Integral Equations, Calculus of Variations and their Applications

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Lecture 54

Variational Problem Involving a Conditional Extremum-2

Hello friends welcome to today's lecture and in today's lecture we will continue our discussion.

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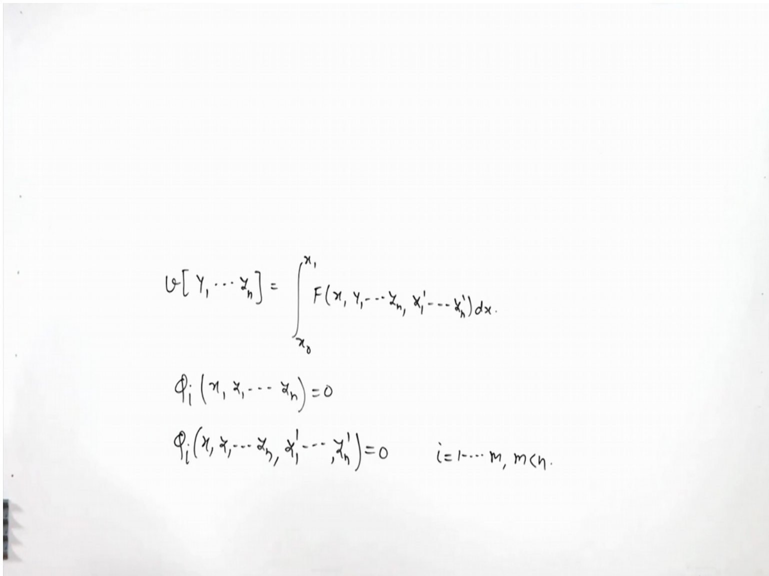

In previous lecture we have considered the variational problem with constraint this. Now in todays class we extend our discussion when this constraint is not only this finite equation but set of equations here. So we have y 1 dash to say y n dash is equal to 0 here, i is from 1 to say m, here m is less than n and try to see that how to find out the extremal functions of this variational problem provided this constraints are given.

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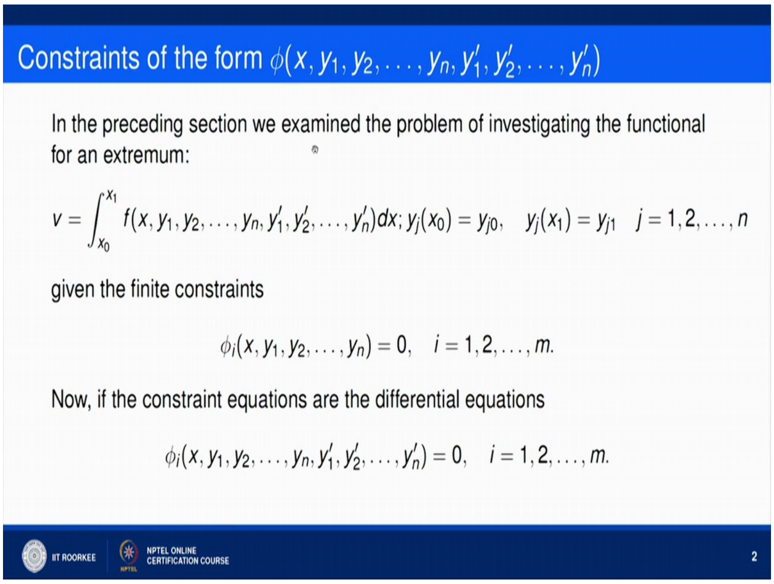

So let us consider here. So as I pointed out in the preceding section we examined the problem of investigating the functional for an extremum v equal to x not to x 1 f of x, y 1 to y n, y 1 dash to y n dash d of x satisfying the boundary condition given at the end x not and given at the end x 1 as y j x not equal to j j not and y j x 1 is equal to y j 1 for each j is equal to 1 to n and here in we have discussed that we have constraint given in this form y i x, y 1 to y n equal to 0, i is from 1 to m.

But now let us consider the constraint in place of finite constraint let us consider the set of differential equations. So here constraints are set of differential equation phi i x, y 1 to y n, y 1 dash to y n dash equal to 0 for i equal to 1 to m, here m is strictly less than n.

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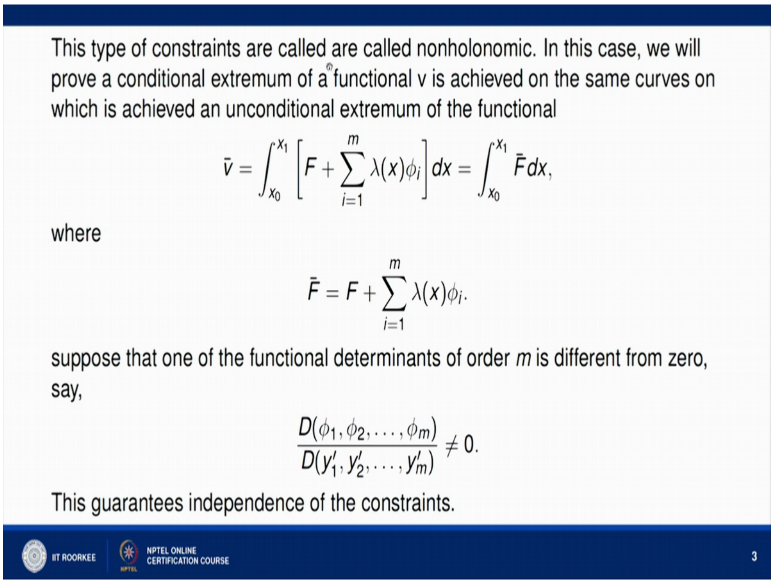

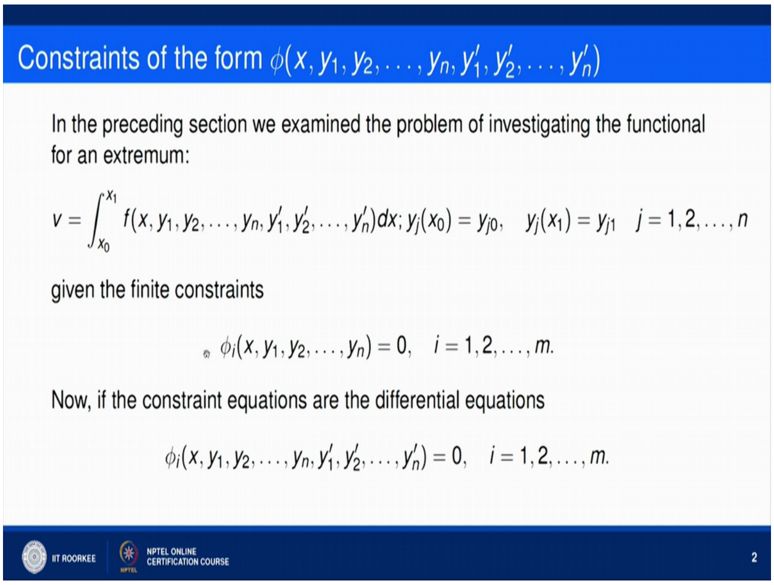

So this type of constraint are called nonholonomic and the constraint defined here are called holonomic. So this we call as holonomic constraint and this we call as nonholonomic constraint. So in this case we will prove a conditional extremum of a functional v is achieved on the same curve on which it is achieved an unconditional extremum of the functional here. So as we discussed in previous case we also show that the extremal conditional extremum of v achieved on conditional extremum of this v bar.

So here we defined a new function F bar as F plus i equal to 1 to m lambda x phi of i and here we assume that these phi i are are independent to each other it means that we are assuming that Jacobian of phi 1 to phi m with respect to y 1 dash to y m dash is not equal to 0. So here we are relabeling our variable in the sense that this Jacobian of phi 1 to phi m with respect to y 1 dash to y m dash are nonzero and this also guarantees the independence of the constraint.

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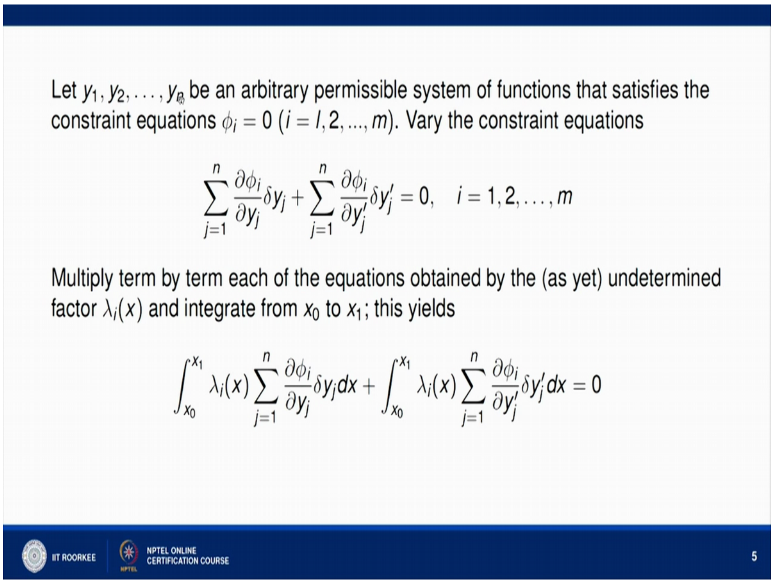

So here we try to proceed in a similar way as we proceeded in previous section here. So here let us take y 1 to y n be an arbitrary permissible system of function that satisfy the constraint equation phi i equal to 0 for each i equal to 1 to m. Now here we vary the constraint equation and we have this j equal to 1 to n deba phi i by deba y j delta y j plus summation j equal to 1 deba phi i by deba y j dash delta y j dash equal to 0 for each i equal to 1 to m.

If you remember in previous case this term is not coming so so this term was not there in previous discussion. So as we discussed there what we try to do here we multiply by say each term by lambda i and integrate from x not to x 1 and we have this x not to x 1 lambda i x summation j equal to 1 to n deba phi i by deba y j delta y j dx plus x not to x 1 lambda i x summation j equal to 1 to n deba phi i by deba y j dash delta y j dash dx equal to 0. So again we have then addition of this term. So here we simplify the second term by transferring the derivative on this term.

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integrating each term of the second integral by parts and taking into consideration that $\delta y_j' = (\delta y_j)'$ and $(\delta y_j)_{x=x_0} = (\delta y_j)_{x=x_1} = 0$, we will have

$$\int_{x_0}^{x_1} \sum_{j=1}^{n} \left[\lambda_i(x) \frac{\partial \phi_i}{\partial y_j} \delta y_j - \frac{d}{dx} \left(\lambda_i(x) \frac{\partial \phi_i}{\partial y_j'} \right) \right] \delta y_j dx = 0. \qquad (1)$$

From the basic necessary condition for an extremum, $\delta v = 0$, we have

$$\int_{x_0}^{x_1} \sum_{j=1}^{n} \left(F_{y_j} - \frac{d}{dx} F_{y_j'} \right) \delta y_j dx = 0, \qquad (2)$$

Since

$$\delta v = \int_{x_0}^{x_1} \sum_{j=1}^{n} (F_{y_j} \delta y_j + F_{y_j'} \delta y_j') dx = \int_{x_0}^{x_1} \sum_{j=1}^{n} \left(F_{y_j} - \frac{d}{dx} F_{y_j'} \right) \delta y_j dx$$

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Adding termwise all the equations (1) and equation (2) and introducing the notation

$$\bar{F} = F + \sum_{i=1}^{m} \lambda_i(x) \phi_i$$

we have

$$\int_{x_0}^{x_1} \sum_{j=1}^{n} \left(\bar{F}_{y_j} - \frac{d}{dx} \bar{F}_{y_j'} \right) \delta y_j dx = 0 \qquad (3)$$

Since the variations $\delta y_j, j = 1, 2, \ldots, n$ are not arbitrary, we cannot yet use the fundamental lemma. Choose m factors $\lambda_1(x), \lambda_2(x), \ldots, \lambda_m(x)$ so that they satisfy the equations

$$\bar{F}_{y_j} - \frac{d}{dx} \bar{F}_{y_j'} = 0, \quad j = 1, 2, \ldots, m$$

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So for this we integrate each term of the second integral by parts and taking into considering that delta y j dash equal to delta y j whole dash and variation at the point x not and x 1 is simply 0. So by doing this we have this condition x not to x 1 j equal to 1 to n lambda i x deba phi i by deba y j delta y i and delta y j minus d by dx of lambda i x deba phi i by deba y j dash into delta y j dx equal to 0.

Now we will proceed further so now let us consider the functional here from the basic necessary condition for an extremum delta v has to be 0 which reduces to this condition x not to x 1 j equal to 1 to n F of y j minus d by dx F of y j dash delta y j dx is equal to 0. This we obtain from this that delta y is equal to this x not to x 1 j equal to 1 to n F of y j delta y j plus

F of y j dash delta y j dash d of x. So by transferring this derivative on this we are getting this equation number 2. So adding term vice all the equations 1 and equation 2.

So here we have this equation 1 which we obtain from the constraints and this which we obtain by necessary condition for extremum that is delta v equal to 0 so here we add term by term and we get this notation F bar equal to F plus summation i equal to 1 to m lambda i x phi i. Then this equation 1 and 2 can be written in a concise form as 3. So here 3 is what x not to x 1 j equal to 1 to n F bar y j minus d by dx of F bar y j dash delta y j dx equal to 0, where F bar is defined as F plus summation i equal to 1 lambda i x phi of i. Now again this variation delta y j, j is from 1 to n are not arbitrary, we cannot use the fundamental lemma. So here as we done in a previous case we choose m factor lambda 1 to lambda m so that they satisfy the equation F bar y j minus d by dx of F bar y j dash equal to 0 for each j equal to 1 to m.

So this we did in previous case also the only difference between this case and the previous case is that in previous case this is resulting into system of a linear equation.

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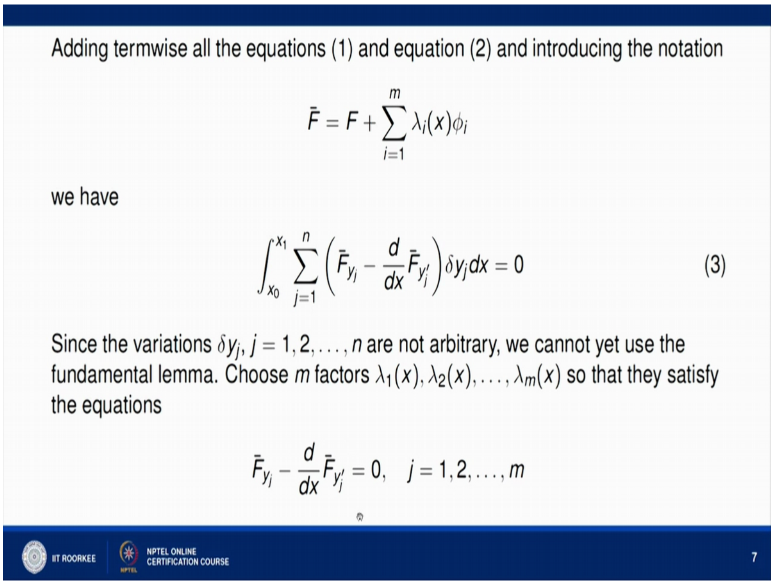

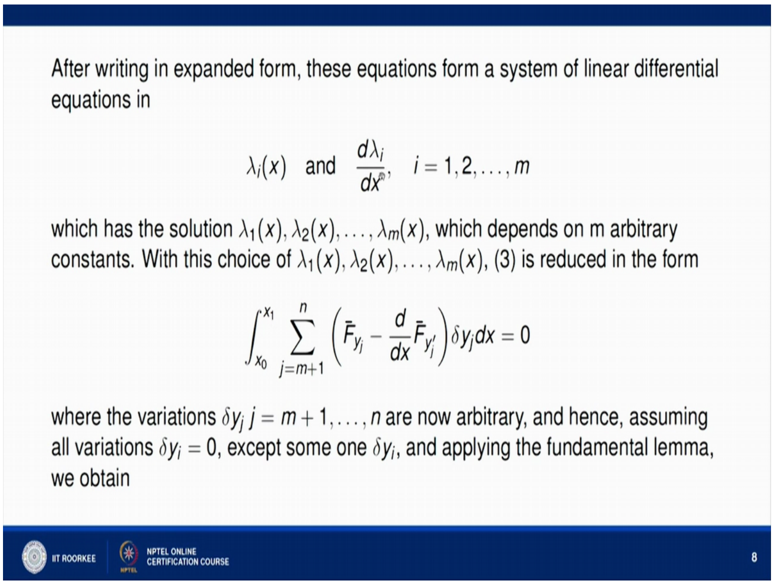

But in this particular case this will reduce to system of an linear differential equation in lambda i x and d lambda i by d of x. So this is the only difference between the previous case and this case and since it is a linear differential equation and the involved functions F are phi all are continuous. So we have existence guaranteed so and we can say that which has the solution lambda 1 to lambda m which depend on m arbitrary constants. Now with this choice of lambda 1 to lambda m.

Now this previous equation number 3 is reduced to from m plus 1 to n and we can say that it is x not to x 1 summation m plus 1 to n F bar y j minus d by dx F bar y j dash delta y j dx equal to 0. And here the variation delta y j, j from m plus 1 to n are now arbitrary and hence by assuming all variation except one as 0 we can apply the fundamental lemma.

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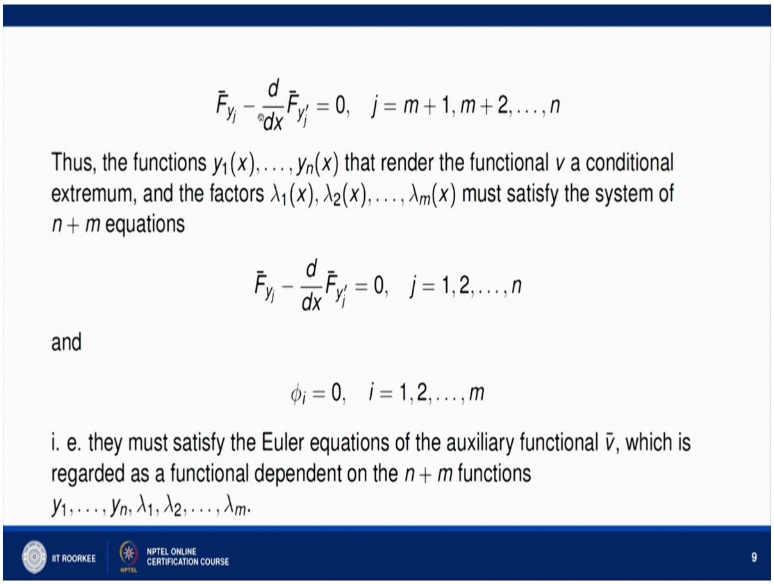

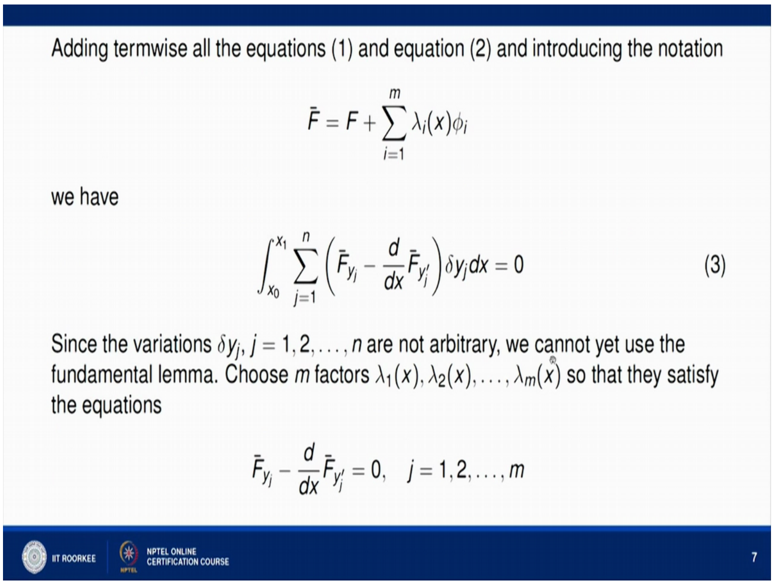

And we can get the from the Euler equation that F bar we can get equation F bar y j minus d by dx F bar y j dash equal to 0 for j equal to m plus 1 to n. And so if we combine equation like this which we have obtained by choosing factors lambda 1 to lambda m with this condition here F bar y j minus d by dx F bar y j dash equal to 0 which we obtain by applying the fundamental lemma. So now this will reduce to this this will extend to this F bar y j minus d by dx of F bar y j dash equal to 0 for j equal to 1 to n along with the condition phi i equal to 0 for i equal to 1 to m. Now these are n plus m equation which is sufficient to provide as the solution y 1 to y n and lambda 1 to lambda m.

So it means that the function y 1 to y n x that render the functional v a conditional extremum and the factor lambda 1 to lambda m x must satisfy the system of n plus m equation which is given by this and this.

It means that they must satisfy the Eulers Equation of the auxiliary functional v bar, which is regarded as functional dependent on the n plus m functions. So if we consider v bar as a function of this y 1 to y n and lambda 1 to lambda m then these n plus m equation can be obtained through the Eulers Equation applied on this v bar. So let us go to new problem and if you look at these this analysis is similar to the analysis which we have discussed in previous case the only change from previous study and this study is only these points let me remind you.

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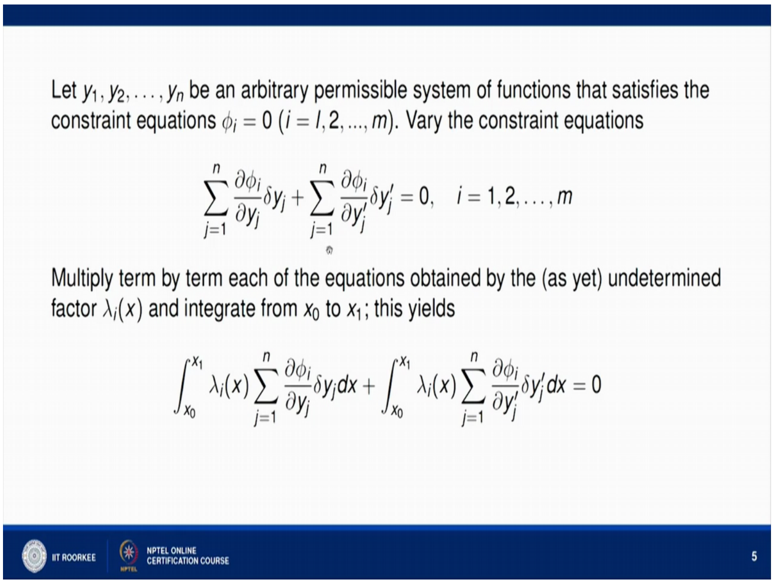

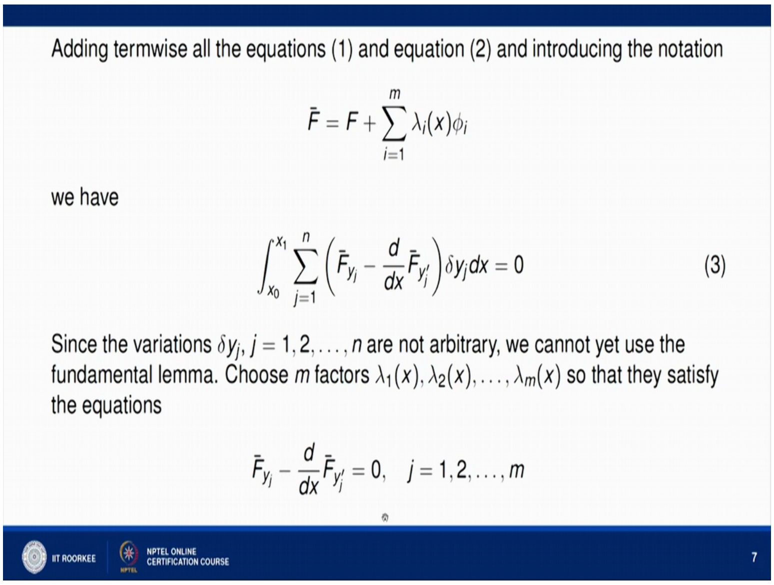

One addition is here, right another addition is here in the in terms of equation so here we are getting a differential equation rather than simple finite a linear system of linear equation and rest are almost same.

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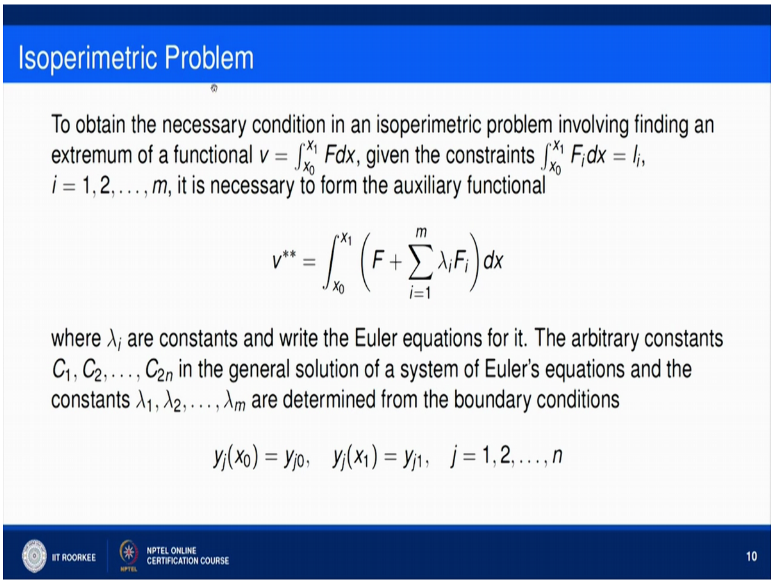

So now let us move to new problem that is isoperimetric problem and this we have already studied.

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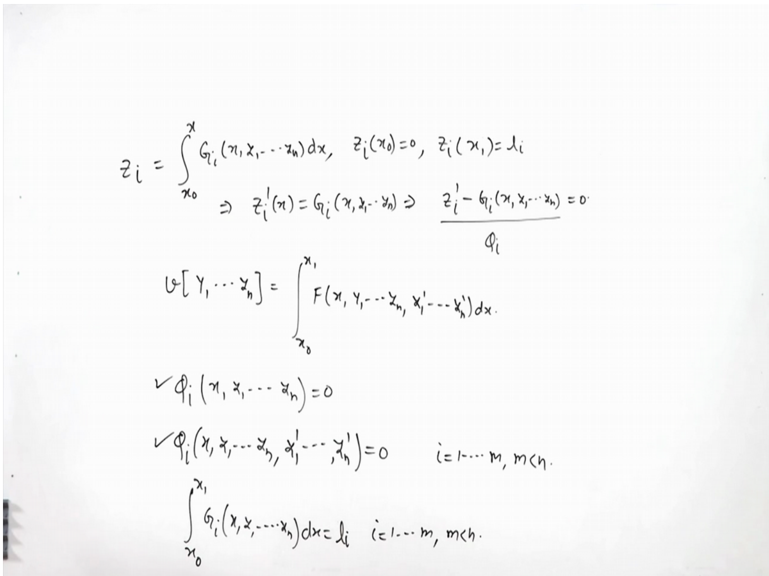

The only difference between these previously discussed problem here that in first we have considered that finite equation conditions are given in terms of finite equation and now we have already discussed the case when constraints are nothing but system of system of differential equation. Now in case of isoperimetric problem your conditions are given in terms of integral equation. So here we have x not to x 1 we have G i say x, y 1 to y n d of x equal to 0 for each i equal to 1 to m, m is less than n. So this also can be reduced to previous case by just assuming not 0 some values say l i.

So this can be reduced to this so problem of extremum of this along with this condition are known as isoperimetric problem which we have already discussed but here we can observe that this problem can be reduced to the problem which we have just discussed by assuming a new variable that is z i as x not to x, say G i x, y 1 to say y n d of x. So by here we can say that z i x not is simply 0 and z i your x 1 is x 1 is given as l of i.

So what is the advantage of considering this z i? That if we differentiate with respect to x here then we have z i dash x is equal to G of i x, y 1 to y n. This I can write it as z i dash minus j of i x, y 1 to say y n is equal to 0. So if we denote this as say phi i then this isoperimetric problem is reduced to the problem of finding the extremum of this functional along with this condition extremum, okay. So this we can solve. So taking this as phi i we can apply our discussion which we have just done just over and we can find out say the solution of the isoperimetric problem.

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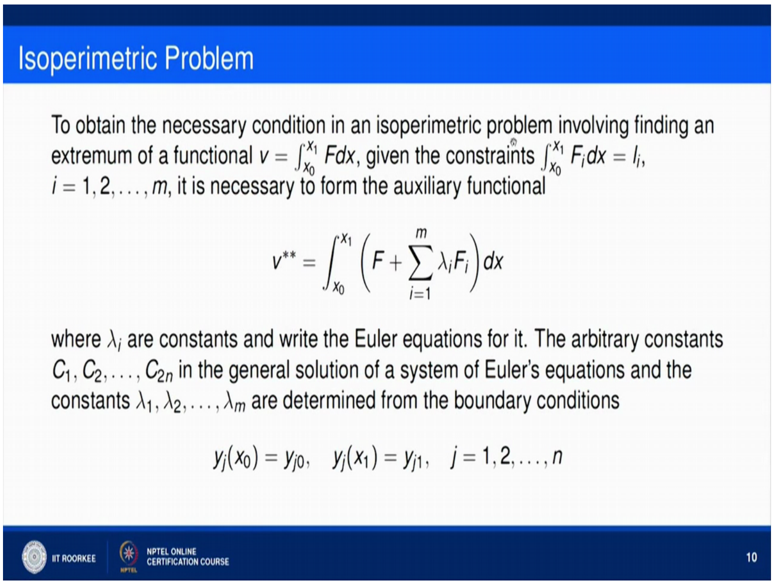

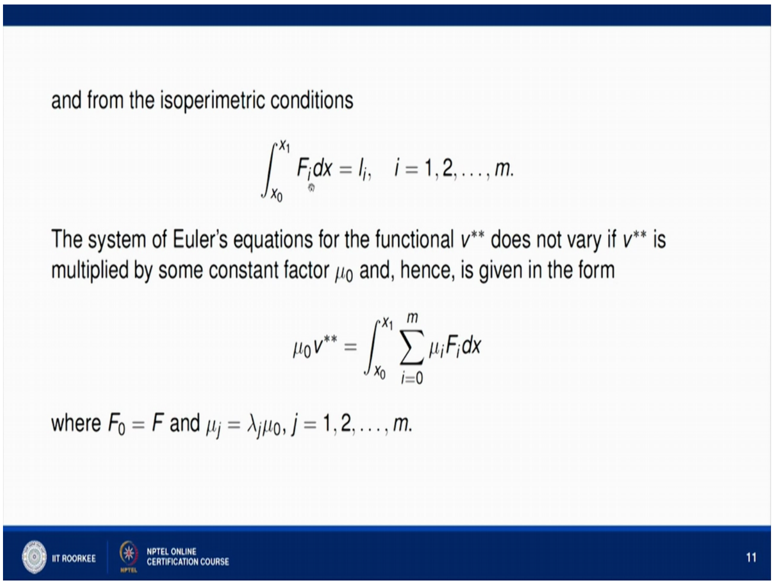

So we can see that to obtain the necessary condition in an isoperimetric problem involving finding an extremum of a functional v which is given as x not to x 1 F dx, given the constraints x not to x 1 F i dx equal to l i, i equal to 1 to m, it is necessary to form the auxiliary functional v double star that is x not to x 1 F plus summation i equal to 1 to m lambda i F i d of x. Where lambda i are constants and we can write the Euler Equation for this functional v double star this we can obtain from the previous discussion.

The arbitrary constants C 1 to C 2n in the general solution of system of Eulers Equation and the constants lambda 1 to lambda m we can obtain these by boundary condition and boundary condition defined at both the end. And from the isoperimetric condition that is x not to x 1 F i dx equal to l i, from i equal to 1 to m. And the system of Eulers Equation for the functional v double star does not vary if v double star is multiplied by some constant factor.

So it means that if we already find out say extremal functional for v star with the v double star then extremal curve will not change if we multiply by some nonzero constant that is Mu not and we can say that Mu not v double star equal to x not to x 1 summation i equal to 0 to m Mu i F i dx. So what we try to do here let us consider here.

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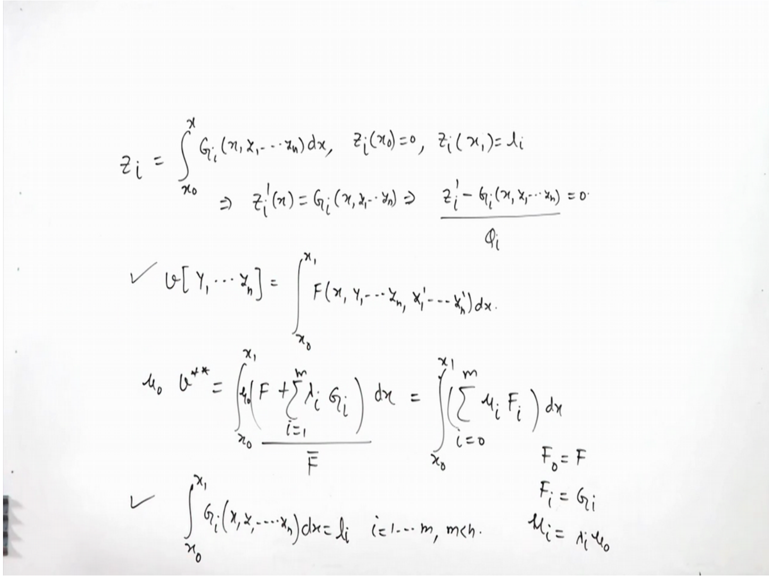

So what we are doing here we are having a problem this we are considering this problem along with this condition. So for this we consider v double star as x not to x 1 we call this as F plus here we have lambda i summation lambda i you can write it G i, i is from 1 to m d of x. So here we consider this as F bar and we simply find out the Eulers Equation for this. Now if we multiply both side by any constraint say Mu not then also there is no change in solution of Eulers Equation.

So what we can do here? We rewrite this as integral x not to x 1 and we can write this as i equal to 0 to m rather than from starting from 1 to m and we can write this as say Mu i and we can write it your F i and d of x. So here your F 0 is your F and your F i is your G i and your Mu i is equal to lambda i Mu not. So here we can simply write that Mu not Mu not v double star is given as x not to x 1 summation i equal to 0 to m Mu i F i d of x the purpose behind behind doing this that now this is symmetric in all F i and we can say that and we can say that.

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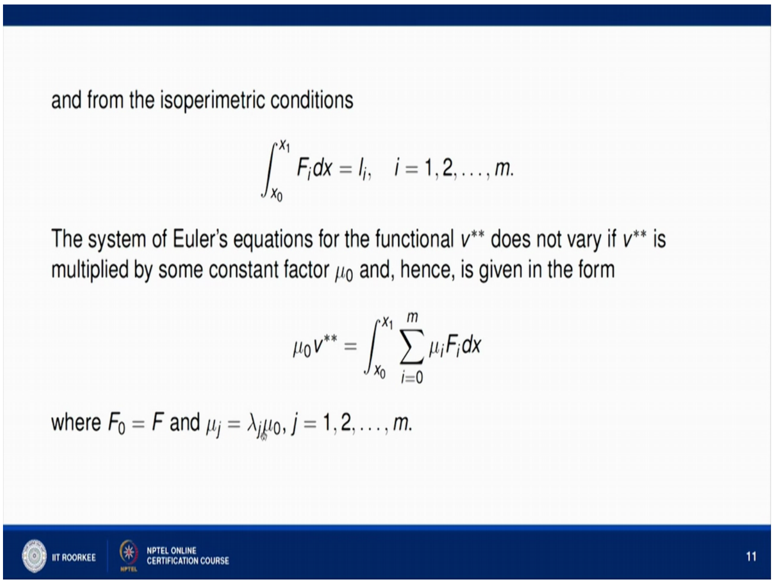

So here F not is F and Mu j is lambda j Mu not for j equal to 1 to m.

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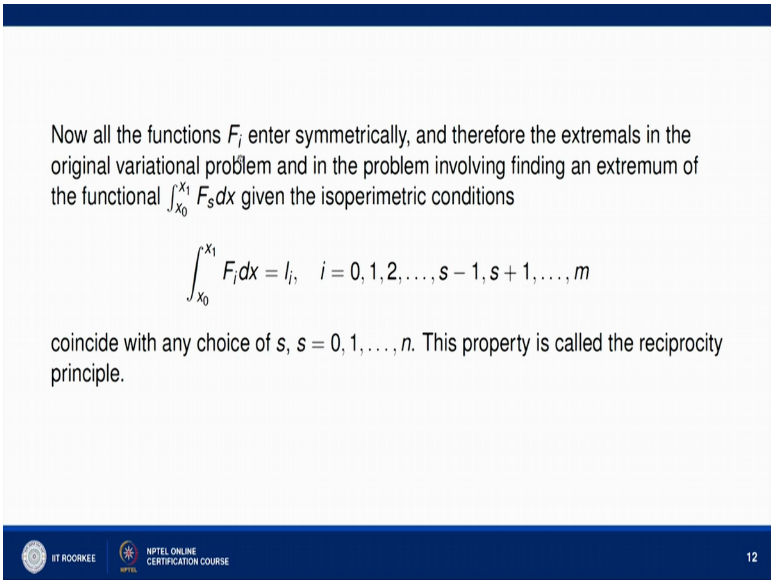

The purpose is that now all the functions F i enter in a symmetric manner and therefore the extremal in the original variational problem and in the problem involving finding an extremum of the functional x not to x 1 F of s d of x given the isoperimetric condition x not to x 1 F i dx equal to l i, where i is from 0 to s minus 1, comma s plus 1 to m coincide.

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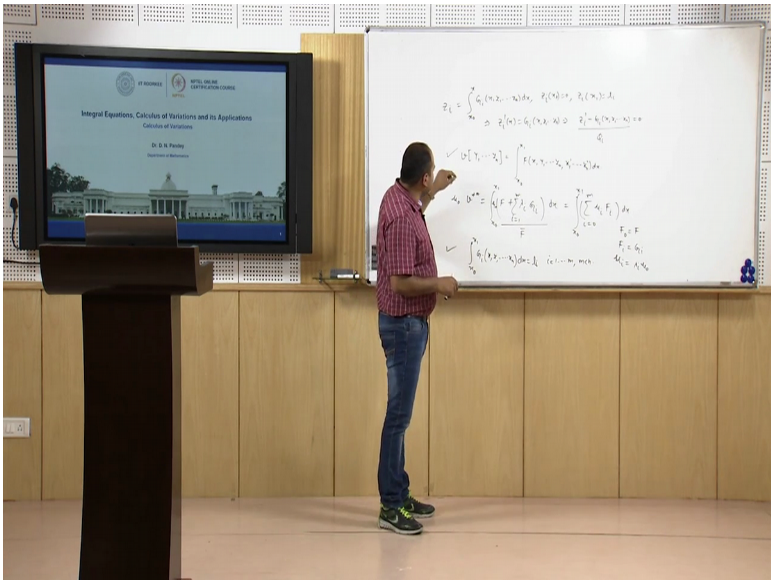

It means that coming back to this the purpose behind doing this is that if you look at we want to extremize this functional along with this condition and for that we have seen that the extremal functional basically satisfy the Eulers Equation corresponding to this variational problem this functional.

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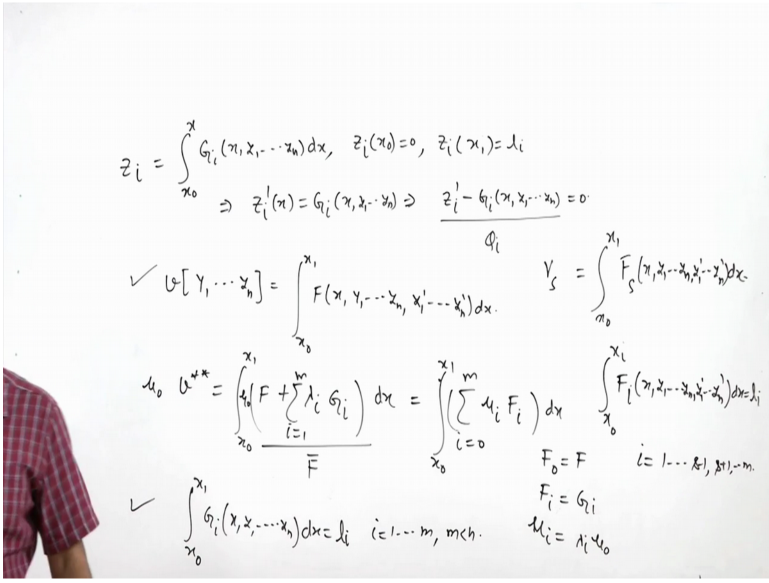

So here we can say that that code of these m plus 1 functionals we can say that let us extremize say let us say extremize x not to x 1 we can take out of these F i from 0 to m here, you can take any say function let us say that you take F of s x, y 1 to say y n and y 1 dash to y n dash d of x we want to say let us extremize this let us call this as V of s here keeping others as condition. So we can say that now conditions are reduced as (x not to x 1) x not to x n F of i x, y 1 to say y n and y 1 dash to y n dash d of x equal to l i, where here i is running from 1 to say s minus 1 leaving this s out, s plus 1 to say m.

So it means that here you can take you can take you can extremize any of the integral keeping all other integral as constant then also your extremal functional will not change and it will in all these cases your you have to form this kind of function and your extremal functional functions will has to satisfy the Eulers Equation corresponding to this functional Mu not v double star. So this property is known as reciprocity principle.

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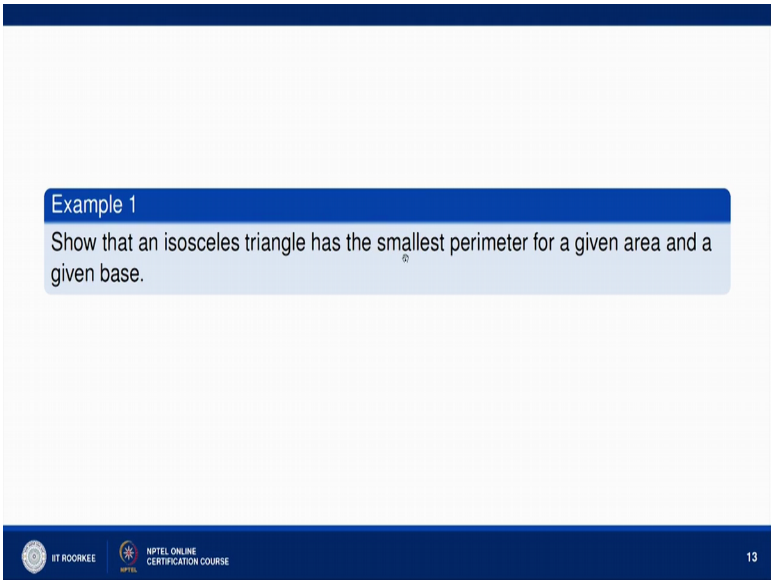

So here we discuss the first example which says that show that an isosceles triangle has the smallest parameter for a given area and a given base. So here we try to use the reciprocity principle to solve this particular problem. So we try to solve this problem by saying that a triangle isosceles triangle having the given perimeter and a given base has say maximum area.

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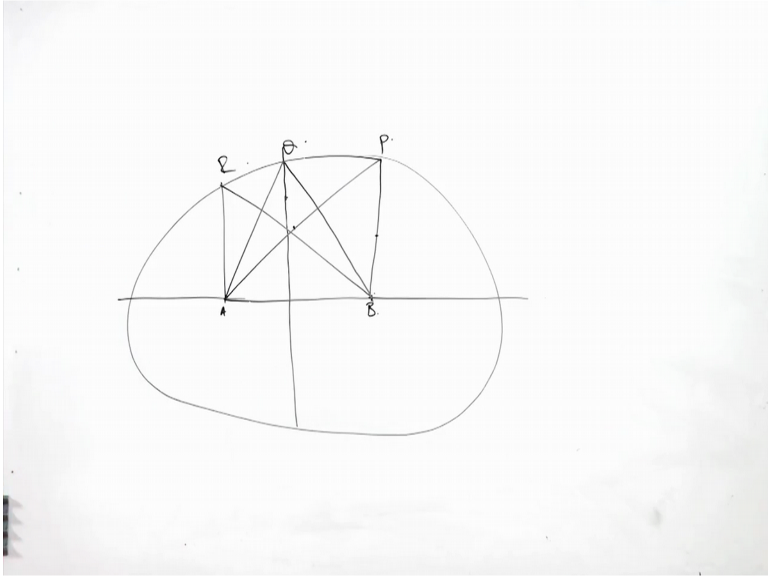

So if you look at here let us take the base, base is already given let us say that this base is given as A and B and perimeter is fixed here. So let us say that perimeter is some kind of l so and we want to find out say triangle having the maximum area and we try to show that that triangle is an isosceles triangle. So what we try to do here? Let us consider an ellipse whose focus are situated here and keeping this A, B as major axis and let us consider this as y axis here and let us consider this as the perimeter, this plus this is the perimeter here and vary this and we can have a an ellipse kind of figure here.

So here you can say that by the property of an ellipse if we take any triangle keeping the base point as A, B then the area for perimeter is going same. So we can consider this as P, we can say that Q and we can consider this R. So we can say that the perimeter of PAB, QAB and RAB will be the same that is that can be shown in a with the property of an ellipse. But the area of this AQB is going to be the highest the reason being that this is the having the highest say height. So this AQB will the highest AQB will be the will be a triangle having the maximum area because of it is having the highest height.

So we can say that this AQB is the triangle having the same perimeter but having the maximum area. So this can be considered with the help of reciprocity principle that given a given that isosceles triangle has the smallest perimeter for a given area and a given base. So with the help of reciprocity principle we can say that isosceles triangle has the smallest perimeter for a given area and a given base.

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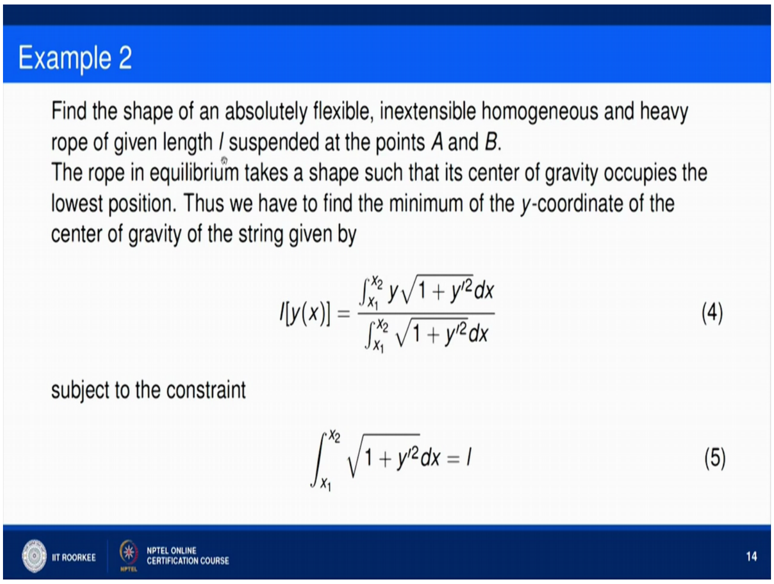

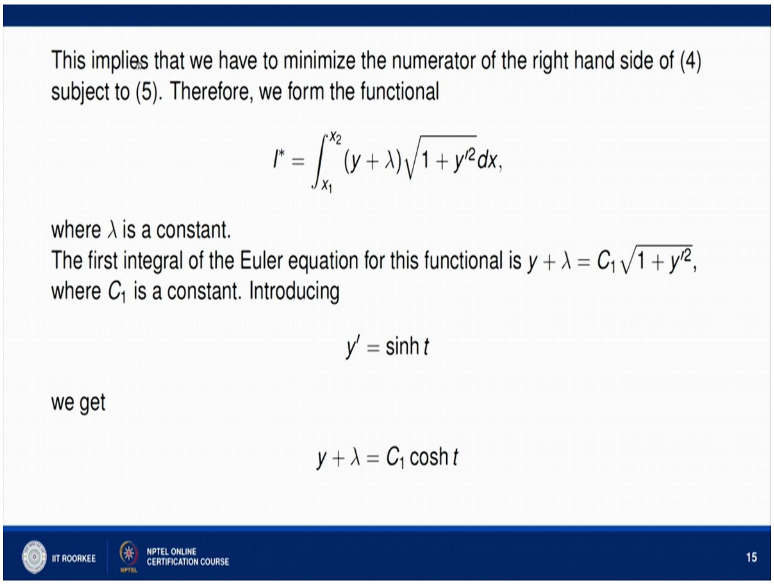

Now moving on to second example here we want to find out the shape of an absolute absolutely flexible, inextensible homogeneous and heavy rope of a given length l suspended at the points A and B. And the rope in equilibrium takes a shape such that its center of gravity occupies the lowest position. So we try to find out the shape of such a rope. So here we use a condition that in equilibrium position the shape it will any rope will take a shape such that its center of gravity occupies the lowest position.

So we need to find out the minimum of the y coordinate of the center of gravity of the string given by this. So I of y of x is given by (x 1 to x not) x 1 to x 2 y of under root 1 plus y dash square d of x divided by x 1 to x 2 under root 1 plus y dash square d of x keeping the constraint x 1 to x 2 under root 1 plus y dash square d of x equal to l. So this constraint tells you that the length is constant and it is given as l. So we want to find extremize the functional this keeping this constraint in mind.

So here we want to this implies that we have to minimize the numerator of the right hand side of 4 subject to 5. So we want to minimize the this numerator keeping this thing as constant that is l. So here we form the functional I star that is x 1 to x 2 y plus lambda under root 1 plus y dash square d of x and here lambda is a constant.

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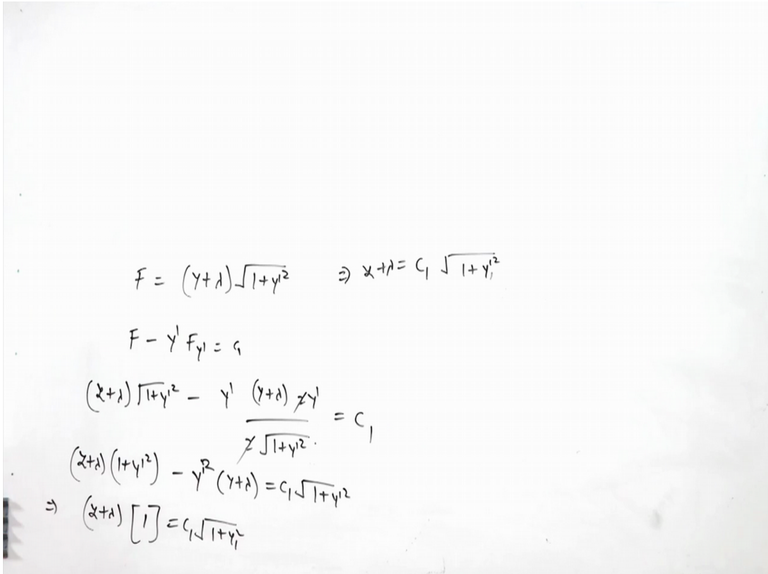

So if you look at if you form the Euler Equation for this then F is given as y plus lambda under root 1 plus y dash square y dash square. Now here F is independent of independent variable x. So here your Eulers Equation reduce to F minus y dash F of y dash equal to constant, constant let us say C 1. The F is y plus lambda under root 1 plus y dash square minus y dash. Now F of y dash will be what y plus lambda and here we will get 2 into 1 plus y dash square 2 of y dash is equal to C 1, so 2, 2 will be cancelled out.

And here we can say that y plus lambda then multiply here so we have 1 plus y dash square minus y dash y dash square y plus lambda equal to C 1 under root 1 plus y dash square. So here we can take this y plus lambda out, so y plus lambda out and inside you can say that it is 1 equal to C 1 under root 1 plus y 1 dash square y dash square. So here we can say that y plus lambda is equal to C 1 under root 1 plus y 1 y dash square. So it is y dash square. So we want to simplify this for simplifying this we can introduce a new parameter t so by saying that y dash equal to sin hyperbolic t then y plus lambda can be written as C 1 under root 1 plus sin hyperbolic square t can be written as y plus lambda equal to C 1 cos hyperbolic t.

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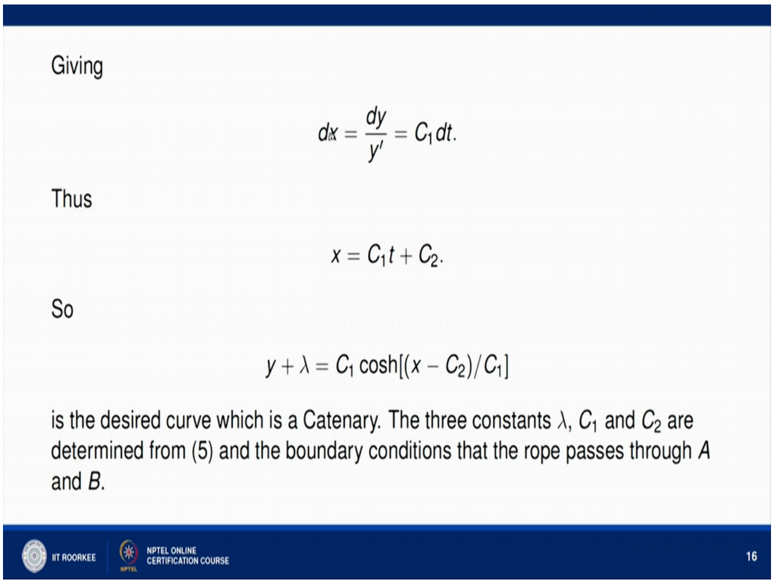

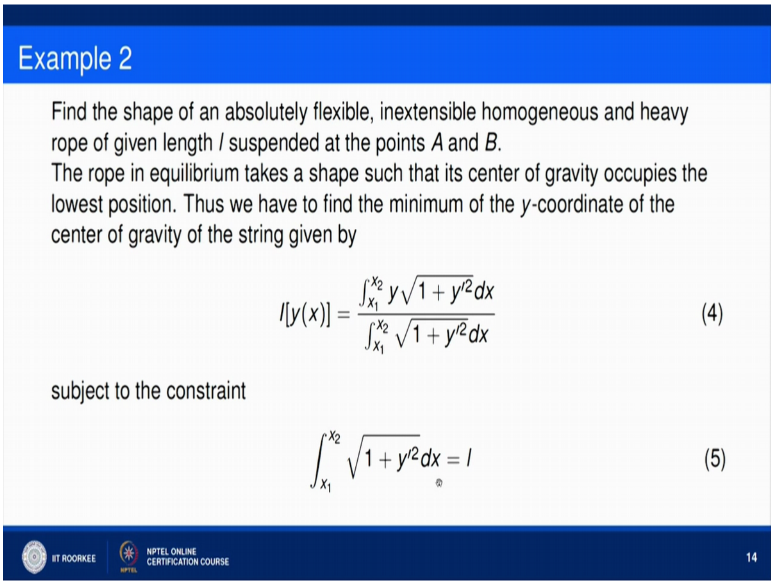

So now we can find out say parameter representation of x that is d of x equal to dy upon y dash that is C 1 dt and we can say that x is equal to C 1 t plus C 2, so taking t from this equation we can say that t can be written as x minus C 2 of divided by C 1. So y plus lambda can be written as C 1 cos hyperbolic x minus C 2 upon C 1. So the shape of the hanging rope can be will be satisfying this equation which is the curve which is known as catenary equation. And the constant lambda C 1 and C 2 can be obtained from the condition this and the boundary condition satisfy by the problem, okay.

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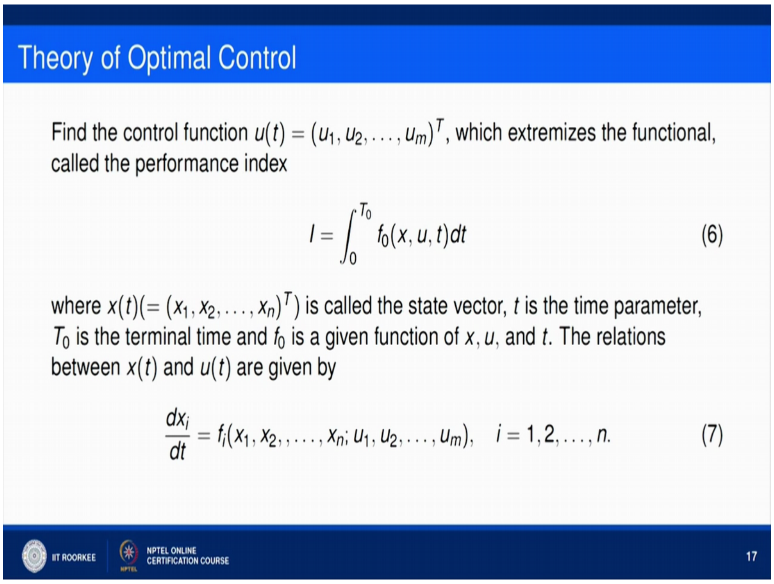

So now let us also discuss a new problem which is known as finding the control which optimize a certain functional. So here the problem is as follows. Find the control function u t which is u 1 to u m transpose which extremizes the functional which is known as performance index I which is defined as 0 to t not f not x, comma u, comma t dt. Here x is a state vector given as x 1 to x n transpose and t is the time parameter and t not is the terminal time and f not is a given function of x, comma u and t.

And the relation between the control vector and the state vector is given by this equation which is given as differential equation and it is defined as dx i by dt equal to f i x 1 to x n, u 1 to u m. So your control variable and state variables are given by this and we want to find out the control which extremizes the functional. So this is a very common problem in theory of optimal control and we want to extremize the the corresponding functional this.

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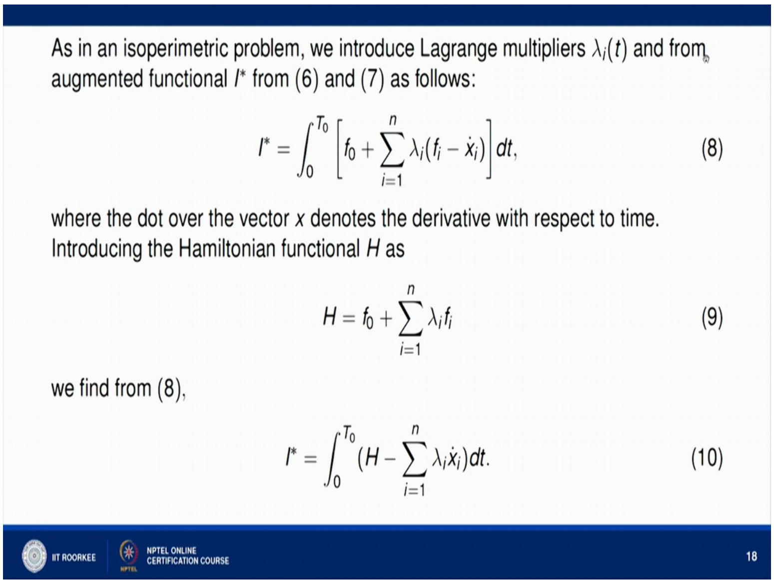

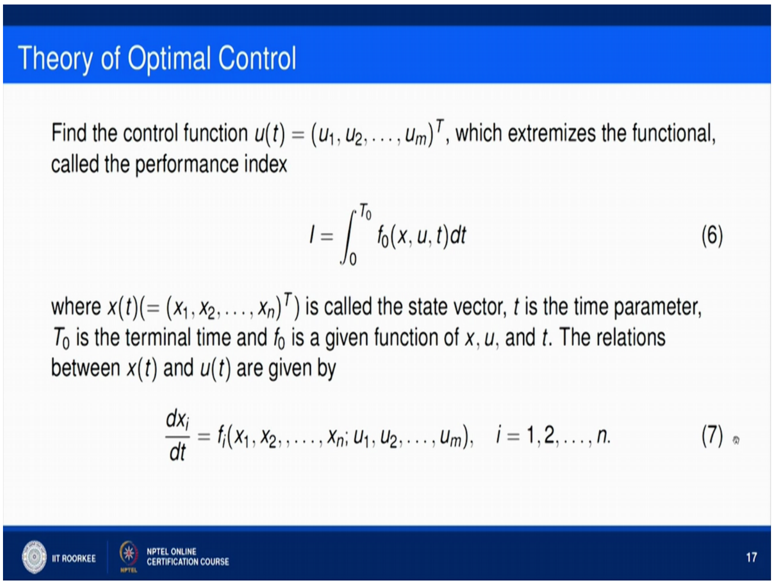

So as in a isoperimetric problem we introduce a Lagrange multiplier lambda i t and from augmented functional I star that is given as 0 to t not f not plus summation i equal to 1 to n lambda i f i minus x i dash. So previous equation number 7 can be written as f i minus x i dash is equal to 0, so this can be considered as constraint given in terms of differential equation.

So here we form the functional I star which is given by this. Now here x i dash denote the derivate with respect to time t. Now here we denote this portion f not plus summation i equal to 1 to n lambda i f i as H which is known as Hamiltonian functional H. Now in term software Hamiltonian functional H we can rewrite this I star as 0 to t not H minus summation i equal to 1 to n lambda i x i dot dt.

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The Euler equations are

$$\frac{\partial H}{\partial x_i} = -\lambda_i', \qquad i = 1, 2, \ldots, n, \tag{11}$$

$$\frac{\partial H}{\partial u_j} = 0, \qquad j = 1, 2, \ldots, m. \tag{12}$$

The optimal solutions for x, u and λ are obtained by solving (7), (11) and (12). In fact, these are $2n + m$ equations for $x_i's, u_i's$ and $\lambda_i's$. If the initial conditions $x_i(0) = 0, (i = 1, 2, \ldots, n)$ and the terminal conditions $x_j(T_0), (j = 1, 2, \ldots, l), l < n$ are known, then the terminal values $x_j(T_0)$ for $j = l+1, l+2, \ldots, n$ are free. In this case we use the free end conditions

$$\lambda_j(T) = 0, \qquad j = l+1, l+2, \ldots, n. \tag{13}$$

The equations (13) are known as the transversality conditions.

As in an isoperimetric problem, we introduce Lagrange multipliers $\lambda_i(t)$ and from augmented functional I^* from (6) and (7) as follows:

$$I^* = \int_0^{T_0} \left[f_0 + \sum_{i=1}^{n} \lambda_i (f_i - \dot{x}_i) \right] dt, \tag{8}$$

where the dot over the vector x denotes the derivative with respect to time. Introducing the Hamiltonian functional H as

$$H = f_0 + \sum_{i=1}^{n} \lambda_i f_i \tag{9}$$

we find from (8),

$$I^* = \int_0^{T_0} (H - \sum_{i=1}^{n} \lambda_i \dot{x}_i) dt. \tag{10}$$

Theory of Optimal Control

Find the control function $u(t) = (u_1, u_2, \ldots, u_m)^T$, which extremizes the functional, called the performance index

$$I = \int_0^{T_0} f_0(x, u, t) dt \tag{6}$$

where $x(t) (= (x_1, x_2, \ldots, x_n)^T)$ is called the state vector, t is the time parameter, T_0 is the terminal time and f_0 is a given function of x, u, and t. The relations between $x(t)$ and $u(t)$ are given by

$$\frac{dx_i}{dt} = f_i(x_1, x_2, \ldots, x_n; u_1, u_2, \ldots, u_m), \quad i = 1, 2, \ldots, n. \tag{7}$$

So as we pointed out we find out say Eulers Equation and Eulers Equation is nothing but deba H by deba x i equal to minus lambda i dash, for i equal to 1 to n and deba H by deba u j equal to 0 that is j equal to 1 to m. So what we did we find out say Eulers Equation for this and which result in this n plus m equations. So the optimal solution for x, comma u and lambda are obtained by solving the set of equation that is 7 and the set of equation which we have obtained through the Eulers Equation and so these are in fact these are 2n plus m equations for which x, u and lambda i can be obtained.

So if the initial condition x i is are given at and and 0 and the terminal condition x j given at t not are known and if the conditions are less in number we can always define lambda j at terminal say T as 0 and these kind of conditions are known as transversality conditions.

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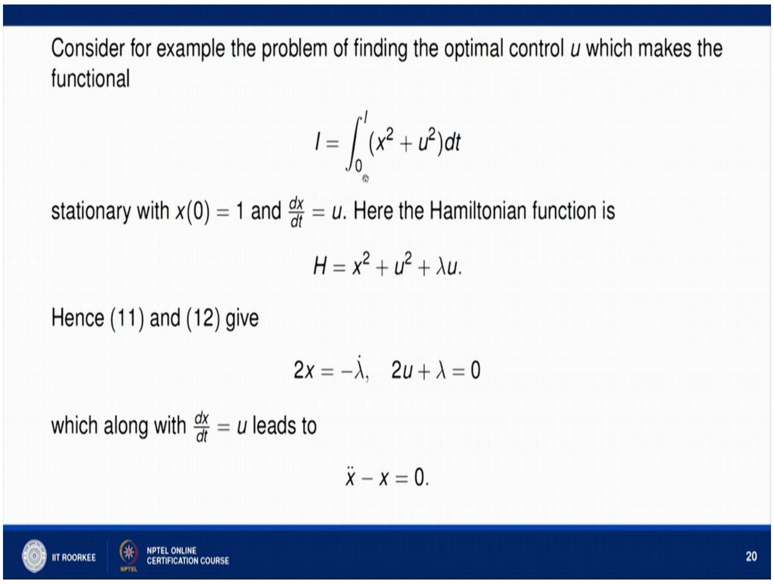

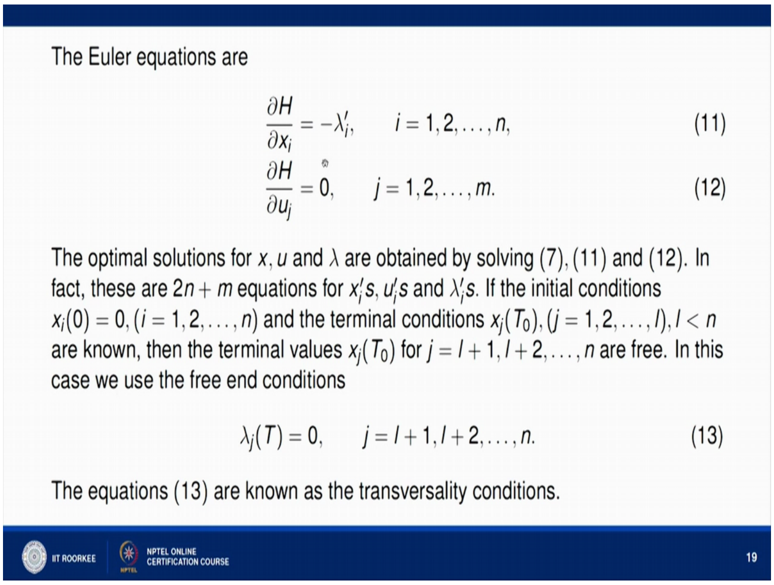

So let us consider one quick example on this, so here we consider the example of finding the optimal control u which makes the functional stationary with the condition that x of 0 is equal to 1 and dx by dt is equal to u. So control and state variable is related with this differential equation dx by dt equal to u. So here as pointed out we can form the Hamiltonian function as H equals to x square plus u square plus lambda u and we can apply the Eulers Equation which is given in deba H by deba x i equal to minus lambda i dash and deba H by deba u j equal to 0.

So if we do this we have this equation 2x equal to minus lambda dash which we get from equation number 11 and deba H by deba u j equal to 0 we will get this 2u plus lambda equal to 0. So here u is given as dx by dt, so if you use this and you can take out lambda here and put it here and we can say that equation that state vector satisfy is this second order differential equation.

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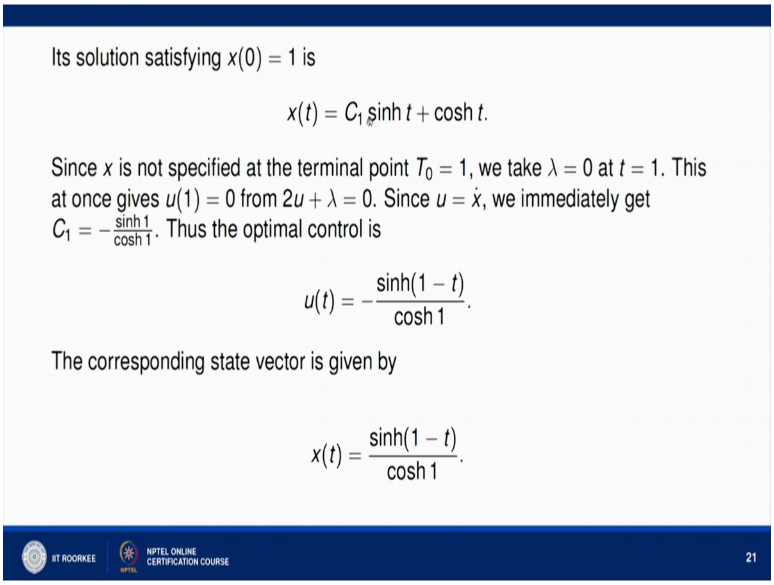

Consider for example the problem of finding the optimal control u which makes the functional

$$I = \int_0^1 (x^2 + u^2)dt$$

stationary with $x(0) = 1$ and $\frac{dx}{dt} = u$. Here the Hamiltonian function is

$$H = x^2 + u^2 + \lambda u.$$

Hence (11) and (12) give

$$2x = -\dot{\lambda}, \quad 2u + \lambda = 0$$

which along with $\frac{dx}{dt} = u$ leads to

$$\ddot{x} - x = 0.$$

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And whose solution is given x t of C 1 sin hyperbolic t plus C 2 cos hyperbolic t. Now C 2 we can obtain by the condition which is given at x of 0 is equal to 1. So we can obtain C 2 as 1 so your x t is given as C 1 sin hyperbolic t plus cos hyperbolic t. Now we do not have any way to find out the C 1 because x is not specified at the terminal point T not. So for that let us take lambda equal to 0 at t equal to 1 so this at once gives u 1 equal to 0. Because we have this equation 2u plus lambda equal to 0, so at t equal to 1 if lambda 1 is equal to 0 so u at 1 is equal to 0. So u at 1 equal to 0 means dx by dt is equal to 0 at x equal to 1.

So we can say that u equal to x dot and we immediately get C 1 equal to minus sin hyperbolic 1 divided by cos hyperbolic 1. So using this value of C 1 you can write down the value of x t and hence we can find out the optimal control like this. So u t is defined as minus sin hyperbolic 1 minus t upon cos hyperbolic 1 and the corresponding state vector is given by x of t equal to sin hyperbolic 1 minus t upon cos hyperbolic 1.

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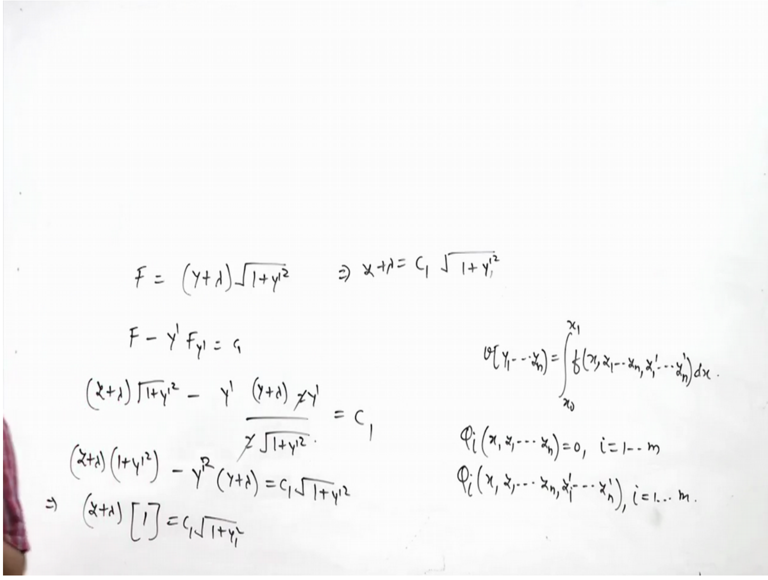

So here in todays lecture we have seen few things first thing we consider the variational problem with so variational problem is we have say y 1 to say y n which is given as say x not to x 1 f of x, comma y 1 to say y n and y 1 dash to say y n dash d of x and we have considered the case when conditions are constant given in terms of finite equation that is x, comma y 1 to y n is equal to 0, i is from 1 to say m that we have discussed in previous lecture.

And in todays lecture we have discussed the constraint of the form given in terms of differential equation like this i equal to 1 to say m and also we have discussed the principle known as reciprocity principle and with the help of reciprocity principle we have discussed certain problem and then as an example of this case we have discussed a case of optimal control.

And here it is to be pointed out that this optimal control is not lying on the boundary of the set of admissible control, if it is then we have a separate theory not this theory will not work for the control which lie on the boundary of the set of admissible control. So here I stop and we will discuss in next lecture, so thank you very much for listening us meet in next lecture, thank you.