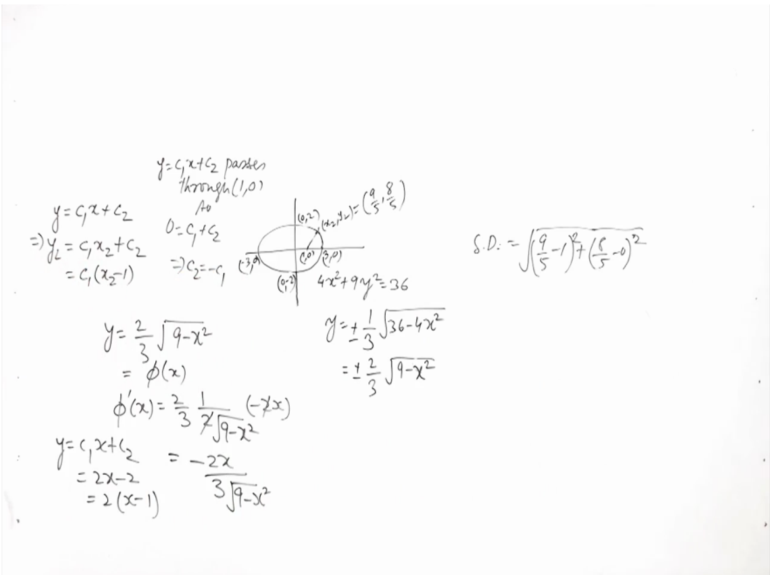

We are taking positive value of x2 here, if you take negative value of x2, the corresponding value of that and y2 you get will not give you the shortest distance, so we are taking x2 equal to 9 by 5. Now from 4, 3 times 9 - x2 square raised to the power half equal to 2 x2 C1. What we get is on putting the value of x2 is 9 by 5, we get C1 equal to 2 and since C2 is - E1, we get C2 equal to - 2, C2 equal to -2. So since y2 equal to C1 times x2 -1, we can put the value of C1 as 2 and x2 9 by 5 to arrive at the value of y2 which is 8 by 5.

Thus we can say that the shortest distance is attained along the line y equal to C1 x + C2 which is, C1 is 2, so 2x - 2 or 2 times x -1. So the shortest distance is attained along the line y equal to 2 times x -1 and the required shortest distance is the point x2, y2 is coming out to be 9 by 5, 8 by 5, so the shortest distance is under root 9 by 5 - 1 whole square, 8 by 5 - 0 whole square. So which is equal to 4 square root 5 divided by 5. So this is example 1 where we have calculated the shortest distance between the fixed point and the other point which x2, y2 which varies along the curve which is an ellipse.

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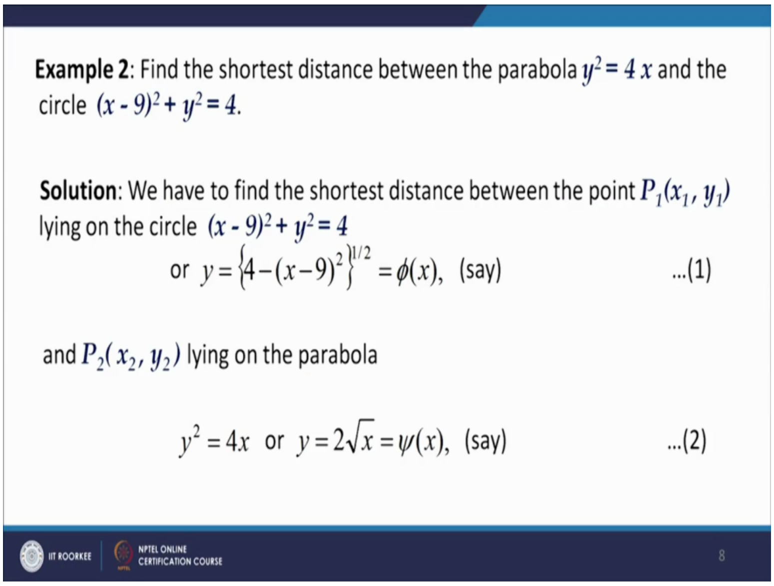

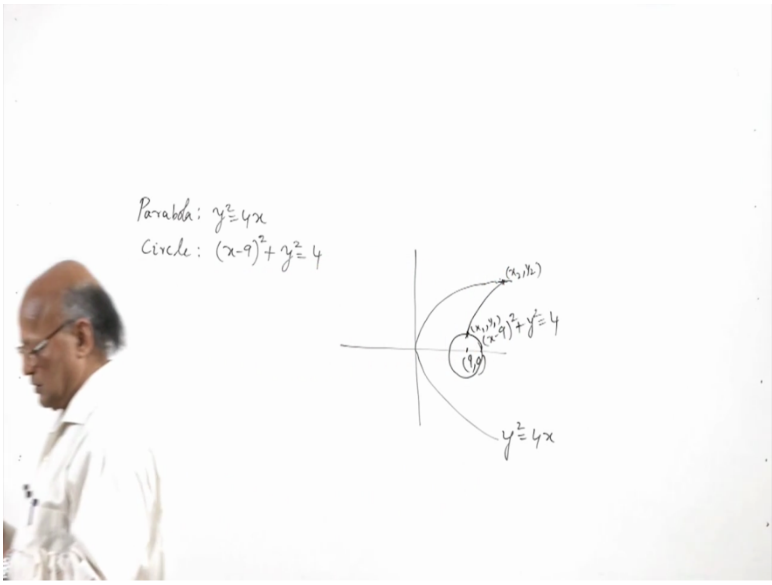

Now let us take another problem where we are going to find out the shortest distance between the parabola y square equal to 4x and the circle x -9 whole square + y square equal to 4. So here both the points x1, y1 and x2, y2 will be unique, such that the distance between the 2 points is, the distance between the 2 curves is the shortest distance. So let us see, draw the figure 1st. We have to find the shortest distance between the parabola y square equal to 4x and the circle. So parabola here is y square equal to 4x and circle is x -9 whole square + y square equal to 4. So let us draw the figure, let us say this is your parabola y square equal to 4x, the circle has Centre had 3, 0, sorry 9, 0, circle has Centre at 9, 0 and radius is 2.

So let us say this is your circle and radius is 2. So this is your circle x -9 whole square + y square equal to 4. Let us say x1, y1 is a point here on the circle and x2, y2 is a point here on the parabola and we are finding the shortest distance between the 2 points between the 2 curves. So, so we have to find the shortest distance between a point x1, y1 which lies on the circle x -9 whole square + y square equal to 4 and x2, y2 which lies on the parabola y square equal to 4x. Now the equation of the circle can be put as y equal to 4 - x -9 whole square raised to the power half, we are taking positive value of y here.

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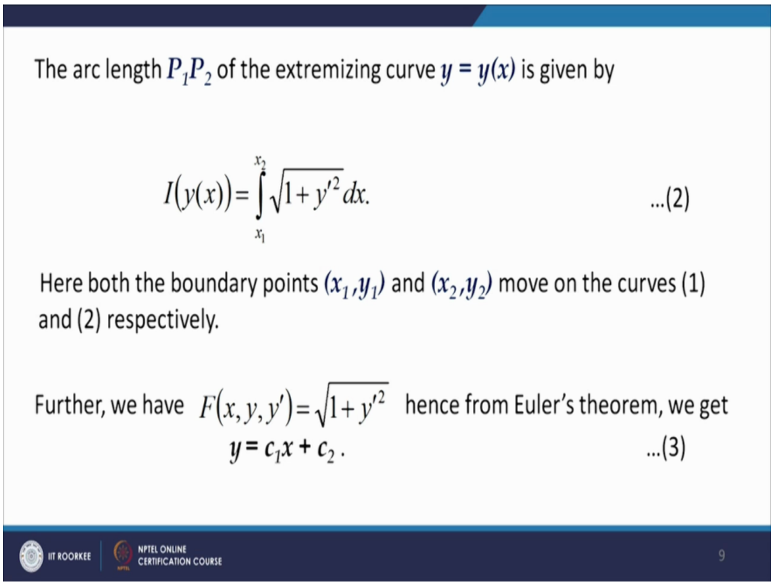

And similarly y equal to, y square equal to 4x, which we are writing as y equal to 2 root x, we are finding the distance, shortest distance in the positive quadrant. So now say y equal to under root 4 - x -9 whole square under root, under root 4 - x -9 whole square we write as Phi x and y equal to 2 root x we write as psi x. Then the arc length P1, P2 of the extremizing curve, this is extremizing curve. So the arc length P1 P2 of the extremizing curve will be given by integral x1 to x2 under root 1+ y dash square, under root 1+ y dash square dx.

Here we note that both the endpoints x1, y1 and x2, y2 are moving, x1, y1 is moving on the circle while x2, y2 is moving on the parabola. Now here again F of x, y, y dash is under root 1+ y dash square and therefore from the Euler's theorem we get that y equal to C1 x + C2.

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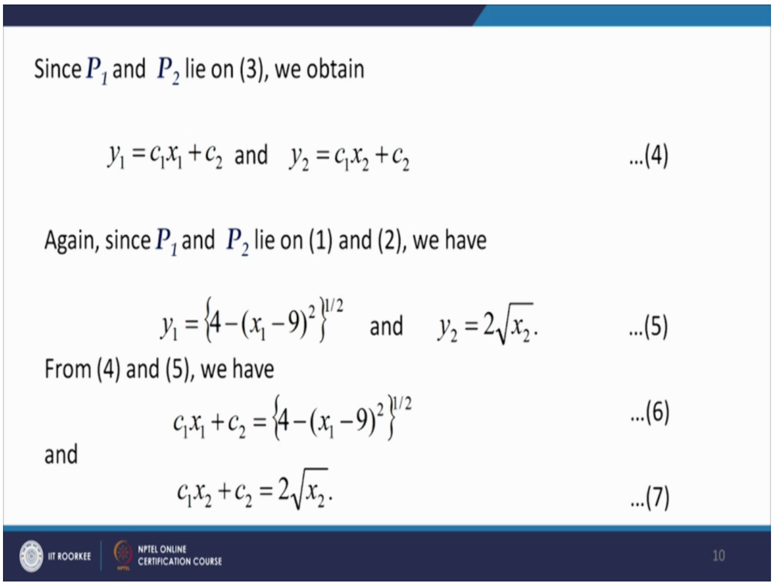

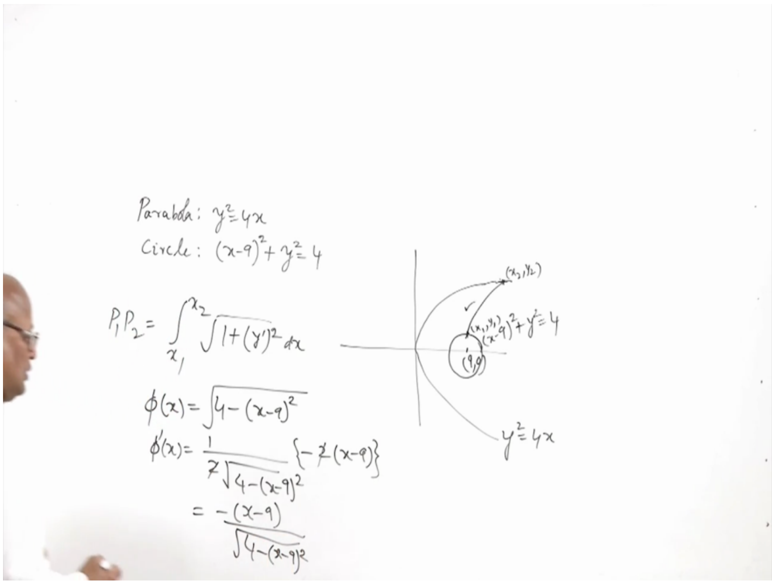

Now since P1 and P2 lie on the lie on 3, P1 and P2 lie on the 3 which is the equation of a straight line, we get y1 equal to C1 x1 + C2 and y2 equal to C1 x2 + C2. And moreover P1 and P2 lie on the circle and parabola respectively, so x1, y1 lie on the circle, therefore y1 equal to 4 - x1 -9 whole square to the power half. And x2, y2 lie on the parabola, so y2 is equal to 2 square root x2. So now from the equations 4 and 5 we have C1 x1 + C2 equal to y1, y1 is equal to 4 - x1 -9 whole square raised the power half and similarly C1 x2 + C2 which is y2 is equal to 2 root x2. Now since P1 and P2 are both moving, we will have the, we will have to use the transversality conditions at P1 and P2 for the curve y equal to Phi x and for the curves y equal to psi x.

So at the transversality condition at the point P1 will be F last Phi dash - y dash into F y dash at x equal to x1 equal to 0. And the transversality condition at the point P2 will be F + psi dash - y dash into Fy dash at x equal to x2 equal to 0. Now F is under root 1+ y dash square, so 1+ y dash square to the power half + and Phi dash, Phi is equal to, Phi x is equal to , we are writing Phi x equal to 4 -, Phi x equal to 4 - x - the whole square under root. So when you find Phi dash x here, what you get is 1 by 2 times under root 4 - x -9 whole square into -2 times x -9.

So this 2 will cancel and we get - x -9 divided by under root 4 - x -9 whole square. So, so let us put the value of Phi dash as - x -9 over 4 - x -9 whole square and square root. And then - y dash into F y dash, Fy dash is the partial derivative of F x, y, y dash which is under root 1+ y dash square with respect to y dash, so it is y dash over 1+ y dell square to the power half at x equal to x1 equal to 0. And similarly let us write the transversality condition at the point P2, so we have F equal to 1+ y dash square to the power half divided, +1 by psi dash is, psi x is equal to, at, this curve, this curve are writing y equal 2 root x is equal to psi x.

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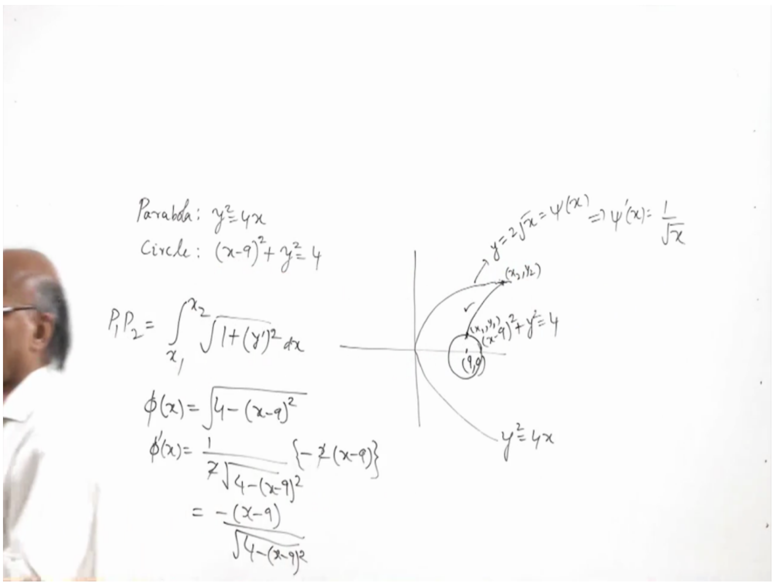

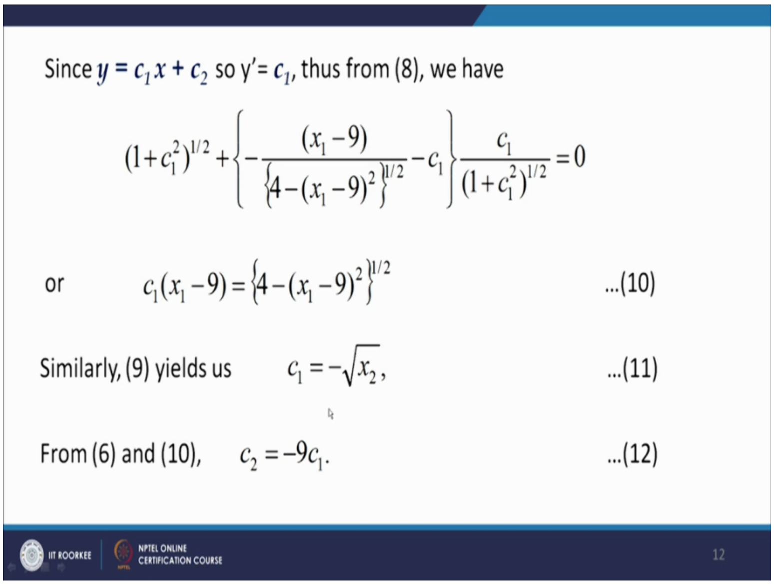

So psi dash x is equal to 1 by root x. So let us put the value of psi dash as one by root x - y dash multiplied by y dash over under root 1+ y dash square at the point x equal to x2 equal to 0. Now y is equal to C1 x + C2 and therefore if you differentiate y with respect to x, you get y dash equal to C1. So let us put C1, y dash equal to C1 in equation 8 and then the equation 8 becomes 1+ C1 square to the power half + inside the curly bracket we have - x1 -9 over 4 - x in -9 the whole square to the power half - C1 into C1 upon 1+ C1 square to the power half equal to 0.

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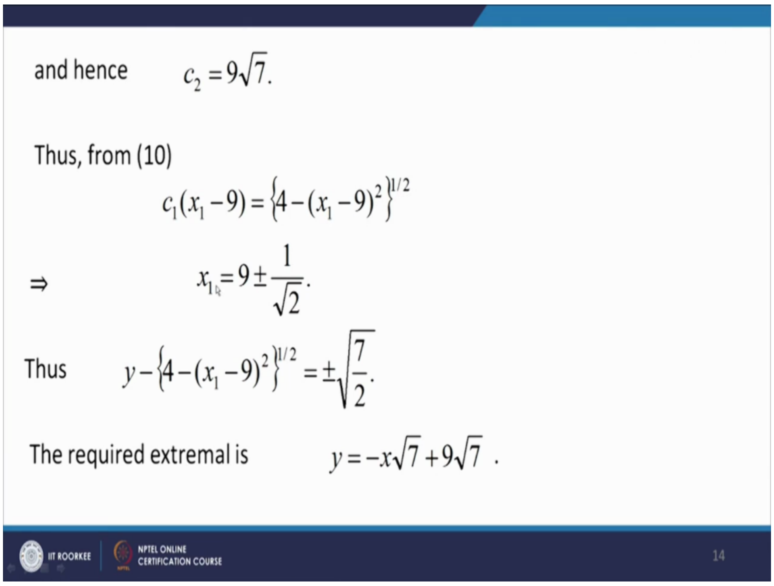

On simplifying this gives you C1 times x1 -9 equal to 4 - x1 - whole square to the power half. Similarly if you put y dash equal to C1 in equation, in equation 9, and then simplify, what you get is C1 equal to - root x2. And from the equation 6 and 10 and let us look at the equation 6, equation 6 is C1 x1 + C2, C1 x1 + C2 equal to 4 - x1 -9 whole square under root , this is 6 and what is 10, 10 is C1 times x1 -9. So this is equal to C1 times x1 -9. So C1 times x1 - equal to 4 - x1 -1 square to the power half. So what we will get, C1 x1 will cancel and we will get C1 x1, C1 x1 will cancel, we will get C2 equal to -9 C1.

So we get C2 equal to - C1 and from 11 and 12, let us look at 11 and 12, 11th equation is C1 equal to - root x2, 12th equation is C2 equal to -9 C1. So C2 is equal to -9 root x2 we get. And putting the values of C1 and C2 in equation 3, so we have got the value of C1 as - root x2 and C2 as 9 root x2, let us put them in equation 7 , 7 is this one, C1 x2 + C2 equal to 2 root x2. We put the values of C1, C2 in terms of x2 and simplify this equation, we will get the value of x2 as 7. And hence from 5, from 5 we get y2 equal to 2 root x2, so we get y2 equal to 2 root 7.

And thus the coordinates of the point P2 are, P2, this is P2, this is P2 point, so it is root 7, 2 root 7. So x2, y2 point is root 7, x2, y2 point is 7, sorry, 7, 7, 2 root 7 and since x2 is equal to 7, C1 is equal to , C1 was equal to - root x2, so we get - root 7. So C1 is - root 7 and C2 is 9 root 7. C2 is 9 root 7, so now let us simplify this equation, C1 times x1 - 9, C1 times x1 - is equal to 4 - x1 -9 the whole square to the power half, we get x1 equal to 9+ - 1 by root 2. So 9 x1, x1 here, x1 comes out to be 9+ - 1 by root 2. Okay, so 9+ -1 by root 2 and the corresponding value of y comes out to be + - root 7 by 2.

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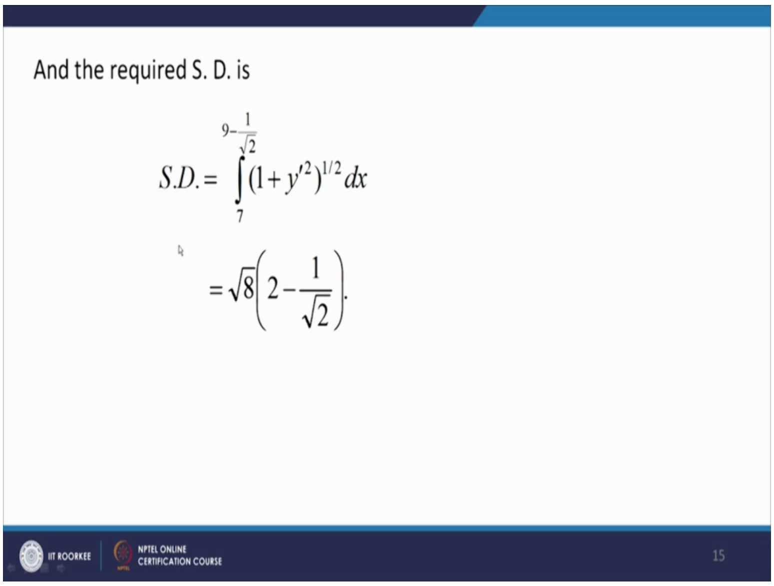

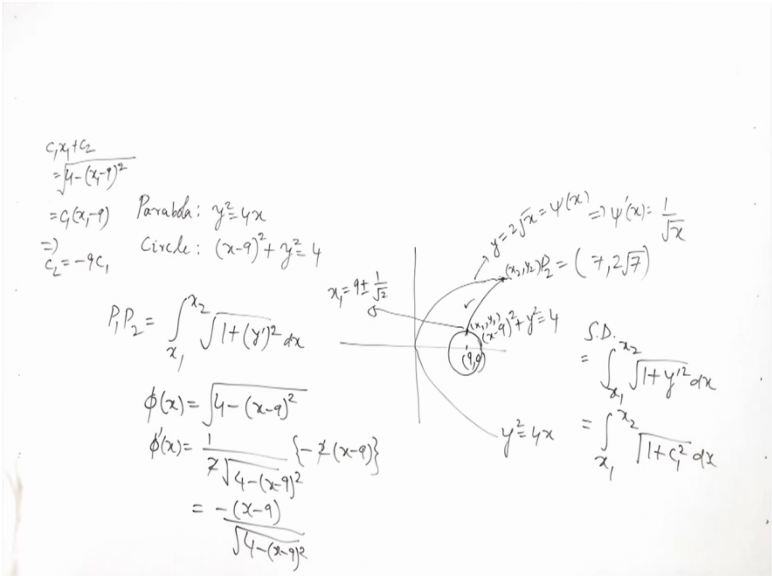

And so the required extremal is y equal to C1 x + C2 that is - x root 7 + 9 root 7 and the required SD is, so here we will see that x1 is equal to 7, sorry x1 is equal to, this is 7 and then here we have 9 - root, 1 by root 2, okay. So, yah x2, y2 was equal to 7 and 2 root 7 and here we have x1 equal to 9+ -1 by root 2 and we have y2, y1 equal to + - root 7 by 2. So here we get SD as integral of 7 to - 9 - 1 by root 2, 1+ y dash square, y dash is 1+ C1 square to the power half. And this is, shorter distance is equal to integral under x1 to x2, under root 1+ y dash square dx. Y dash is equal to C1, so integral x1 to x2 under root 1+ C1 square dx. And C1 is equal to - root 7.

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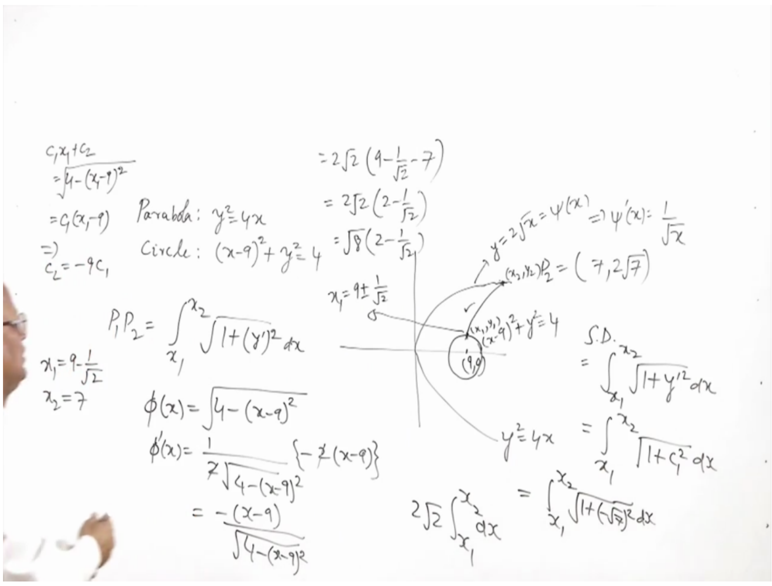

So this is x1 to x2 under root 1+ - root 7 whole square dx. So 7 +1 means 8, so root 8, that means 2 root 2 and then x1 - x2 dx. And x1 - x2 means we are getting x2 equal to 7. So, so this will be equal to 2 root 2, x2 is equal to 7 - x1 is equal to 9 - 1 by root 2 we are getting. This will be coming out to be negative, so that is why we are writing 9 - 1 by root 2, okay, so 2 root 2, 7 - x1, y1 to x2, y2. Okay. So this will be, this will be 9 -1 by root 2 -, here we are seeing that we have to take x2 equal to, we have to take x1 as 9 - this this upper limit we have to take as 9 -1 by root 2, lower limit will have to be taken as 7 because this is how much, this is 2 root 2 into 2 - 1 by root 2.

So that means we have to take the modulus here because it turns out that if we take x1 as 7 and x2 as 9 -1 by root 2, then, okay, x1 is 9 -1 by root 2, yes, so here x1 actually x1 is 9 -1 by root 2 and x2 is 7. So x2 - x1 comes out to be negative, so we will have to, because the distance is always positive, so we have to take 9 - 1 by root 2 - 7 and this will give you 2 root 2 into 2 -1 by 8 or you can say root 8 into 2 - 1 by root 2. This is, this is because x2 is 7 and x1 is 9 -1 by root 2. So 7 - 9 -1 by root 2 comes out to be negative, so we have to consider, and distance is always positive, so we have to consider 9 - 1 by root 2 - 7, so that will give us the shortest distance.

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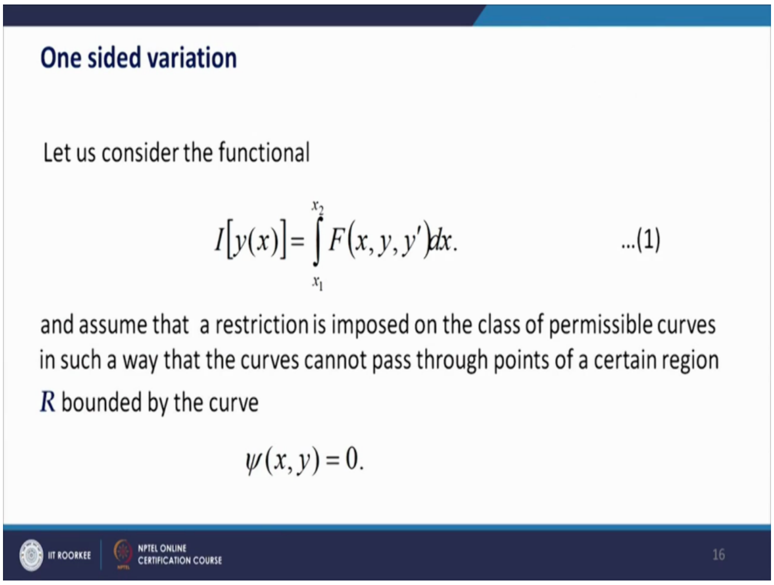

Now, now we will consider the one-sided variation, let us consider the functional I yx is equal to integral x1 to x2 F x, y, y dash dx. And let us assume that a restriction is imposed on the class of permissible curves in such a way that the curves cannot pass through points of a certain region R bounded by the curve psi x, y equal to 0. So let us say we have this situation. Let say we have this situation, we have a region R, we have a point A here and another point P here . This curve R, this region R is bounded by the curve psi x, y equal to 0.

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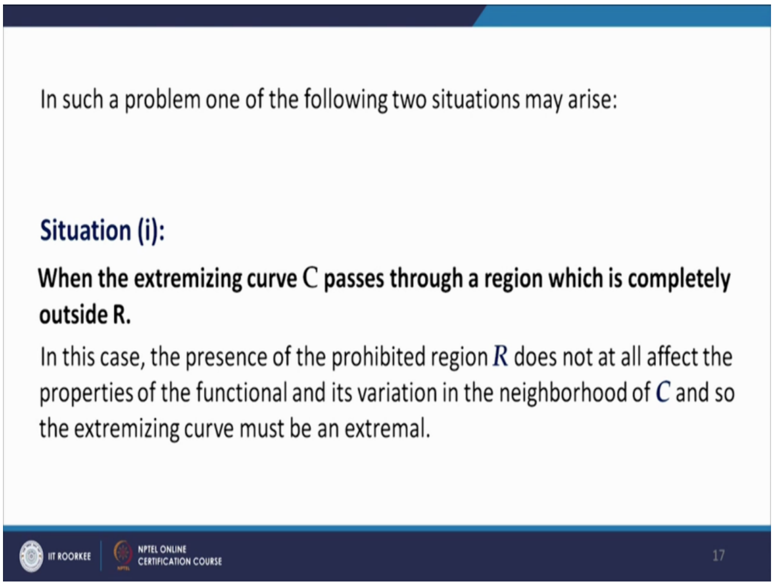

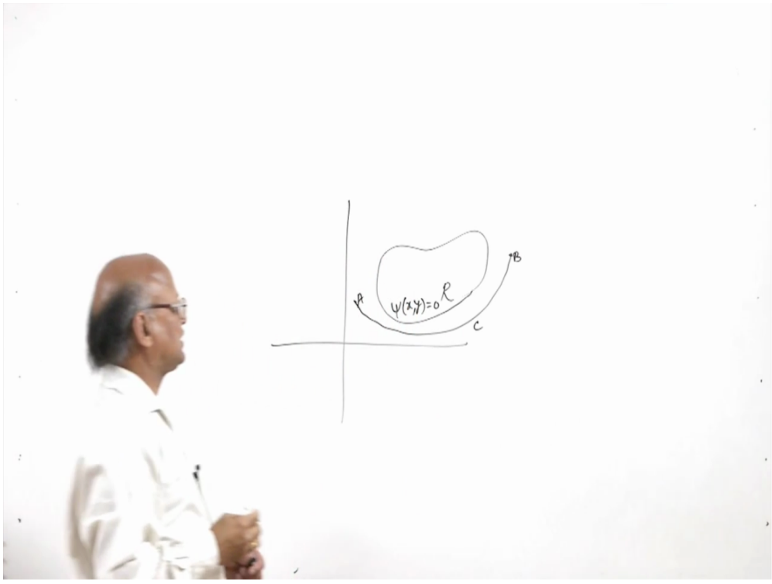

So the region R is bounded by the curve psi x, y is equal to 0 and we want that function or that extremal on which the restriction is imposed that it cannot pass through the points of the region R. So what we will have, in such a problem one of the following 2 situations may arise. 1st situation is when the extremizing curve C passes through a region which is completely outside R, for example like this. So this curve C completely lies outside the region R, okay. So when the extremizing curve C passes through origin which is completely outside R, in this case the presence of the prohibited region R, prohibited region is R, does not at all affect the properties of the functional and its variation in the neighbourhood of C. And so the extremizing curve must be an extremal.

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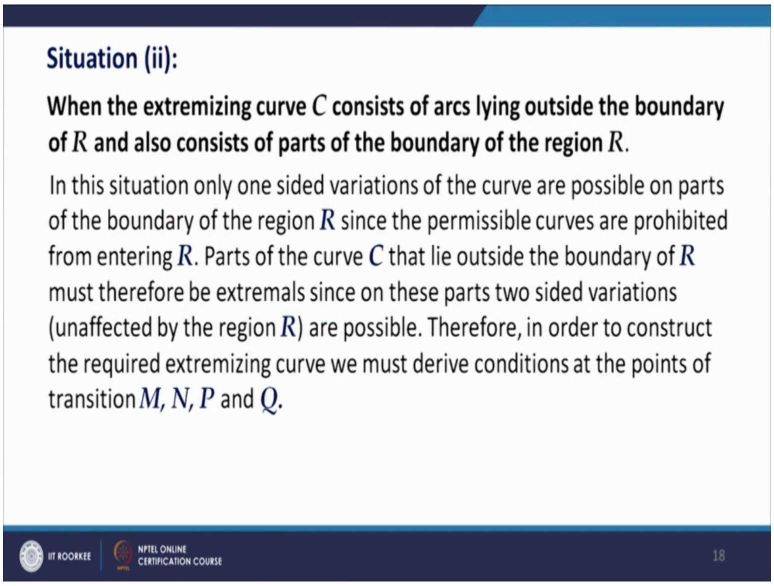

Now let us see the situation 2. In the 2nd situation the extremizing curve C consists of arcs lying outside the boundary of R and also consists of parts of the boundary of the region R. So we may have a situation like this. So we may have a situation like this where the extremizing curve C consists of arcs which lie outside the boundary of the region R, that is M and then, then M and q B and also consists of the parts of the boundary of the region R, that is M, N and P and Pq. In this situation only one-sided variations of the curve are possible on parts the boundary of the region R. Because the region, this region is prohibited, this region is prohibited, okay, so only one-sided variations are possible on the parts of the boundary of the region R.

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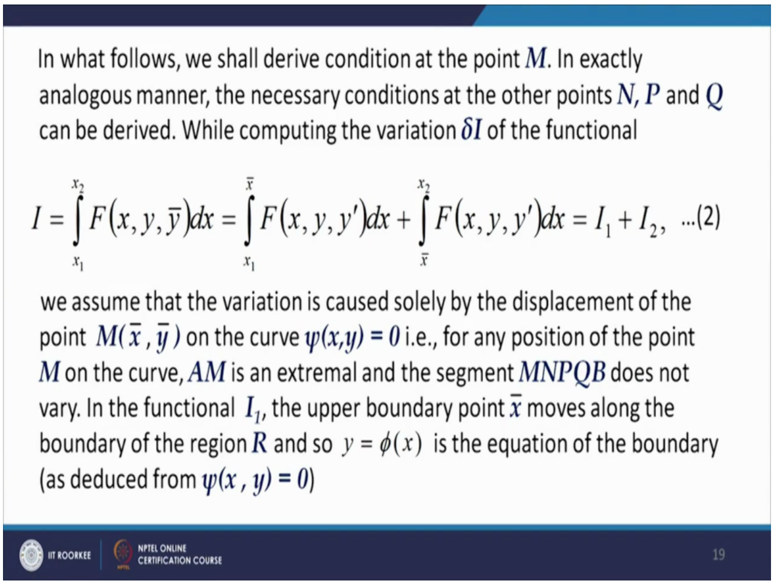

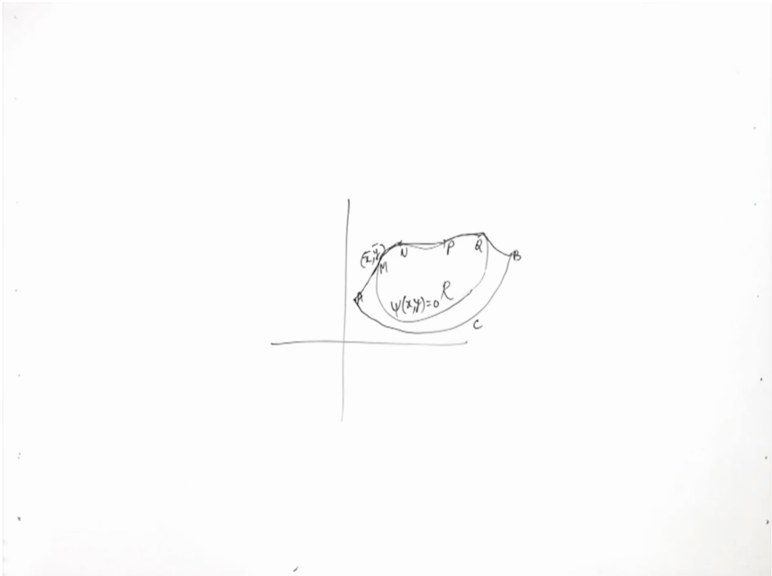

Now since the permissible curves are prohibited from entering R, parts of the curve C that lies outside the boundary of R must therefore be extremal. M, the parts of the curve that lie outside the boundary, so that is M and qb, therefore they must be extremal. And since on these parts 2 sided variations are possible and therefore in order to construct the required extremizing curve we must derive the conditions at the point, point of transition M, N, P and q. And what follows, we shall derived the condition at the point M in exactly analogous manner, the necessary conditions at the other points N, P and q can be derived. While computing the variations, delta I of the functional, I equal to x1 to x2 F, x, y, y dash dx.

We assume that the variation is caused only by the displacement of the point M. The point M is having coordinates x bar, y bar. So by the displacement of the point M x bar, y bar on the curve psi x, y equal to 0, that is for any position of the point M on the curve, M is an extremal and the segment mnpqb does not vary. So in the functional I1, that is x1 to x bar F x, y, y dash dx, I1, the upper boundary point x bar moves along the boundary of the region R and so y is equal to Phi x is the equation of the boundary as deduce from psi x, y equal to 0.

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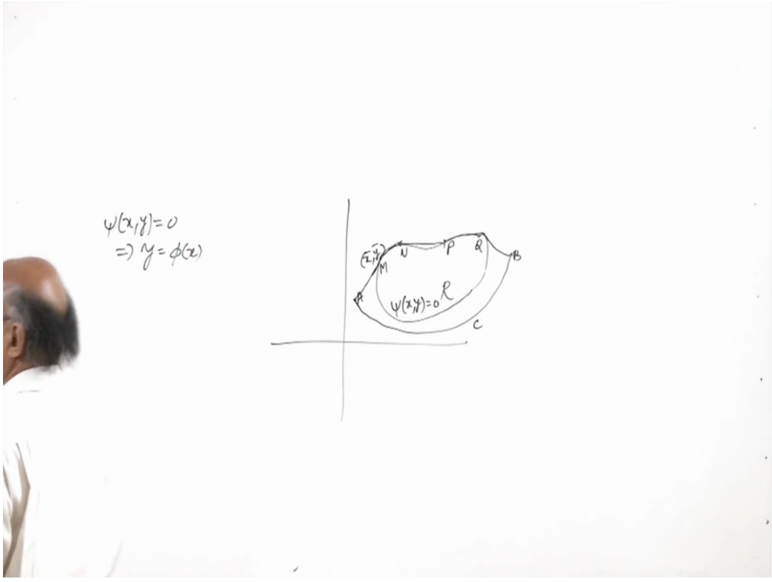

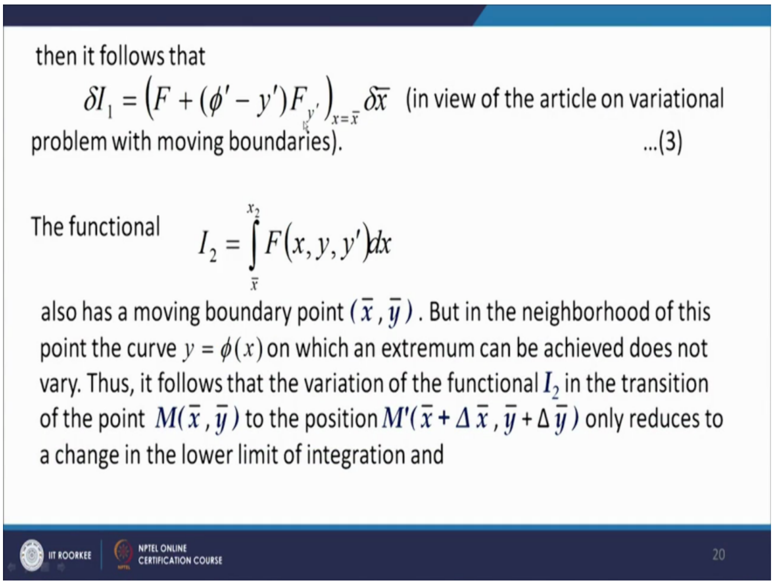

So psi x, y equal to 0, we are writing as y is equal to Phi x. So x bar, y bar point moves on the boundary of the region R and so y equal to Phi x, which is the equation of the boundary gives you y bar is equal to that of Phi x bar. Now then it follows that delta I1, dash I1 equal to F + Phi dash - y dash, Fy dash at x equal to x bar into delta x bar which is, which follows from the article on variational problem with moving boundaries. So delta I1 is this and the functional I2 equal to x bar to x2 F x, y, y dash dx also has a moving boundary point which is your x bar, y bar. But in the neighbour neighbourhood of the point, neighbourhood of the point x bar, y bar, the curve y equal to Phi x1 which extremum can be achieved does not vary.

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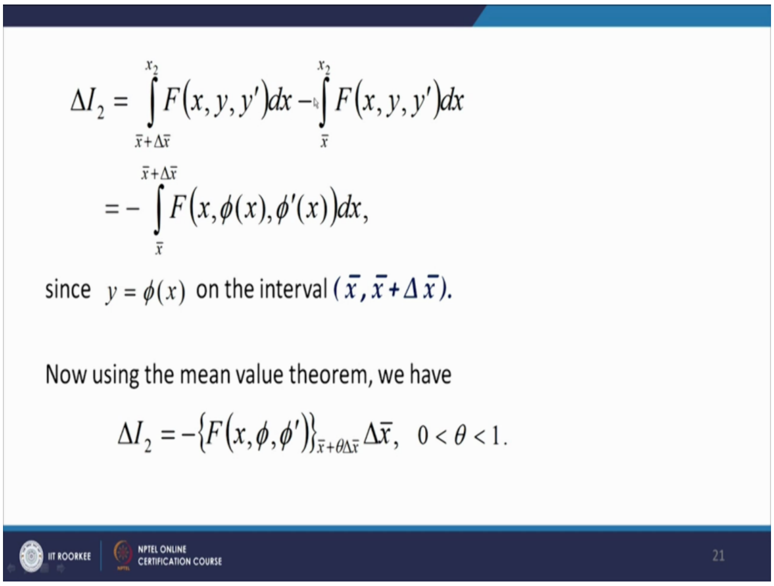

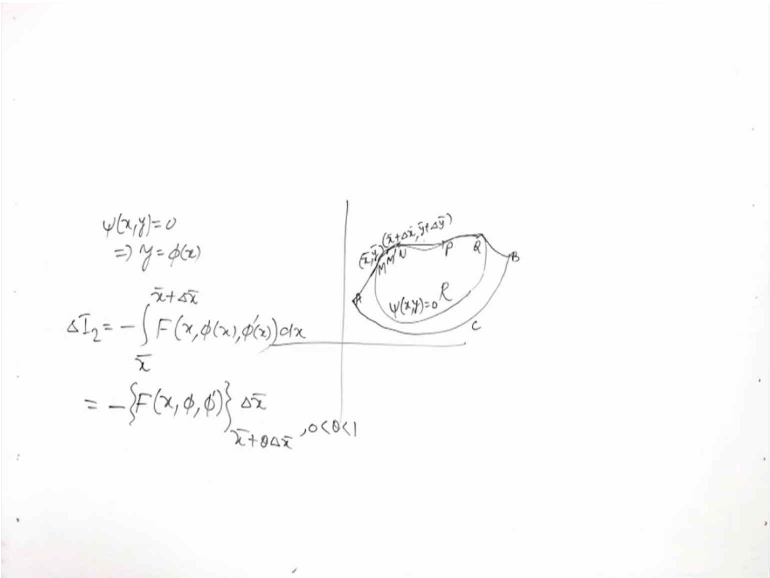

And so it follows that the very set of the force like to in the transition of the point Mx bar, y bar to the position M dash x bar + delta x, so let us say M, M dash is here, M dash is x bar + delta x bar, y bar + delta y bar, only reduces to a change in the lower limit of integration. And we have Delta I2 equal to x bar + delta x bar to x2 F x, y, y dash dx - x bar to x2 F x, y, y dash dx. This is the change in the functional, in the value of I2, when the point x bar, y bar moves to x1 + delta x bar y bar, y bar + delta y bar. Now this difference of the 2 integrals is nothing but - x bar to x bar + delta x bar, F x, y, Phi, F x, y, y dash dx but y is equal to Phi x. So it is F x, Phi, Phi dash dx.

Since y equal to Phi x lies on by, since y equal to Phi x lies on the interval x bar to x bar + x bar + delta x bar. Okay. Now, now let us use the mean value theorem, so by using the mean value theorem delta I2 which can be written as, see delta I2 is - x bar to x bar + delta x bar F x, Phi x, Phi dash x dx. By mean value theorem we can write it as, this is equal to - F x, Phi, Phi dash at x bar + Theta delta x bar, where Theta lies between 0 and 1, into delta x bar. So by mean value theorem we can write it as - F x, Phi, Phi dash, x1 + delta x bar into delta x bar where Theta lies in the interval of 0, 1. Now by the continuity of F x, Phi, Phi dash, F x Phi, Phi dash x bar + Theta delta x bar is equal to F x, Phi, Phi dash at x equal to x bar - Alpha, where Alpha goes to 0 as delta x bar goes to 0.

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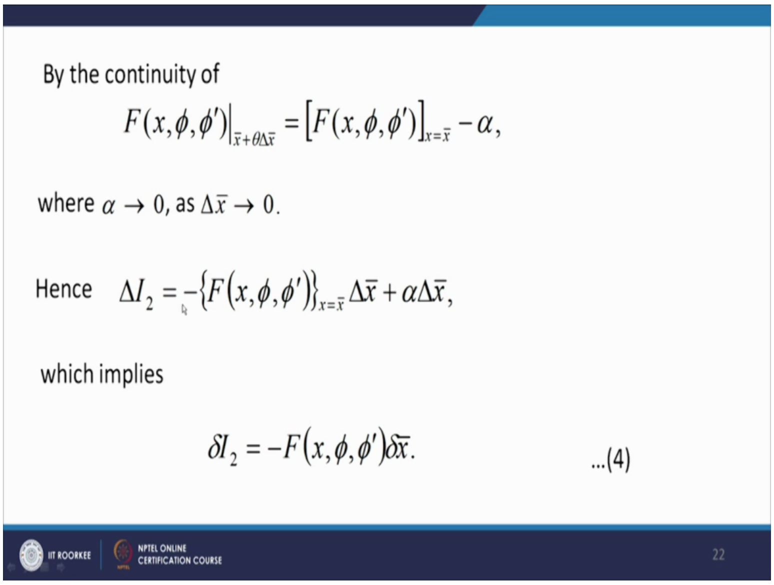

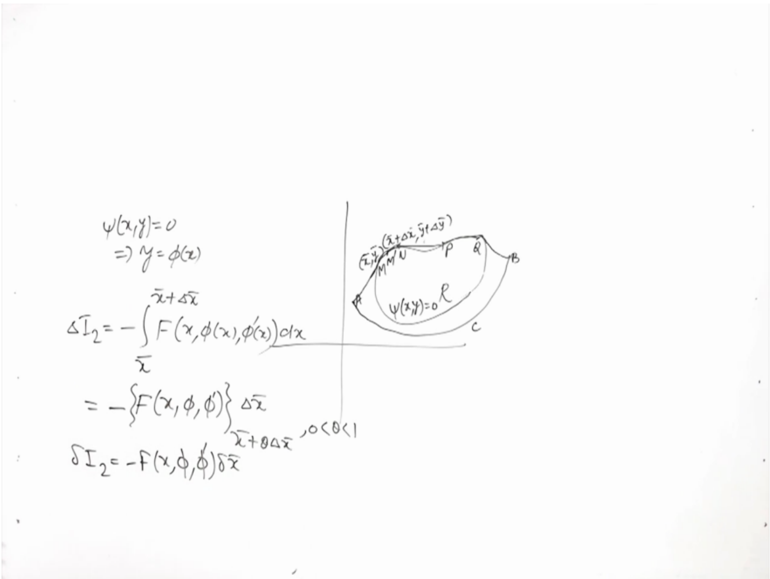

And therefore I2, delta I2 is equal to - F x, Phi, Phi dash at x equal to x bar into delta x bar + Alpha delta x bar. Now which implies that this delta I2, this delta I2, dell I2 or delta I2, this is equal to - F x Phi, Phi dash into delta x bar. So this is delta I2 and from 3 and 4 we find delta

I equal to, so let us see, yes, this 3 gives you delta I1 is equal to F + pie dash - y dash Fy dash at x equal to x bar delta x bar and we have just seen the equation for which is delta I2 equal to - F x Phi, Phi dash delta x, let us put these values to get this delta I, variation in I.

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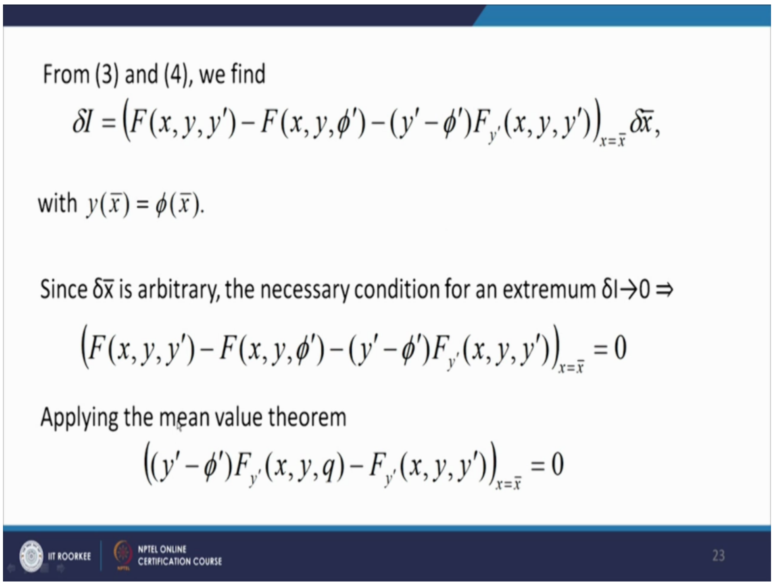

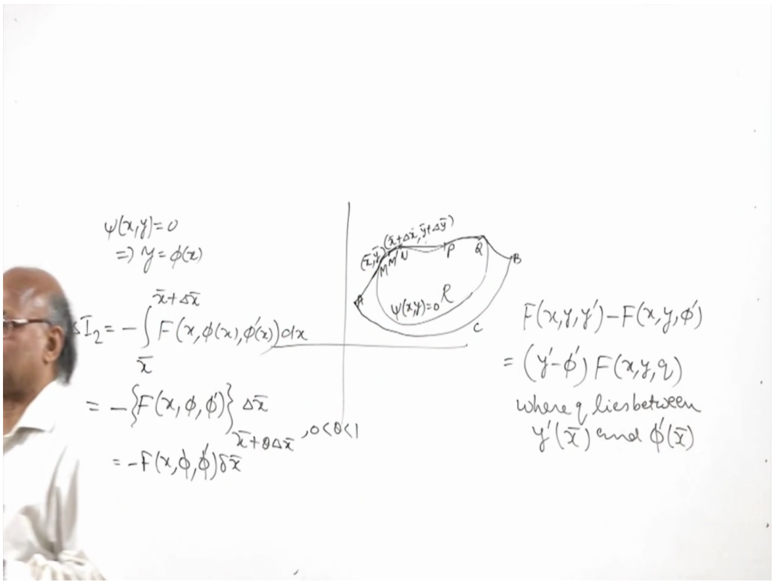

So delta is equal to delta I1 + delta I2 and what we get is delta I equal to this quantity, by combining 3 and 4. So yx bar is equal to Phi x bar. So since delta x bar is arbitrary, the necessary condition for an extremum delta I goes to 0 will then imply that F x, y, y dash - F x, y, Phi dash - y dash - Phi dash into Fy dash x, y, y dash at x equal to x bar equal to 0. Now let us apply the mean value theorem here, in the 1st 2 terms F x, y, y dash - F x, y, Phi dash. So by applying mean value theorem we will get this as y dash - Phi dash into Fy dash x, y, q where q lies between y dash x bar and Phi dash x bar.

Now let us apply the mean value theorem 2 F x x, y dash - F x, y Phi dash, F x y, y dash - F x y, Phi dash. So by mean value theorem this will be equal to y dash - Phi dash into F x, y, q where q lies between y dash, x bar and Phi dash, x bar. So the 1st 2 terms here, the 1st 2 terms here F x, y, y dash - F x, y, Phi dash can be expressed as y dash - Phi dash into Fy dash x, y, q. So we can take y dash - Phi dash common and we have y dash - y dash times F y dash x, y, q, this is Fy dash. Fy dash x, y, q - Fy dash x, y, y dash at x equal to x bar is equal to 0.

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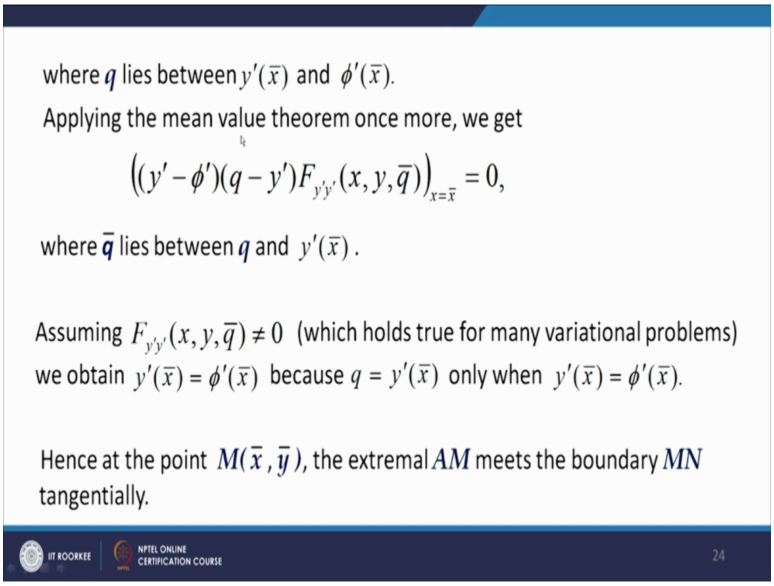

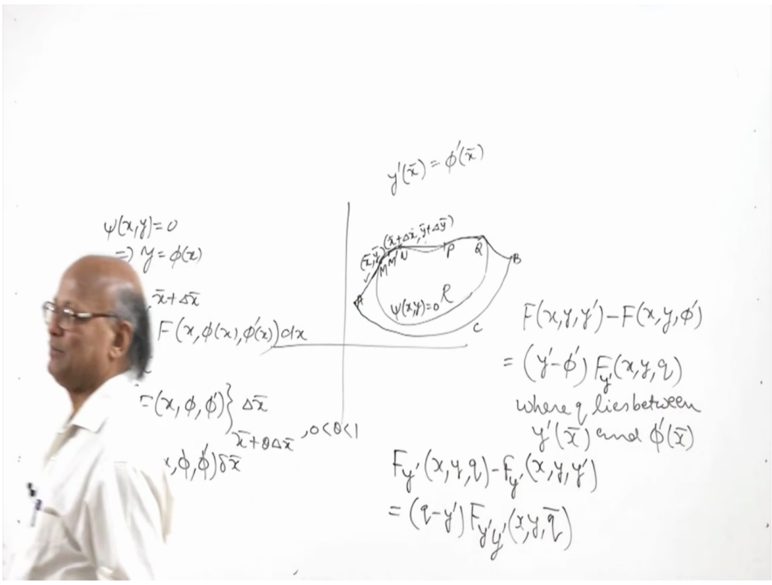

And applying the mean value theorem again, what we will have, let us see. So Fy dash, x, y, q - Fy dash x, y, y dash, let us apply mean value theorem again. So this will be q - y dash into Fy dash y dash, Fy dash y dash, F x, y, y dash, q bar. So we will have this, so we have y dash - Phi dash into q - y dash into Fy dash y dash x, y, q bar at x equal to x where q bar lies between q and y dash x bar. Now let us assume that Fy dash y dash x, y, q bar is not equal to 0 which holds true for many variational problems, then we shall obtain y dash x bar equal to Phi dash x bar.

Because if q equal to y dash x bar will hold only when y dash x bar equal to Phi dash x bar because q lies between y dash x bar and Phi dash x bar. So q can be equal to y dash x bar only when y dash x bar is equal to Phi dash x bar. But y dash x bar equal to Phi dash x bar means what? Y dash x bar is equal to Phi dash x bar means that at the point x bar, y bar, the slope of the extremal, okay, slope of the extremal which is y dash, x bar is equal to the slope of the curve y equal to Phi x. So this means that the extremal meets the curve tangentially. At the point M which is x bar, y bar, the extremal M meets the boundary MN tangentially. So this is what I have to say in this lecture, thank you very much for your attention.