Integral Equations, Calculus of Variations and their Applications
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Lecture 55
Variational problems with moving boundaries-1

Hello friends welcome to my lecture on variational problems with moving boundaries.

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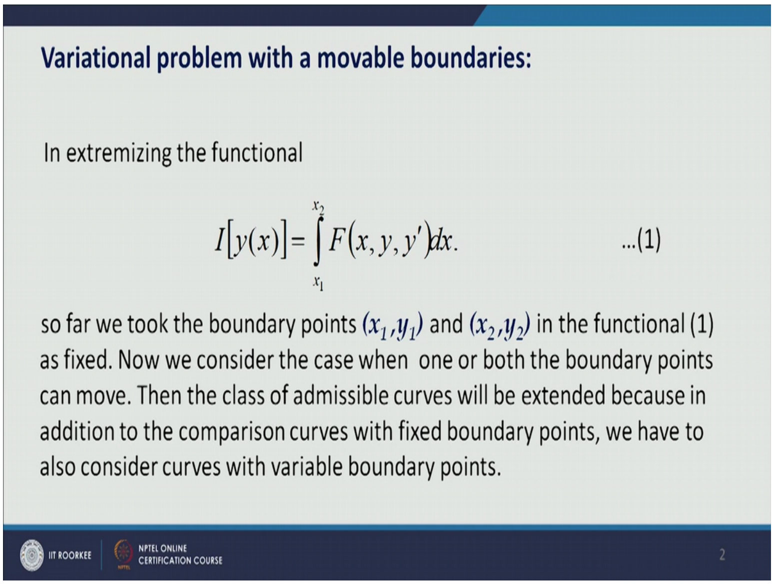

So far we extremize the functional I y x equal to integral x 1 to x 2 F x, y, y dash dx where both the boundary points x 1, y 1 and x 2, y 2 were taken as fixed. Now we shall consider the case where one or both the boundary points x 1, y 1 can move. So the class of admissible curves will be extended because in addition to the comparison curves with fixed boundary points, we have also to consider curves with variable boundary points.

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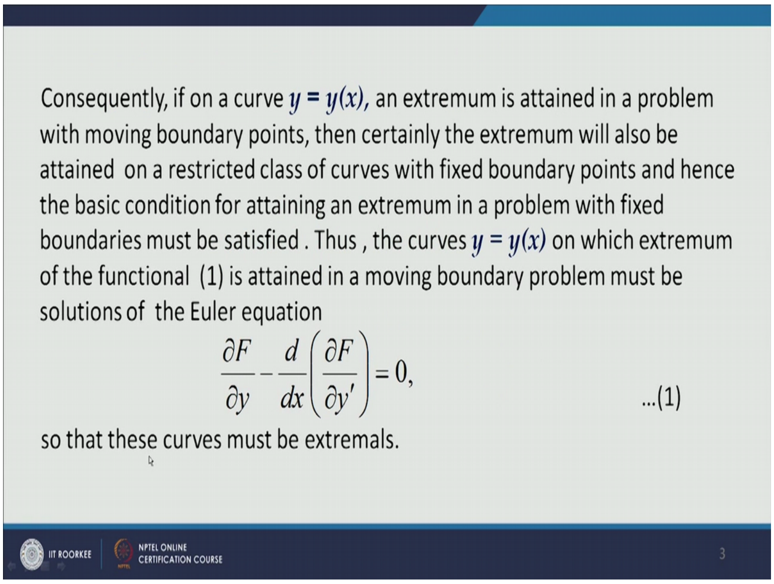

Now consequently, if on a curve y equal to y x, an extremum is attained in a problem with moving boundary points, then certainly the extremum will also be attained on a restricted class of curves with fixed boundary points and hence the basic condition for attaining an extremum in a problem with fixed boundaries must be satisfied. And we know the condition for the extremum in the problem with fixed boundaries it is that the curves y equal to y x on which the extremal of the functional 1 is attained must be solutions of the equation F y minus d over dx F y dash equal to 0, so that these curves must be extremals.

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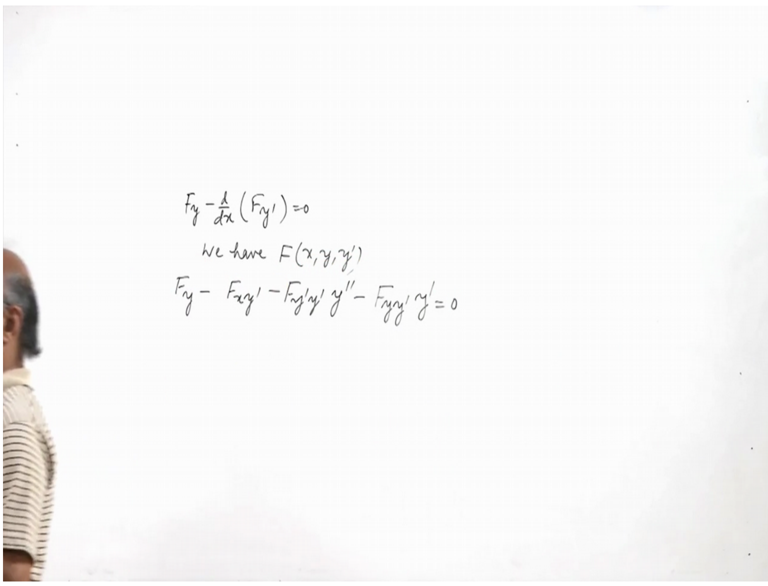

On expanding this equation F y minus d over dx F y dash equal to 0, F y dash is equal to 0. Now we know that we have F as a function of x, y, y dash. So when we differentiate F y dash what we will have with respect to x we shall have derivative of F derivative of F x y dash then we have derivative of F y dash y dash and we also have we will differentiate this with respect to x then we differentiate this with respect to y and we differentiate it with respect y dash and y dash we differentiate with respect to x, so we get y double dash, we differentiate this with respect to x we get F x y dash, we differentiate this with respect to y dash and then y dash is differentiated with respect to x so that we get this. And then we differentiate it with respect to y and then y is differentiated with respect to x.

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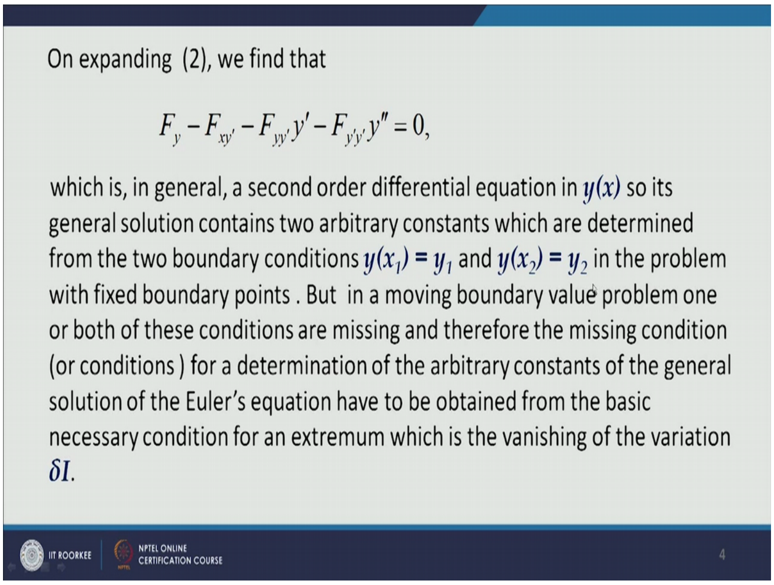

So we get F y minus F x y dash we get F y minus F x y dash minus F y y dash into y dash minus F y y dash F y dash y dash into y double dash equal to 0. Now this is a second order differential equation in y x so its solution will involve two arbitrary constants and these two arbitrary constants can be determined from the two boundary conditions y x 1 equal to y 1 and y x 2 equal to y 2 in the problem with fixed boundary points.

Now in the case of a moving boundary problem one or both of these conditions are missing and therefore the missing condition are conditions if if one boundary point is fixed there will be one mixing condition, if both the boundary points are moving then then we will be missing two conditions. So for a determination of the arbitrary constants of the general solution of the Eulers Equation have to be obtained from the basic necessary condition for an extremum which is vanishing of the variation delta I.

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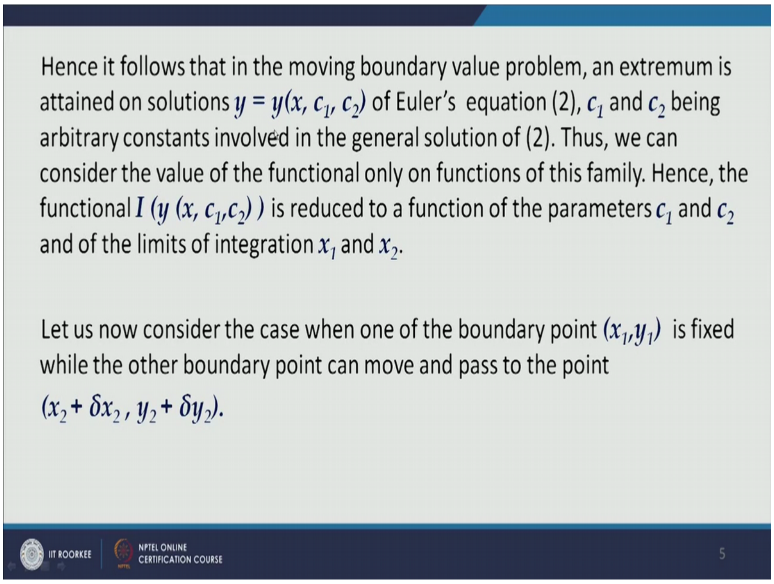

Now, hence it follows that in the moving boundary value problem an extremum is attained on solutions y equal to y x, c 1, c 2 of Eulers Equation the Eulers Equation is F y minus d over dx F y dash equal to 0 where c 1 and c 2 are arbitrary constants involved in the general solution of the Eulers Equation. Now so we can consider the value of the functional only on functions of the family y equal to y x, c 1, c 2.

Now here the functional I y x, c 1, c 2 is then a function of the parameters c 1 and c 2 and of the limits of the integration x 1 and x 2. Now let us consider the case first we discuss a simple case where one of the boundary point x 1, y 1 is fixed while the other boundary point can move and pass to the point x 2 plus delta x 2, y 2 plus delta y 2.

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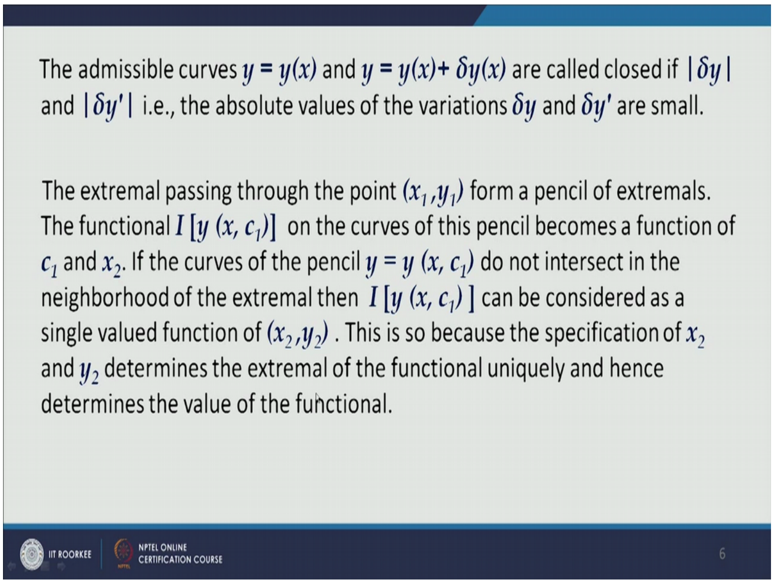

The admissible curves y equal to y x and y equal to y x plus delta y x will be called closed if mod of delta y and mod of delta y dash that is the absolute values of the variations delta y and delta y dash are small. The extremal passing through the point x 1, y 1 form a pencil of extremals.

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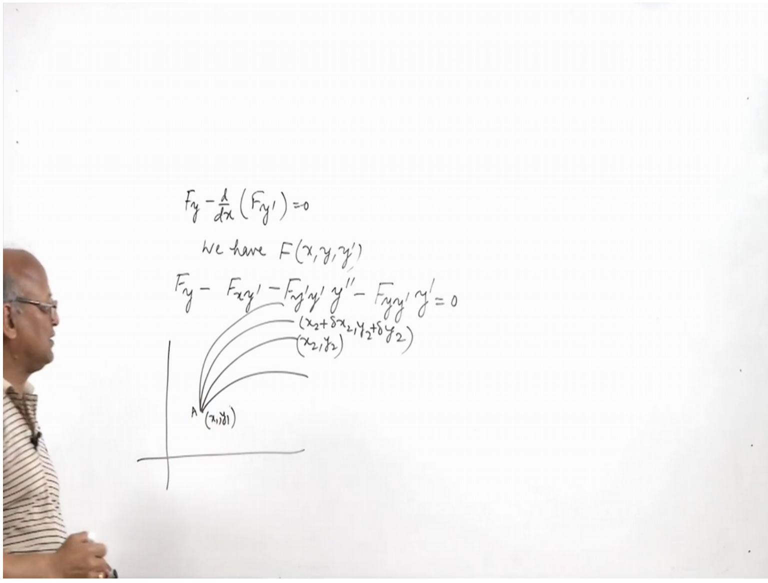

You can see how it forms a pencil of extremals let us say this is our point A x 1, y 1 then we have a pencil of extremals. So this is x 2, y 2, this is x 2 plus delta x 2, y 2 plus delta y 2 we call it a pencil of curves because it is the shape of shape is that of a pencil. Now so the extremal passing through the point x 1, y 1 form a pencil of extremals.

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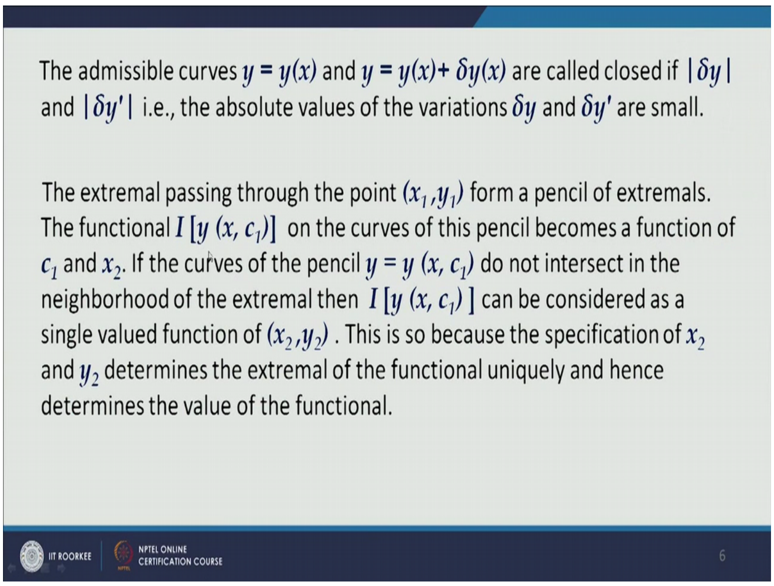

Now the functional I y x, c 1 on the curves of this pencil becomes a function of then c 1 and x 2. If the curves of the family curves of the pencil y equal to y x, c 1 do not intersect in the neighborhood of the extremal then I y x, c 1 can be considered as a single valued function of x 2, y 2. This is because the specification x 2 and y 2 determines the extremal of the functional uniquely and hence determines the value of the functional.

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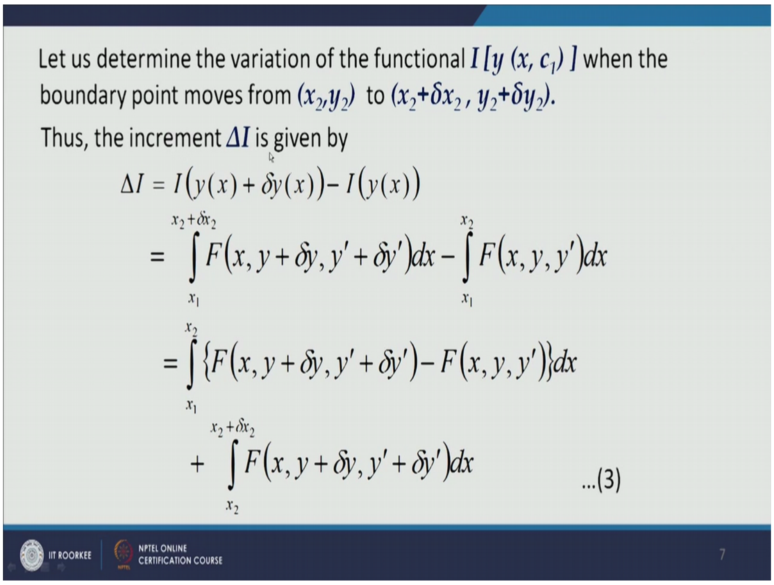

Now let us determine the variation of the functional I y x, c 1 when the boundary point moves from x 2, y 2 to x 2 plus delta x 2, y 2 plus delta y 2. So in this case the increment delta I is then given by delta I equal to I y x plus delta y x minus I y x and I y x plus delta y x will then be integral x 1 to x 2 plus delta x 2 because we are moving from x 2, y 2 to x 2 plus delta x 2, y 2 plus delta y 2, so x 1 to x 2 plus delta x 2 F x, y plus delta y and y dash plus delta y dash dx and I y x will be x 1 to x 2 F x, y, y dash dx. Now this is integral x 1 to x 2 plus delta x 2 can be written as integral x 1 to x 2 plus integral x 2 plus delta x 2 to x 2 plus delta x 2.

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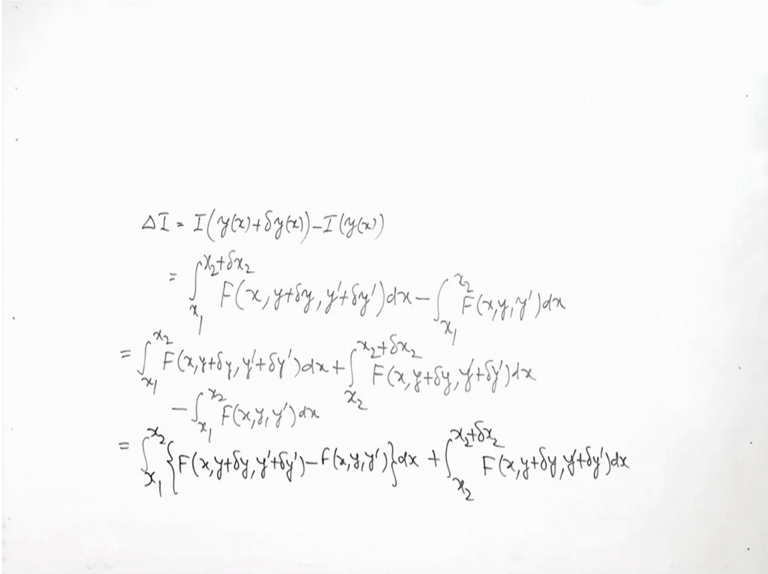

We can write delta I delta I is equal to I y x plus delta y x minus I y x which is equal to integral x 2 to x 2 plus delta x 2 integral x 1 to x 2 plus delta x 2 F x, y plus delta y, y dash plus delta y dash dx minus integral x 1 to x 2 F x, y, y dash dx. Now we can write it as integral x 1 to x 2 F x, y plus delta y, y dash plus delta y dash dx plus integral x 2 to x 2 plus delta x 2 F x, y plus delta y, y dash plus delta y dash dx minus x 1 to x 2 F x, y, y dash dx and this can be combined as integral x 1 to x 2 x 1 to x 2 F x, y plus delta y, y dash plus delta y dash minus F x, y, y dash dx plus integral x 2 to x 2 plus delta x 2 F x, y plus delta y, y dash plus delta y dash dx.

So this is how we get the integral x 1 to x 2 F x, y plus delta y, y dash plus delta y dash minus F x, y, y dash dx plus integral x 2 to x 2 plus delta x 2 F x, y plus delta y, y dash plus delta y dash dx.

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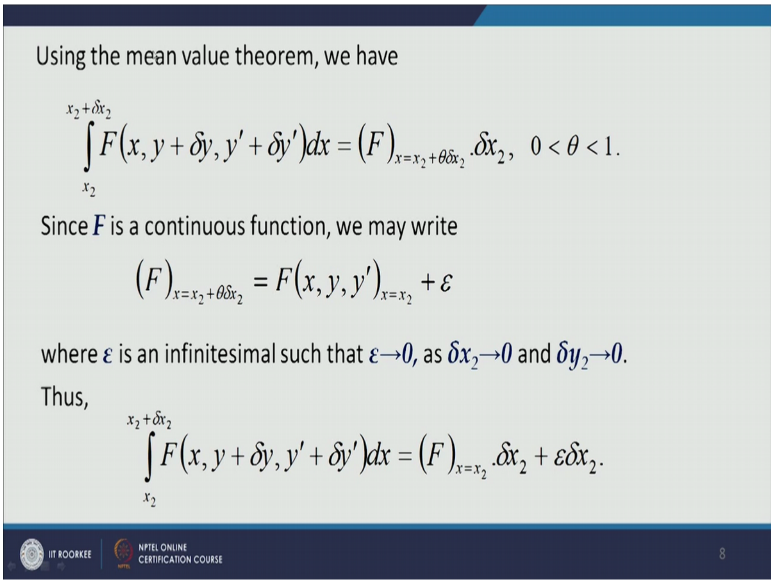

Now let us use the mean value theorem, when we use the mean value theorem the second integral on the right x 2 to x 2 plus delta x 2 F x, y plus delta y, y dash plus delta y dash dx will be equal to the value of F calculated at x equal to x 2 plus theta delta x 2 where theta is between 0 and 1, so this x 2 plus theta delta x 2 is a point in between x 2 and x 2 plus delta x 2 into the length of the intervals. So this is delta x 2.

So using mean value theorem we can write the second integral on the right as F at x equal to x 2 plus theta delta x 2 into delta x 2. Now we know that F is a continuous function, so we may write F at x equal to x 2 plus theta delta x 2 is equal to F x, y, y dash at x equal to x 2 plus epsilon, where epsilon is an infinitesimal such that epsilon goes to 0, as delta x 2 goes to 0 and delta y 2 goes to 0.

Thus we can write integral x 2 to x 2 plus delta x 2 as F x, y plus delta y, y dash plus delta y dash dx equal to F at x equal to x 2 into delta x 2 plus epsilon delta x 2. Let us put the value of this F at x 2 plus theta delta x 2 here, so that we get this.

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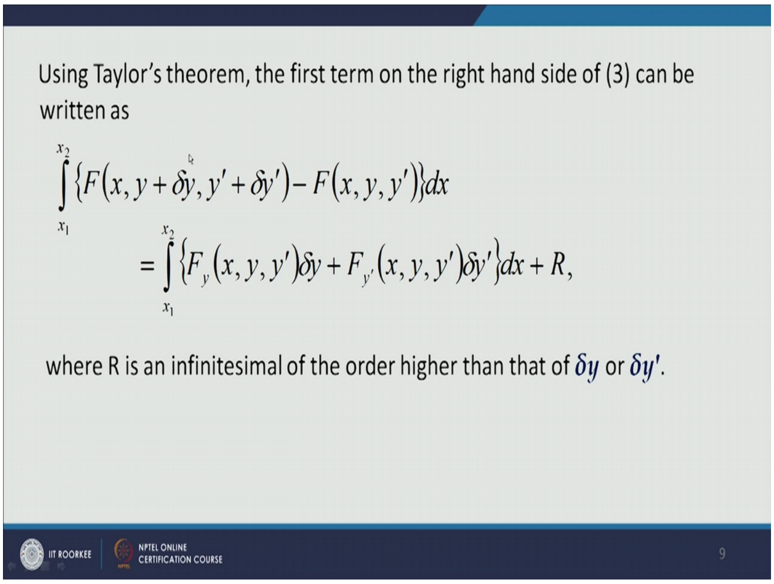

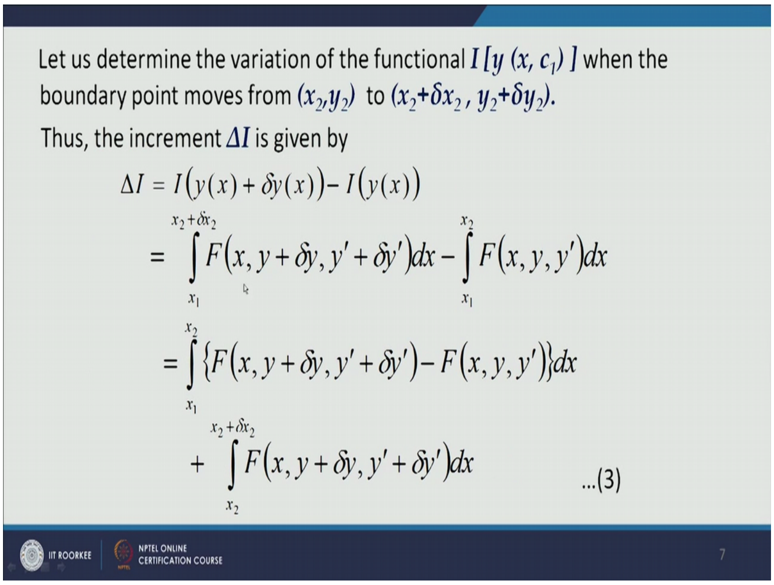

Now using Taylors theorem the first term using Taylors theorem the first term on the right side of this equation, equation number 3 the equation number 3 is this one, this one. So this is the first term on the right side integral x 1 to x 2 F x, y plus delta y, y dash plus delta y dash minus F x, y, y dash F x, y, y dash.

Now this can be written as integral x 1 to x 2 F y x, y, y dash into delta y plus F y dash into x, y, y dash into delta y dash dx plus R where R is an infinitesimal of the order higher than that of delta y or delta y dash.

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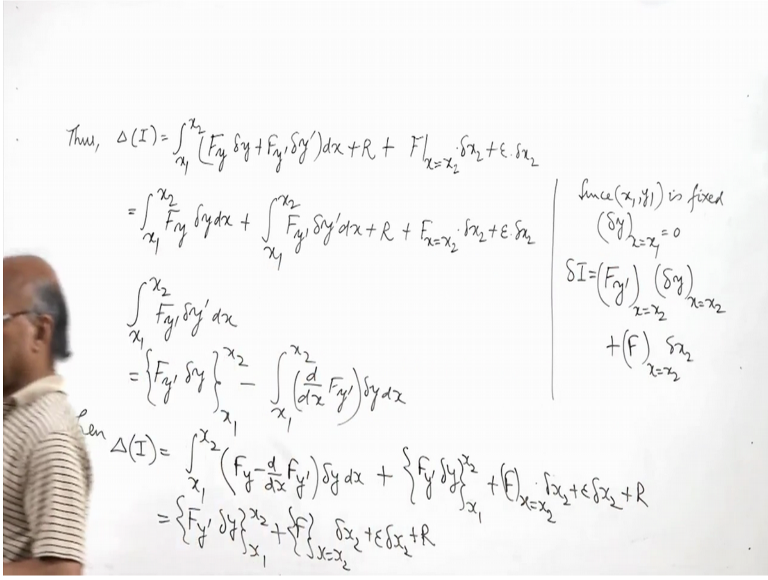

Now integral so what we will have the right hand side will become so thus this we have delta I equal to integral x 1 to x 2 F x, y plus delta y by Taylors theorem we wrote it as derivative with respect to y F y into delta F x, y plus delta y, y dash plus delta y minus F x, y, y dash we wrote as F y delta y plus F y dash delta y dash. So F y delta y plus F y dash delta y dash dx plus R we wrote it as plus R where R is the infinitesimal and x 2 to x 2 plus delta x 2 we wrote as by using the mean value theorem we wrote it as F at x equal to x 2 into delta x 2 plus epsilon into delta x 2.

So we get the so let us now what we do is the let me integrate by parts so we can write it like this x 1 to x 2 F y delta y into dx plus integral x 1 to x 2 F y dash into delta y dash into dx plus R plus F at x equal to x 2 into delta x 2 plus epsilon into delta x 2. Now let us let us evaluate this integral if you evaluate integral x 1 to x 2 by integration by parts let us integrate by parts so we shall have F y dash into delta y minus d over dx of F y dash into delta y into dx.

Now so let us put this here so then delta I is equal to integral x 1 to x 2 F y integral x 1 to x 2 F y delta y delta x we can combine with this and write as F y minus d over dx F y dash into delta y dx. Then we have this one F y dash into delta y x 1 to x 2 and then we have F at x equal to x 2 into delta x 2 plus epsilon delta x 2, so this is what we have.

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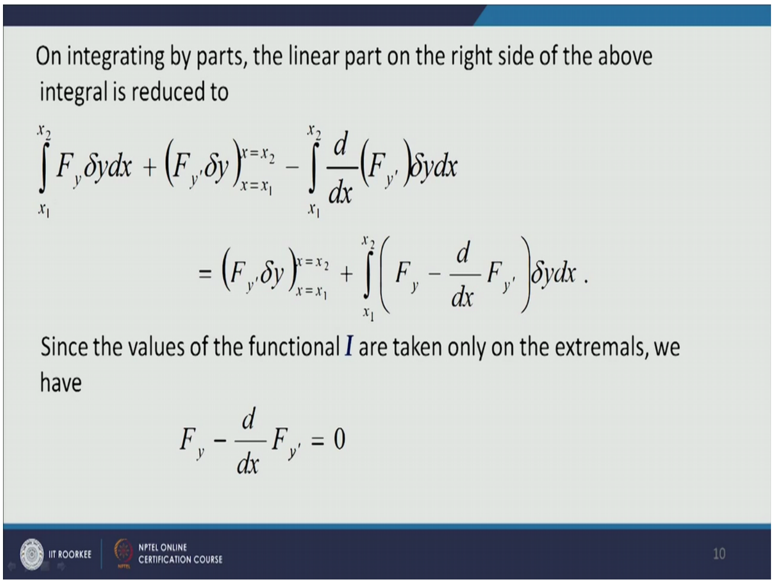

Now since the values of the functional are taken on the extremals therefore by the Eulers theorem we have F y minus d over dx F y dash is equal to 0, so this expression becomes 0 and therefore this becomes F y dash delta y x 1 to x 2 plus F at x equal to x 2 into delta x 2 plus epsilon delta x 2 and we also have R here.

Now now choose the boundary point x 1, y 1 is fixed delta y at x 1, y 1 is equal to 0. So since x 1, y 1 point is fixed delta y at x equal to x 1 is equal to 0 and thus the first expression on the right side becomes F y dash delta y at x equal to x 2. So we and more over one epsilon goes to 0, one delta x 2, delta y 2 goes to 0 and R is also infinitesimally small. So we can write delta I equal to delta I equal to F y dash at x equal to x 2 into delta y at x equal to x 2 plus F at x equal to x 2 delta x 2.

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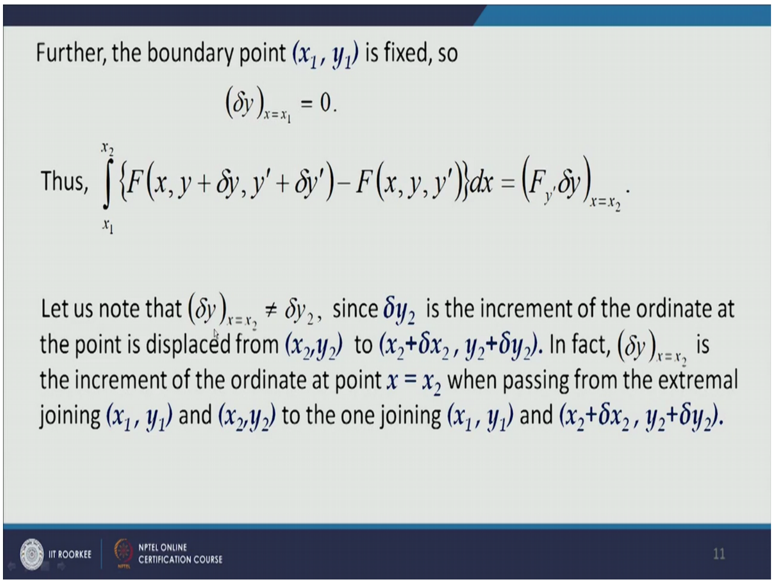

Now let us note that delta y at x equal to x 2 is not equal to delta y 2.

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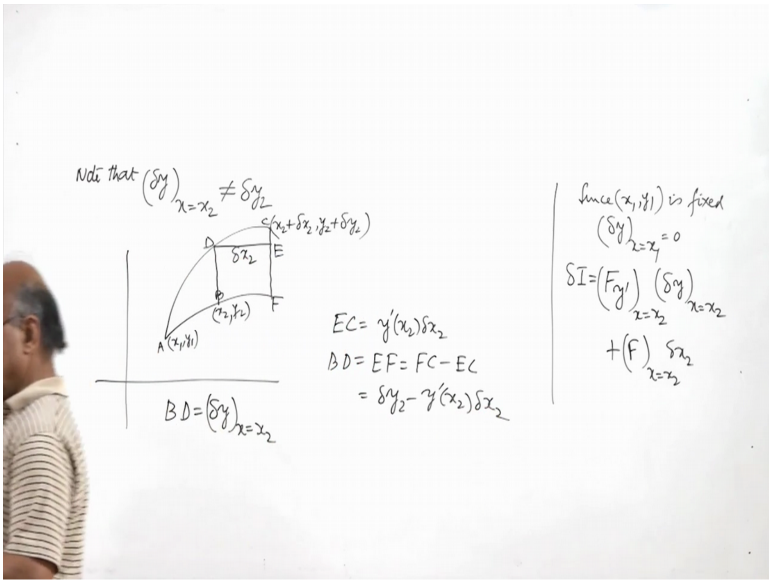

So if we use that delta y note that this is not equal to delta y 2. What we do is let us see how it is not equal say this is your point A, this is your x 2 plus delta x 2, y 2 plus delta y 2, here is the point x 2, y 2, this is your x 1, y 1. Now so this is your point B let us say so now what we do is this is A, D, E let us take this as F and this as C.

Now since delta y 2 is the increment of y 2 when the boundary point is displaced from x 2, y 2 to x 2 plus delta x 2, y 2 plus delta y 2 in fact delta y at x equal to x 2 is the increment of the ordinate at the point x equal to x 2 when passing from the extremal joining x 1, y 1 and x 2, y 2 to the one joining x 1, y 1 and x 2 plus delta x 2, y 2 plus delta y 2.

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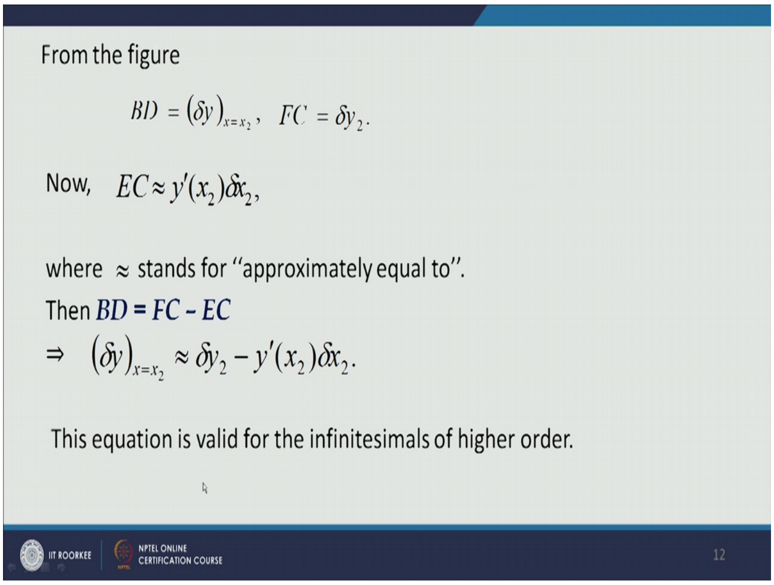

So BD is equal to delta y, x equal to x 2 BD is delta y, x equal to x 2 while FC is equal to delta y 2. So FC is delta y 2 and now EC EC can be approximately equal to y dash x 2 into delta x 2 because this width is delta x 2, y dash at x 2 is the slope of the curve y x at x equal to x 2, so approximately we can assume that EC is nothing but the slope of the curve at x 2 multiplied by the width DE that is delta x 2. So EC is approximately equal to delta x 2 and therefore BD BD is equal to FC BD is equal FC minus BD is equal to EF and EF is equal to FC minus EC. So FC minus EC and so this FC is equal to delta y 2 so delta y 2 minus y dash x 2 into delta x 2. So BD is approximately equal to delta y 2 minus y dash x 2 into delta x 2 this equation is valid for the infinitesimals of higher order.

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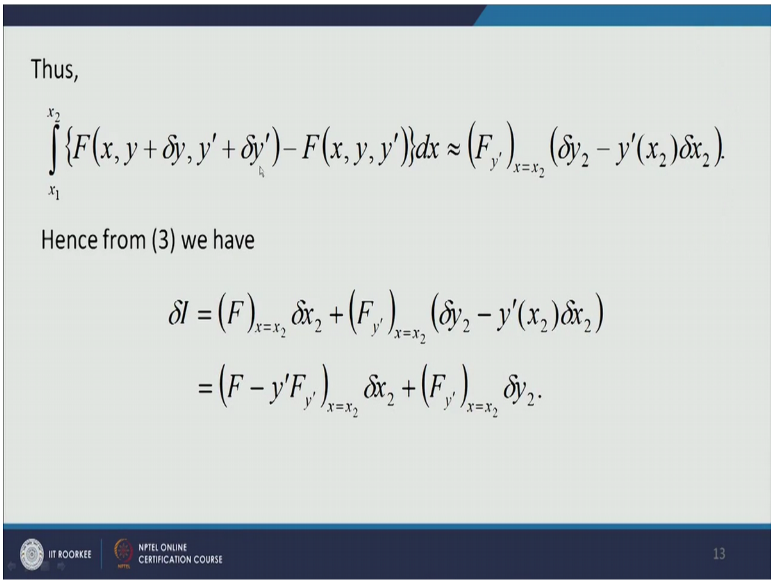

Now thus integral x 1 to x 2 F x, y plus delta y, y dash plus delta y dash minus F x, y, y dash dx which was equal to F y dash at x equal to x 2 into delta y at x equal to x 2, okay.

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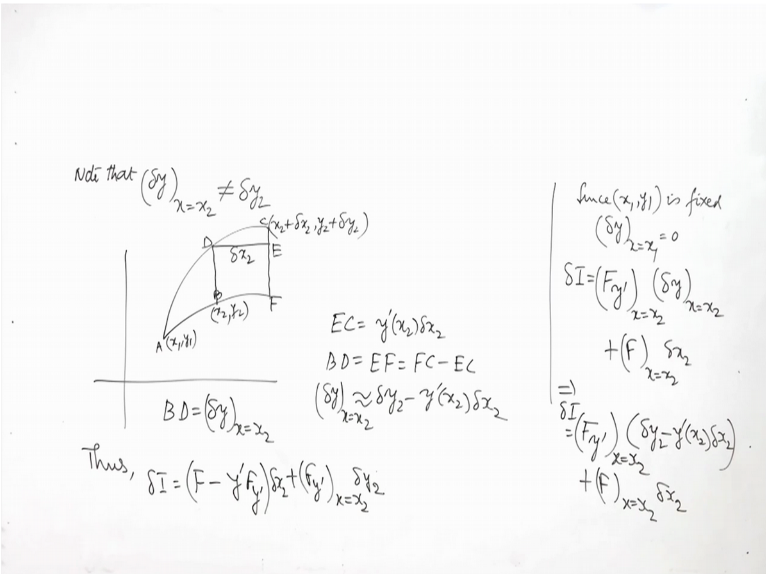

This is so so thus delta I is equal to F y dash at x equal to x 2 into delta y at x equal to x 2 delta y at x equal to x 2 is approximately equal to delta y 2 so we have delta y 2 minus y dash x 2 delta x 2 plus F at x equal to x 2 delta x 2. This is this was equal to F at x equal to x 2 into delta x 2, yes. So we have now combined let us combine the two, so what we have? Delta I is equal to F minus y dash F minus y dash into delta x 2 plus F y dash at x equal to x 2 into delta y 2 F y dash into F y dash F y dash will also come here. So F minus y dash F y dash into delta x 2 plus F y dash at x equal to x 2 into delta y 2.

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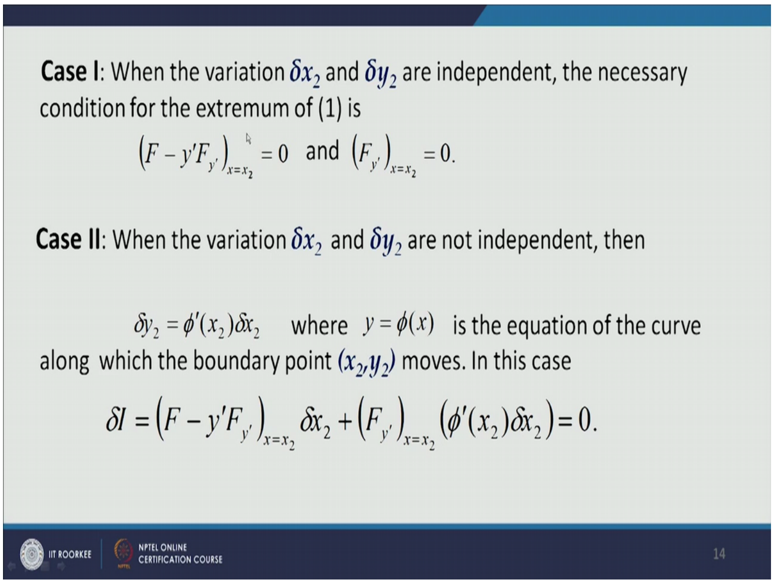

Now now let us discuss the 2 cases, 1 case is the case where the variations delta x 2 and delta y 2 are independent, the necessary then we apply the necessary condition for the extremum of the functional given by 1 so in that case F minus y dash F y dash if delta x 2 delta y 2 is equal to are independent of each other.

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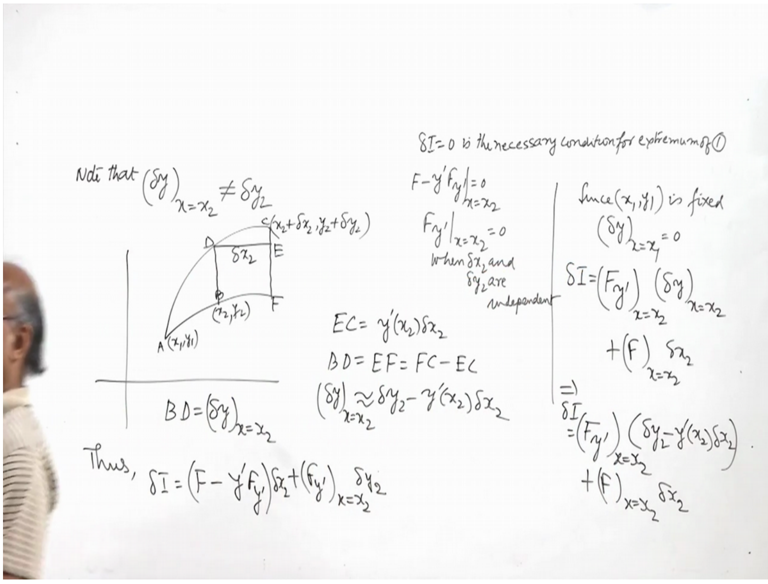

Than delta I equal to 0 will give us F minus y dash F y dash is equal to 0 because the condition for the extremum is delta I equal to 0 this is the necessary condition for the extremum. So if delta x 2, delta y 2 are independent we have F minus y dash F y dash equal to 0 at x equal to x 2 and we also have F y dash at x equal to x 2 equal to 0 when delta x 2, delta y 2 are independent, so this case 1.

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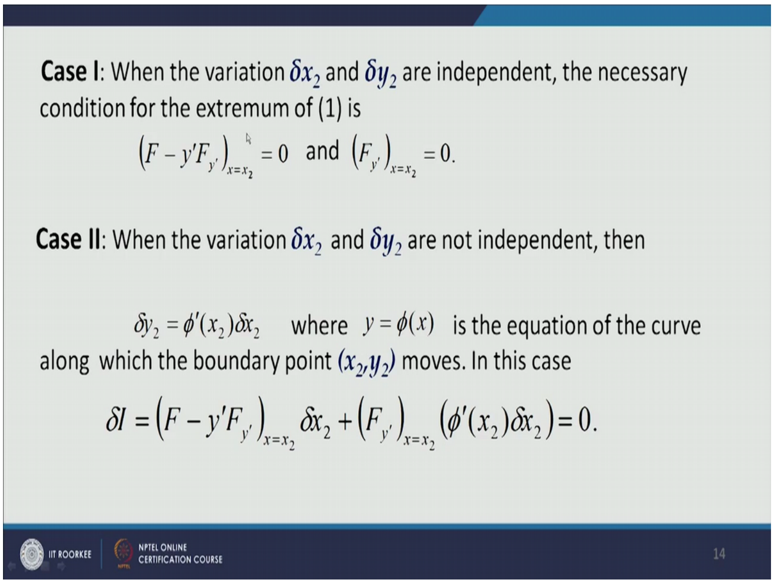

In the case 2 when the variation delta x 2 and delta y 2 are not independent in that case we can write delta y 2 equal to phi dash x 2 into delta x 2 where y equal to phi x is the equation of the curve along which the boundary point x 2, y 2 moves.

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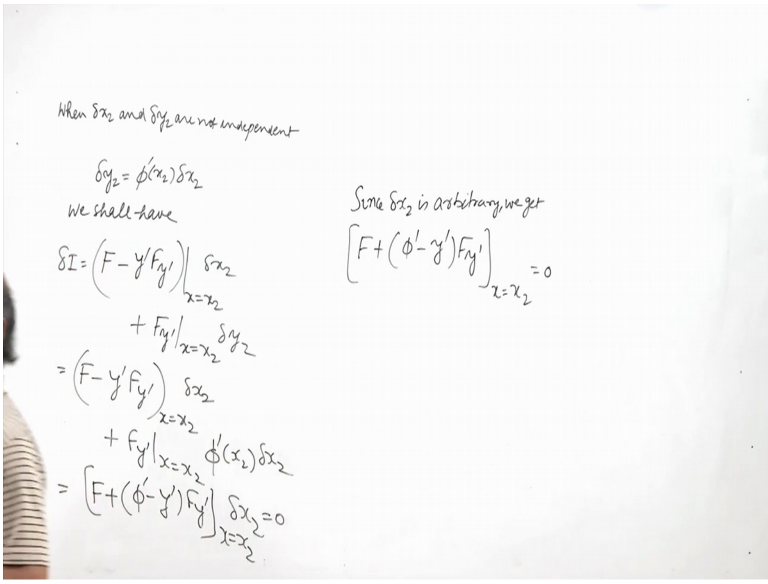

So in this case delta I will be equal to, so when delta x 2 and delta y 2 are not independent. So in this case in this case delta y 2 will be equal to phi dash x 2 into delta x 2 and we therefore will have so we shall have delta I delta I was equal to F minus y dash into F y dash at x equal to x 2 delta x 2 plus F y dash at x equal to x 2 into delta y 2 this will be changed to F minus y dash F y dash at x equal to x 2 into delta x 2 plus F y dash at x equal to x 2, delta y 2 will be replaced by phi dash x 2 delta x 2 and we can combine the term to two terms and we write (F minus) F plus phi dash minus y dash F y dash at x equal to x 2 delta x 2 equal to 0.

So since delta x 2 is arbitrary we shall have so since delta x 2 is arbitrary we get F plus phi dash minus y dash into F y dash at x equal to x 2 is equal to 0. So this condition is to be satisfied at the point x 2, y 2 and this condition is known as the transversality condition.

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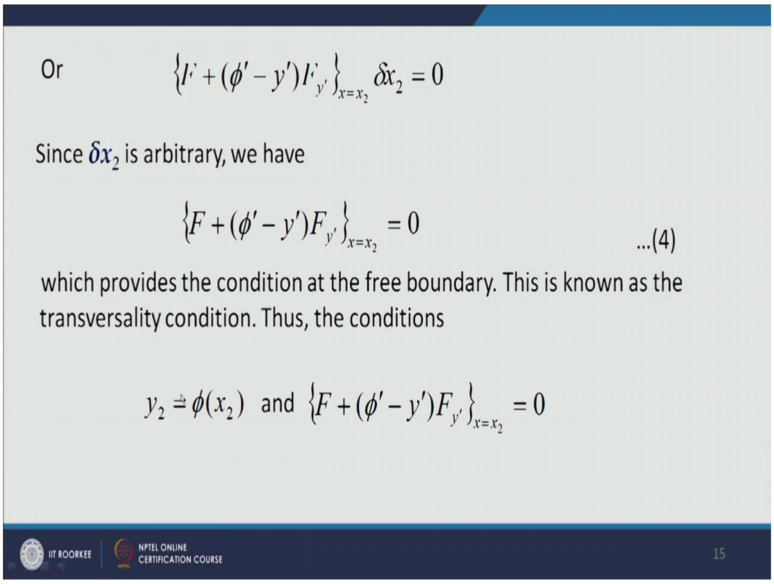

So thus we have two conditions y 2 equal to phi x 2 because x 2, y 2 point lies on the curve y equal to phi x and the transversality condition which is F plus phi dash minus y dash F y dash evaluated at x equal to x 2 is equal to 0.

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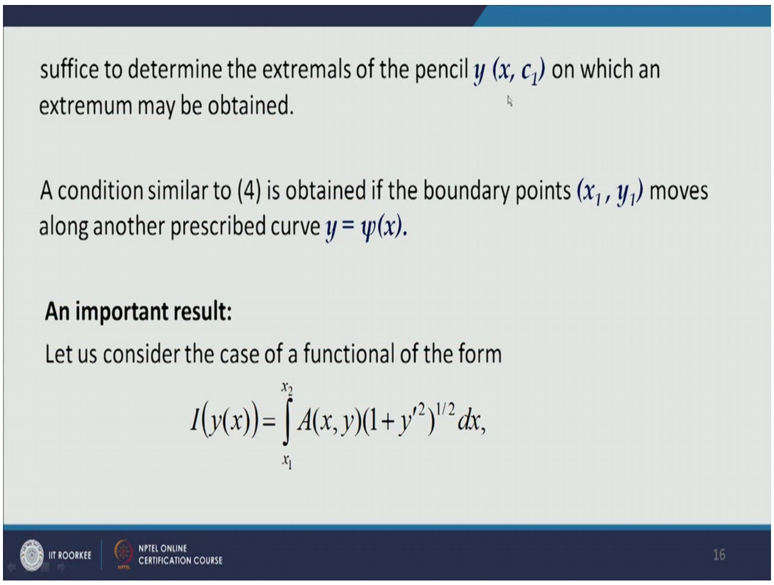

Now these two conditions are sufficient to determine the extremals of the pencil y x, c 1 on which an extremum may be obtained. Now in case the boundary point x 1, y 1 is not fixed it moves on the curve y equal to Psi x then a condition similar to the transversality condition can be derived in a similar manner.

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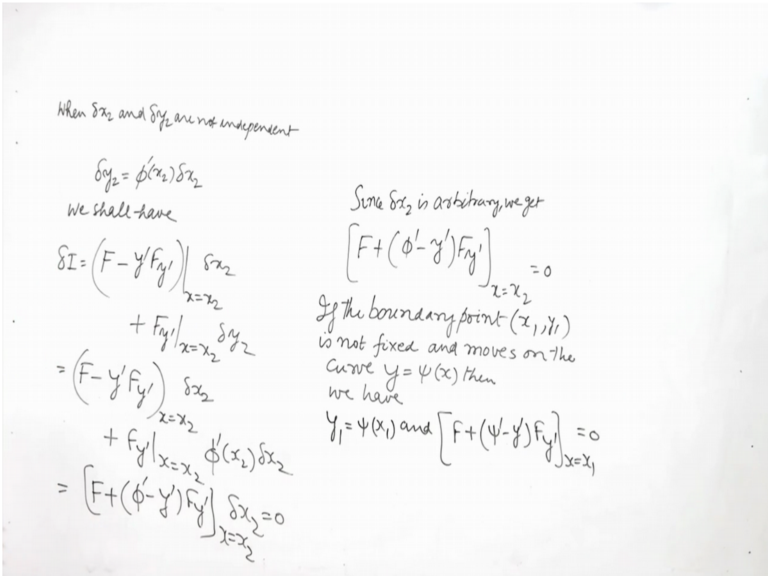

And if we have y equal to Psi x then we shall have the if the boundary point if the boundary point x 1, y 1 is not fixed but moves is not fixed and moves on the curve y equal to Psi x then we have y 1 equal to Psi x 1 y 1 equal to Psi x 1 and the transversality condition F plus Psi dash minus y dash F y dash at x equal to x 1 equal to 0 they can be derived this can be derived in a similar manner.

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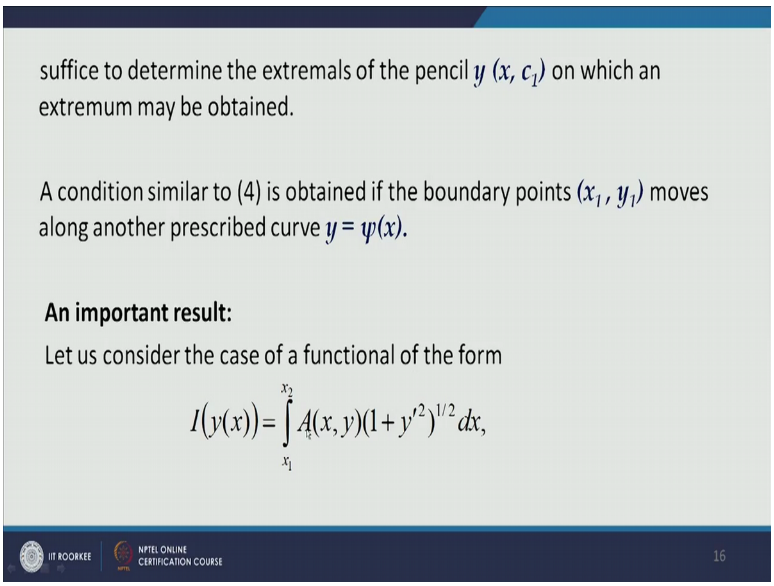

So now let us discuss an important result suppose that we have a functional of the form I y x equal to integral x 1 to x 2 A x, y into 1 plus y dash square raise to the power half dx.

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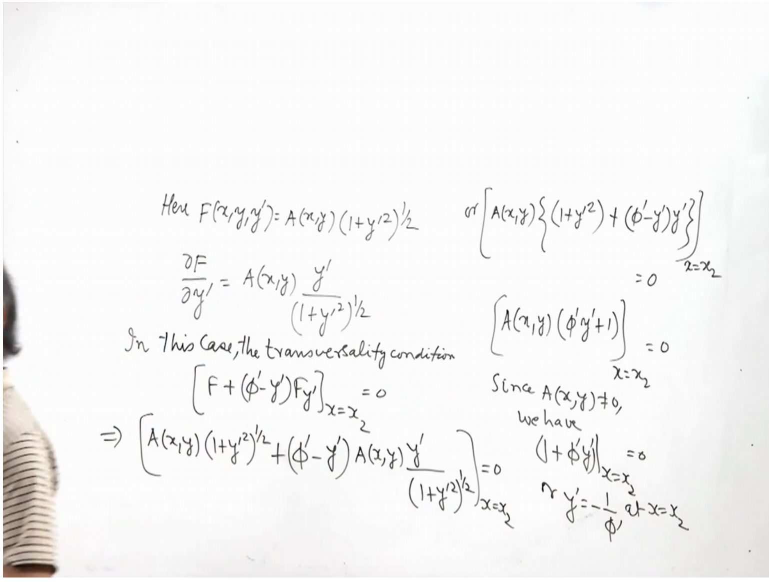

So here F x, y, y dash is equal to A x, y into 1 plus y dash square raise to the power half and we assume that we assume that okay we assume that this A x, y does not vanish we assume that A x, y does not vanish so then let us find the partial derivative of this with respect to y dash. So what we will get A x, y into y dash divided by 1 plus y dash square raise to the power half.

So in this case the transversality condition at x equal to x 2 gives this will imply that we have F is equal to A x, y into 1 plus y dash square raise to the power half and we will have phi dash minus y dash multiplied by F y dash, so F y dash is A x, y into y dash divided by 1 plus y dash square to the power half, or we can write it as or A x, y into 1 plus y dash square when we differentiate this with respect to y dash we get okay we get phi dash minus y dash into y dash equal to 0 at x equal to x 2.

Now what we will get here phi dash y dash, so A x, y into phi dash y dash plus 1 at x equal to x 2 is equal to 0. But A x, y is not equal to 0 since x, y does not vanish we have 1 plus phi dash y dash is equal to 0, 1 plus phi dash y dash at x equal to x 2 equal to 0 or we can say y dash is equal to minus 1 over phi dash at x equal to x 2. This means that the extremal extremizing curve c which is y equal to y x is orthogonal to the curve y equal to phi x on which the point x 2, y 2 moves.

So this is the case where the curve the extremizing curve y equal to y x and the curve y equal to phi x on which the point x 2, y 2 moves they are orthogonal at the point x 2, x equal to x 2, y equal to y 2. So this is the orthogonality condition. So this is a special case where F x, y, y dash is of a particular form, it is A x, y into 1 plus y dash square to the power half where we assume that x, y is not equal to 0. So with this I would like to conclude my lecture, thank you very much.