Integral Equations, Calculus of Variations and their Applications

Professor Dr. D. N. Pandey
Department of Mathematics
Indian Institute of Technology Roorkee

Lecture 53

Variational Problem Involving a Conditional Extremum-1

Hello friends welcome to today's lecture in today's lecture we will discuss the variational problem with the condition extremum.

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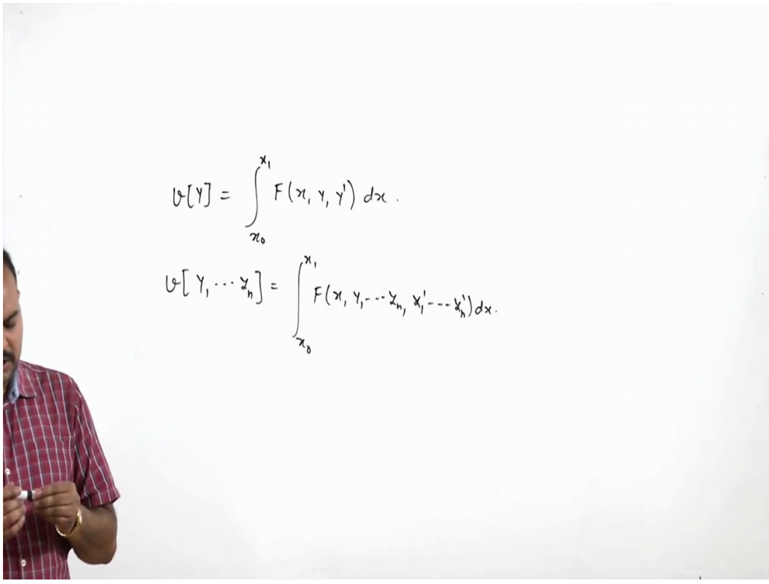

If we recall we have discussed the problem extremum problem for these kind of functional we have x not say x 1 and we have f of x, y, y dash and d of x or you can say that if we have more than 1 curve say y 1 to y n and it is x not to x 1 f of x, y 1 to say y n and y 1 dash to say y n dash and d of x provided some boundary condition defined at the point x not and x 1. So here these problems are known as unconditional extremum. So today we will discuss that what happen if we put certain condition on these functions these variable y and say y1 to y n. So that is the today's topic, so let us discuss here.

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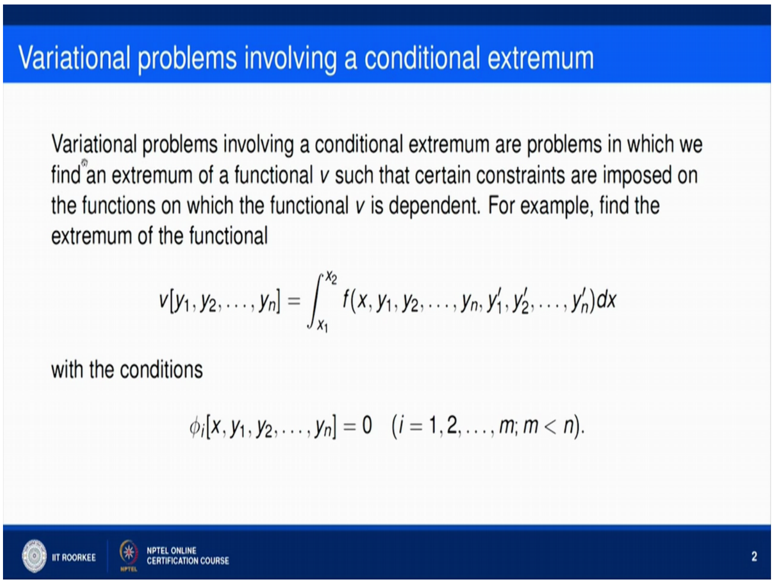

So we want to discuss here a is the variational problem involving a conditional extremum. So variational problem involving a conditional extremum are problems in which we find an extremum of a functional v such that the certain constant are imposed on the function on which the functioanal v is dependent. So here if we consider this functional which is depending on y 1 to y n and define an x 1 to x 2 f of x, y 1 to y n, y 1 dash to y n dash d of x. And the condition define on this variable y 1 to y n are given as phi i x, y 1 to y n equal to 0 for each i equal to 1 to m where m is some number which is strictly less than n.

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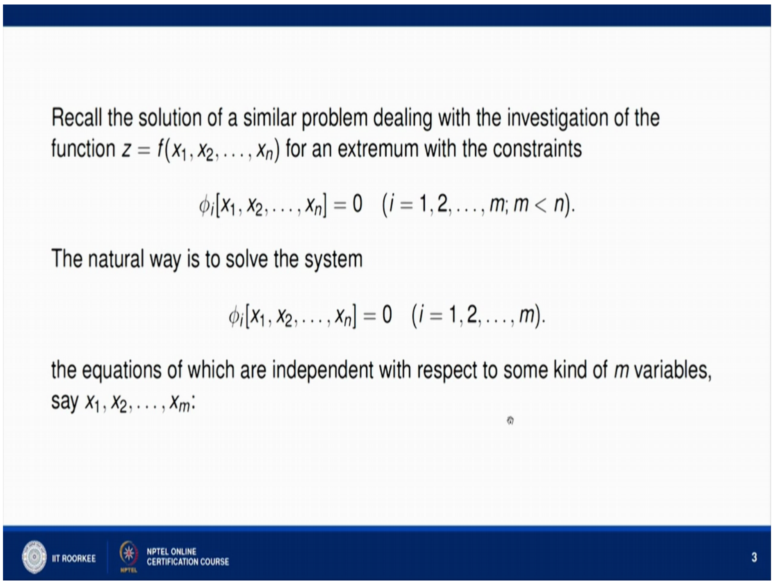

So to discuss this problem let us recall the similar problem and the similar problem is that we try to find out the extremum of the function z equal to f of x 1 to x n where x 1 to x n are variable here and provided that certain constraints are given phi i x 1 to x n equal to 0 where i is equal to 1 to m where m is strictly less than n. So this is the simple problem posed in calculus and to find out this problem to find out the extremum of this problem what we try to do here we try to look at this constraints system of constraint and we try to solve this system of constraint in terms of your x 1 to x n and try and to put that value of x 1 to x n here in the function z equal to f of x 1 to x n and this by doing this your function z is now reduced to a new function which is having a independent variable x m plus 1 to x n and we can simply solve the condition of extremum with no constraint.

So how we can do this? Let us proceed here, so the natural way is to solve the system phi i x 1 to x n equal to 0, where i equal to 1 to m the equation of which are independent. So here we are already assuming that these system of a constraints are independent to each other it means that that we can solve this phi i i equal to 1 to m in terms of x 1 to x m. So without laws of generality I am assuming that we can solve these constraints for x 1 to x m. So if it is not then we can change the relabel the variable and we can always relabel as x 1 to x m.

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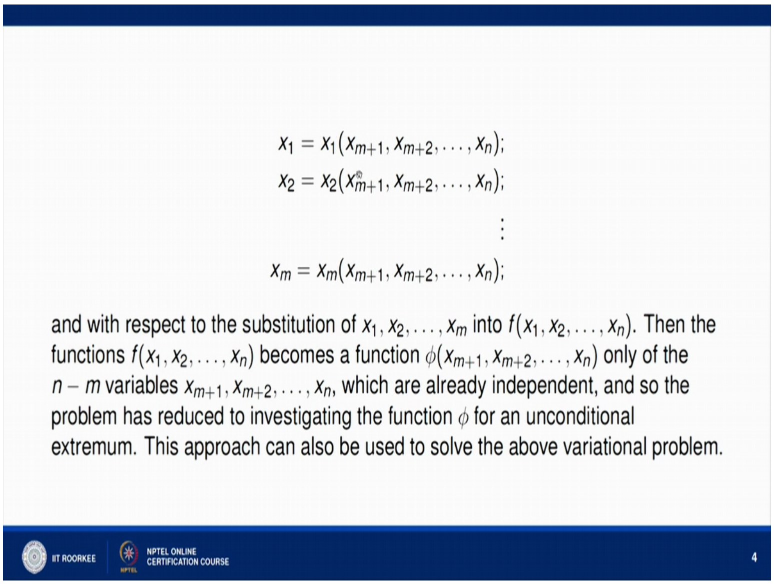

So here if phi i are independent we can solve them as x 1 to as x 1 as function of x m plus 1 to x n and similarly x m to x m as a function of x m plus 1 to x n. So by doing this this function f of x 1 to x n is reduced to a new function phi which is depending now on only n minus m variable x m plus 1, x m plus 2, upto x n. So now we can say that so the problem has reduced to investigating the function phi for an unconditional extremum in the variable x m plus 1 to x n. So we try to apply the same approach to solve the given variational problem.

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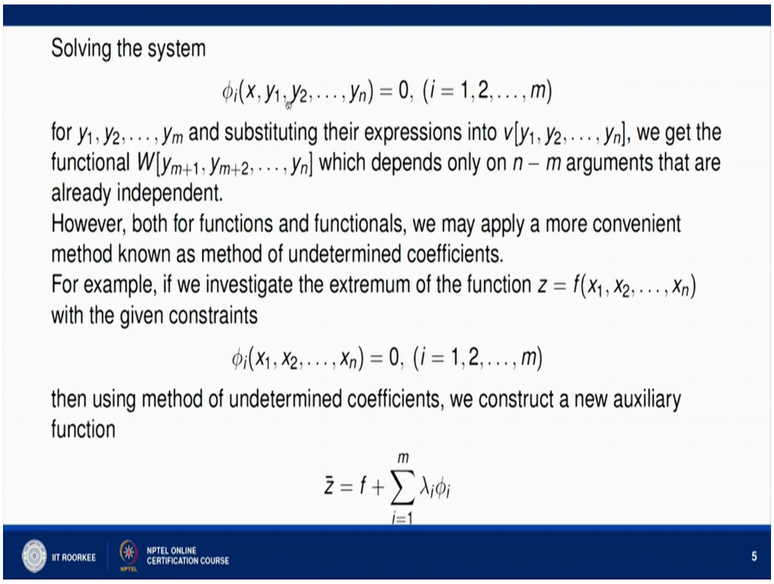

So how we can do that let us see that. So we have a constraint like this phi i x, y 1 to y n equal to 0 where i is equal to running from 1 to m. So we try to solve this and substitute their expression into v functional. So we are solving this for y 1 to y m and after putting this expression y 1 to y m this is functional v which is depending y 1 to y n is now reduced to a new functional W which is now depending on only on y m plus 1 to y n and so it is depending on now n minus m arguments that are already independent.

So this is the classical approach which we apply in calculus and we can apply for functional also. But here in this lecture we try to discuss a new approach or a more convenient approach that is the method of undetermined coefficient. So here to apply the method of undetermined coefficient let us first discuss the case of calculus and then we try to discuss the case for functional.

So here we want to investigate the extremum of the function z equal to f of x 1 to x n with the constraint phi i i is from 1 to m. And using method of undetermined coefficient what we try to do here with the help of this function and the constraint we construct a new auxiliary function z bar which is given as f plus summation i equal to 1 to m lambda i phi i, where lambda i are some unknown parameters here which is known as undetermined coefficients and we try to find out lambda i in a way such that the extremum of this function satisfying this constraint is satisfying the r extremum of this z bar.

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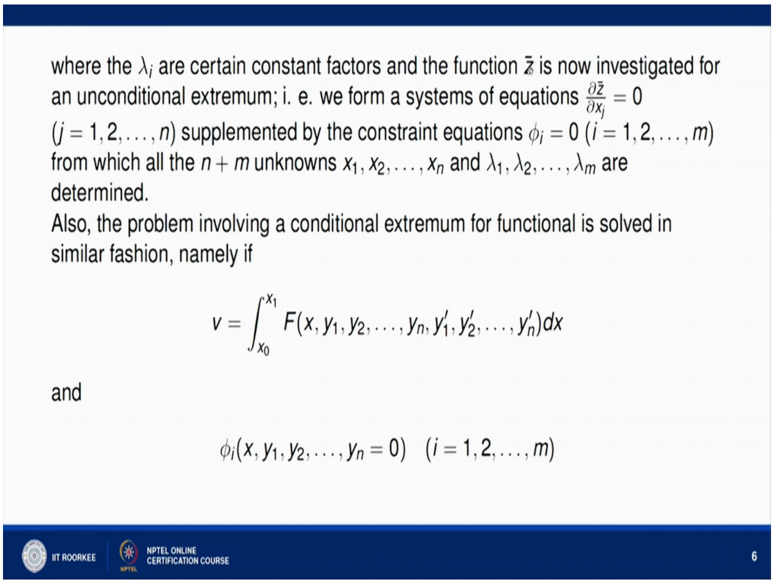

So where the lambda i are certain constant factor and the function z bar is now investigated for an unconditional extremum. So to find out say investigation for z bar for an unconditional extremum we simply say that let us take the partial derivative with respect to (())(7:22) and equate it to 0. So here deba z bar with respect to deba x j is equal to 0 where j is 1 to n, supplemented the constraint equation phi i equal to 0 i is from 1 to m.

So it is basically these are n equations and here we have m equations so we have total n plus m unknown n plus m equation which gives you say n plus m unknowns that is x 1 to x n these constraint lambda 1 to lambda m. So this the method known as Lagrange method of undetermined coefficient which we generally apply in calculus. Now we try to apply the similar method for our variational problem, okay.

So here consider this variational x not to x 1 F of x, y 1 to y n, y 1 dash to y n dash d of x and constraints are given as phi i x is depending on y 1 to y n equal to 0 for i equal to 1 to m. Here this small problem here it is phi i x, y 1 to y n equal to 0. So this bracket is here anyway.

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then the functional

$$\bar{v} = \int_{x_0}^{x_1} \left(F + \sum_{i=1}^{m} \lambda_i(x)\phi_i \right) dx$$

or

$$\bar{v} = \int_{x_0}^{x_1} \bar{F} dx$$

is constructed, where $\bar{F} = F + \sum_{i=1}^{m} \lambda_i(x)\phi_i$ which is now investigated for an unconditional extremum, i.e. we solve the system of Euler's equations

$$\bar{F}_{y_j} - \frac{d}{dx}\bar{F}_{y_j'} = 0 \quad (j = 1, 2, \ldots, n) \qquad (1)$$

with the constraint equations

$$\phi_i = 0 \quad (i = 1, 2, \ldots, m)$$

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where the λ_i are certain constant factors and the function $\bar{z}$ is now investigated for an unconditional extremum; i. e. we form a systems of equations $\frac{\partial \bar{z}}{\partial x_j} = 0$ $(j = 1, 2, \ldots, n)$ supplemented by the constraint equations $\phi_i = 0$ $(i = 1, 2, \ldots, m)$ from which all the $n + m$ unknowns $x_1, x_2, \ldots, x_n$ and $\lambda_1, \lambda_2, \ldots, \lambda_m$ are determined.

Also, the problem involving a conditional extremum for functional is solved in similar fashion, namely if

$$v = \int_{x_0}^{x_1} F(x, y_1, y_2, \ldots, y_n, y_1', y_2', \ldots, y_n') dx$$

and

$$\phi_i(x, y_1, y_2, \ldots, y_n = 0) \quad (i = 1, 2, \ldots, m)$$

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Then the functional v bar which we we consider from this original functional along with the this constraint. So we consider this new functional v bar is defined as x not to x 1 F plus summation i equal to 1 to m lambda i x phi i dx or if we denote this quantity as F bar then we can rewrite this as v bar equal to x not to x 1 F bar d of x, where F bar is denoted as F plus summation i equal to 1 to m lambda i x phi i.

Now we investigate this F bar for unconditional extremum it means that we try to solve the system of Eulers Equation for this functional F bar. It means that F bar y i minus d by dx F bar y j dash equal to 0 for each j equal to 1 to n with the constraint equation that is phi i equal to 0, i is from 1 to m. So here what we try to show here we try to show that the extremum curves extremum functions of this original function satisfying this constraint phi i is satisfying the Eulers Equation corresponding to new functional that is F bar, so that we wanted to prove here.

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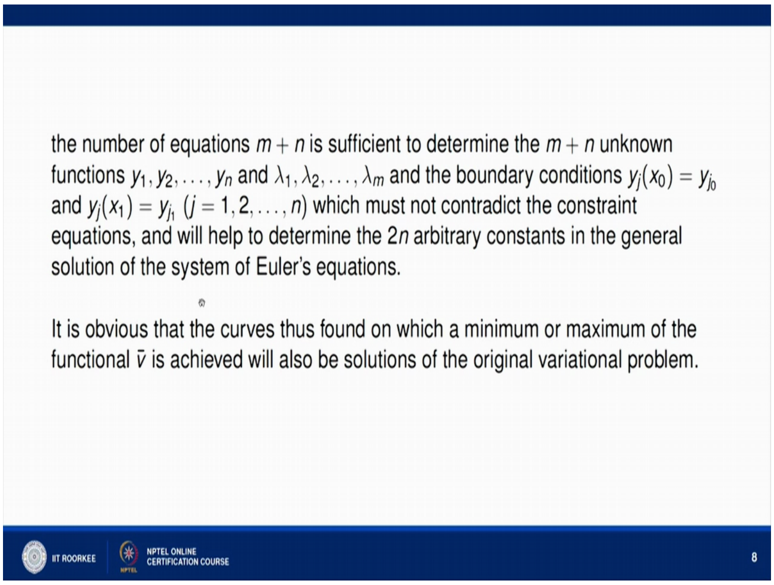

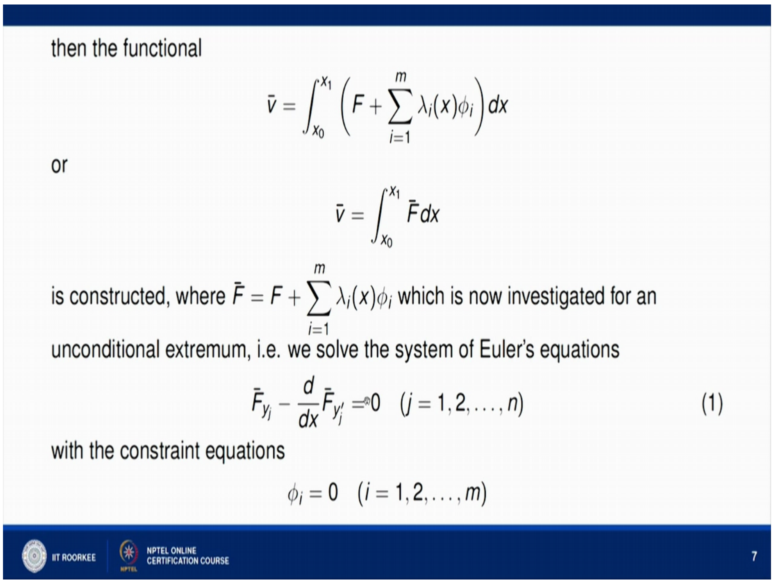

So to proof this we will consider a theorem but before that we here we see that the how many equations we have here we have n equations and here we have m equations given in terms of constraint. So here we have total n plus m equations. So the number of equation m plus n is sufficient to determine the m plus n unknowns that is y 1 to y n and lambda 1 to lambda m and the boundary condition which we which we already know at the point of say boundary point say x not and x 1 which is defined as y j x not equal to y j not and y j x 1 equal to y j 1 of course which must not contradict the constraint equation and will help to determine the 2n arbitrary constants in the general solution of the system of Eulers Equation.

It means that when you solve this system of Eulers Equation you will have generally a second order differential equation and when you solve this you have 2n arbitrary constants which we try to fix with the help of boundary conditions. So now here we try to observe that it is obvious that the curves thus found on which a minimum or maximum of the functional V bar is achieved will also be the solution of the original variational problem.

It means that what we are trying to do here we are considering the functional along with variational problem along with some constants. So it means that we are considering the variational problem with conditional extremum, what we try to do here with the help of method of undetermined coefficient? We try to convert this to a problem of say finding the unconditional extremum of a new functional say a new variational problem that we wanted to prove the equivalence of these two not equivalence of these two in fact we want to show.

So it is obvious that the curves thus found on which a minimum or maximum of the functional v bar is achieved will also be solution of the original variational problem, how lwt us see this?

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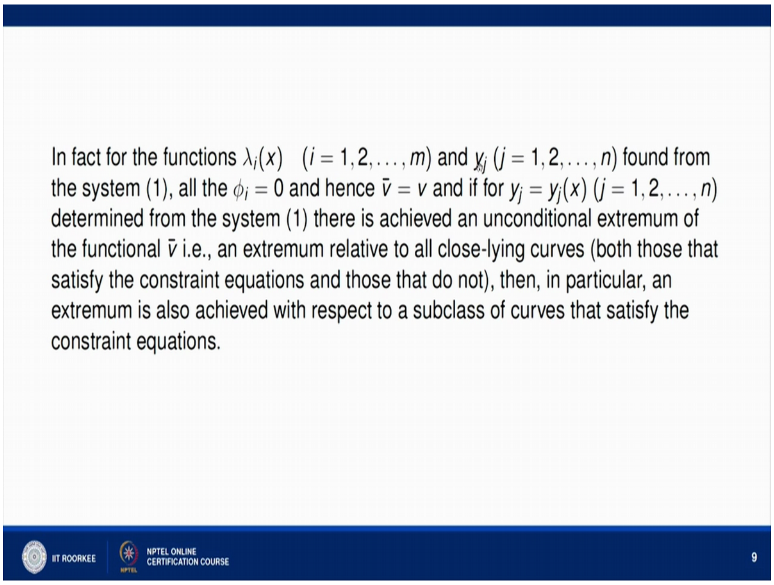

then the functional

$$\bar{v} = \int_{x_0}^{x_1} \left(F + \sum_{i=1}^{m} \lambda_i(x)\phi_i \right) dx$$

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$$\bar{F}_{y_j} - \frac{d}{dx}\bar{F}_{y_j'} = 0 \quad (j = 1, 2, \ldots, n) \qquad (1)$$

with the constraint equations

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In fact the function lambda i x and y j found from the system 1, system 1 is this let me system 1 is this basically it is system of Eulers Equation along with the constraint equation. So this will give you say lambda i and y i. So claim is that the functions lambda i and y j found from the system 1 all the phi i are 0 and if we take phi i are all 0 then v bar is reduced to v, basically v bar is basically what look at here v bar is defined as F plus summation i equal to 1 to m lambda i x phi i, so if phi i are all 0 then v bar is reduced to x not to x 1 F dx which is nothing but v.

So it means that for these functions lambda i and y j your v bar is reduced to v and if for y j, j is from 1 to n determined from the system 1 if we can have a unconditional extremum of the functional v bar it means that extremum related to all close-lying curves means all all curves satisfying the boundary condition they may not satisfy the constraint equation or they may satisfy the constraint equation then in particular an extremum is also achieved with respect to a subclass of curves that satisfy the constraint equation.

So it means that if we solve the system 1 and we can get a lambda i and y j and if y j say extremize the functional v bar then this v bar is reduced to v with this choice and we can say that the extremum extremal curve y j will also be extremal curve for v because if extremum is achieved for v bar which is on the class of all close-lying curves satisfy the constraint equation or may not satisfy the constraint equation then in particular it will then in particular an extremum is also achieved with respect to a subclass of curves that satisfy the constraint equation.

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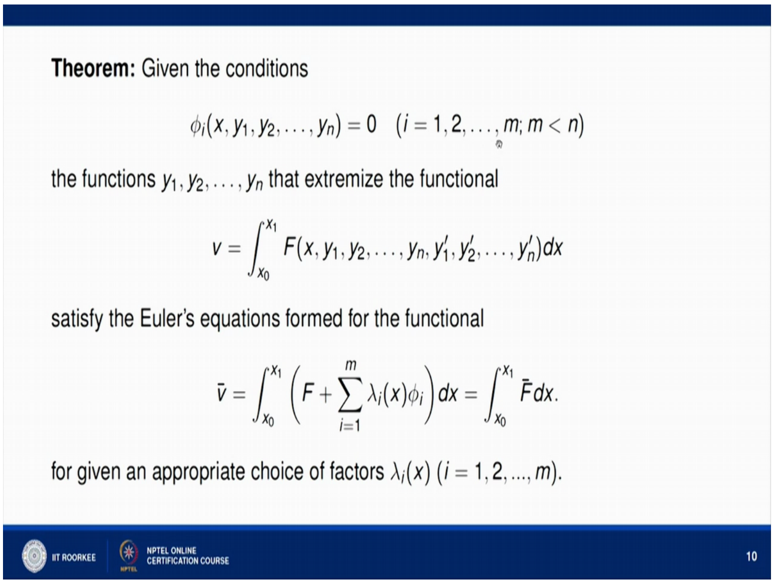

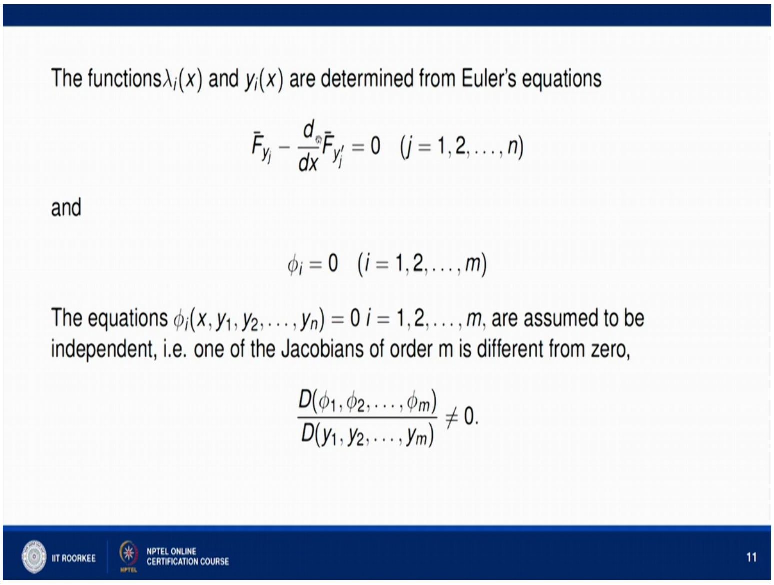

So let us try to prove the just said statement. So here we say that given the condition phi i x from y 1 to y n equal to 0, where i is running from 1 to m, m is strictly less than n. The functions y 1 to y n that extremizes the functional v, so this is the original functional v defined x not to x 1 F of x, y 1 to y n, y 1 dash to y n dash d of x. Then these these extremal function y 1 to y n satisfy the Eulers Equation formed for the functional v bar. So v bar is defined as x not to x F plus i equal to 1 to m lambda i x phi i d of x, we can denote this F as F bar dx. For given an appropriate choice of factors lambda i x, i is from 1 to m.

So and here the factors lambda i x and these extremal curves functions y 1 to y n can be obtained by the Eulers Equation F bar y j minus d by dx of F bar y j dash equal to 0 for j equal to 1 to n and phi i equal to 0 for i equal to 1 to m. In fact if we can consider this F bar which is defined as F plus i equal to 1 to m lambda i x phi i as a function of not only y 1 to y n but also a function of lambda 1 to lambda m then this system of m equation can be cooperated with the Eulers Equation in fact these set of equation is obtained by assuming that F bar is depending only on say y 1 to y n.

But if we assume that F bar is also depending on lambda 1 to lambda m then this equation can also be considered as part of Eulers Equation or otherwise you can simply write down that we have a that lambda i and y i can be obtained by this Eulers Equation and the subsidiary equation phi i from i equal to 1 to m. So and this we can do the equation phi i x from x, y 1 to y n equal to 0, i equal to 1 to m are assumed to be independent.

So here we assume that the constraints are independent constraint, so it means that none of the constraint can be written as in a combination of others. It means that this can be rewrite in the form of Jacobians we can say that Jacobian of phi 1 to phi 1 with respect to some variable y 1 to y m is not equal to 0. So here we are renumbering the variables such that we can assume like this, so without loss of generality I am assuming that the Jacobian of phi 1 to phi m with respect to y 1 to y m is non-zero.

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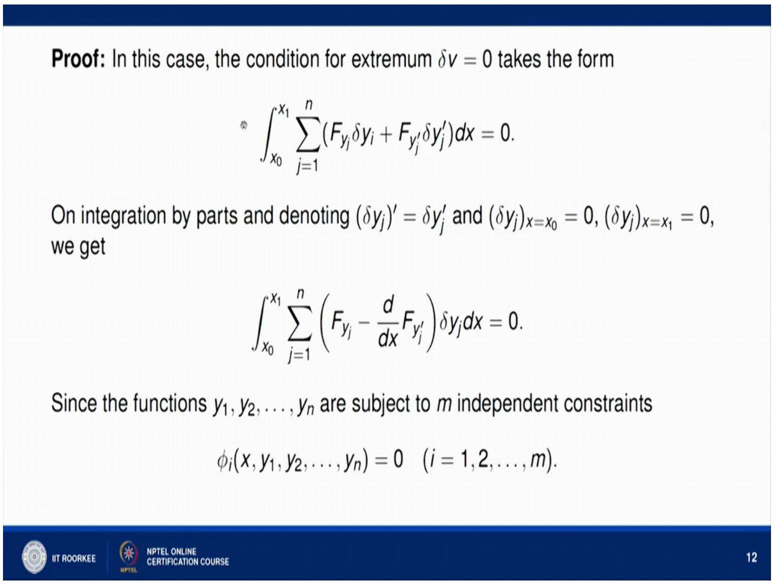

So let us come to the proof here so in this case the condition for extremum delta v equal to 0 takes the form x not to x 1, j is from 1 to n F of y j delta y j plus F y j dash delta y j dash dx equal to 0 and we try to handle this part so here we use integration by parts and denoting delta y j dash equal to delta y j dash and delta y j at x equal to x not is equal to 0 and delta y j x equal to x 1 it equal to 0. So here we are assuming that these functionals are satisfying the boundary conditions, so these are obvious from that fact.

So here we can simplify this and we can write x not to x 1, j equal to 1 to n F of y j minus d by dx of F of y j dash delta y j d of x equal to 0. So here if you recall the proof of the Eulers Equation their we have the similar kind of equation and we say that since delta y j are say arbitrary then we can keep this summation has to be 0. But here we cannot consider right now as delta y j are completely arbitrary because they are related by the the constraint phi i x from 1 to n equal to 0 for i equal to 1 to m. So here we cannot apply the fundamental lambda here to obtain the Eulers Equation for this.

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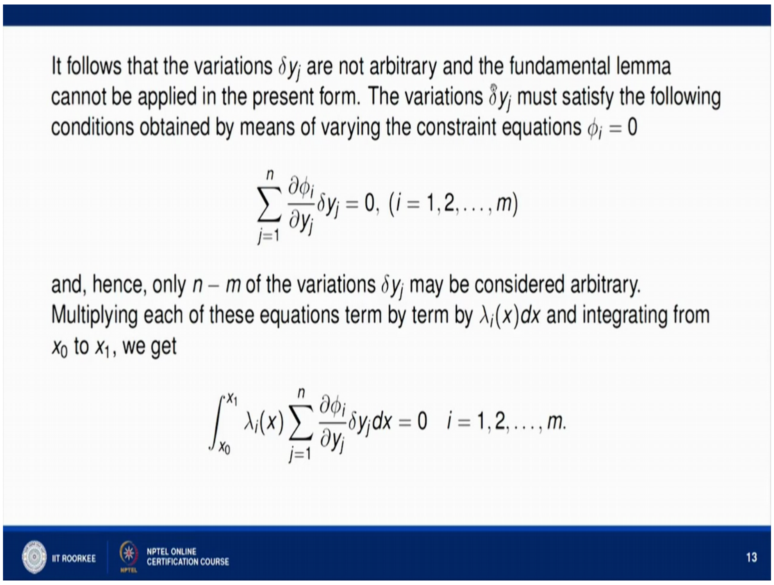

So here we can say that it follows that the variational delta y j are not arbitrary and the fundamental lambda cannot be apply in the present form. So the variation delta delta y j must satisfy the following condition obtained by means of varying the constraint equation phi i equal to 0. So when you discuss y i equal to 0 then it is reduced to this equation j equal to 1 to n deba phi i by deba y j delta y j equal to 0.

So here this delta y j satisfying these m equation and hence we cannot we can say that all delta y j are not arbitrary. So but here we have m equations so we can say that only n minus m of the variation delta y j may be considered arbitrary. So what we try to do here we multiply these equation by say lambda j and integrate from x not to x 1 we get this integral x not to x 1 lambda j x summation j equal to 1 and deba phi i upon deba y j delta y j dx equal to 0, i equal to 1 to m.

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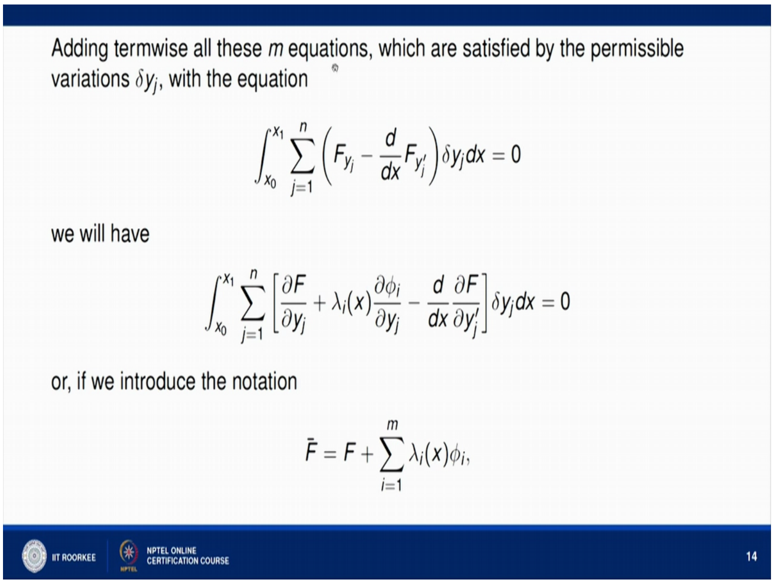

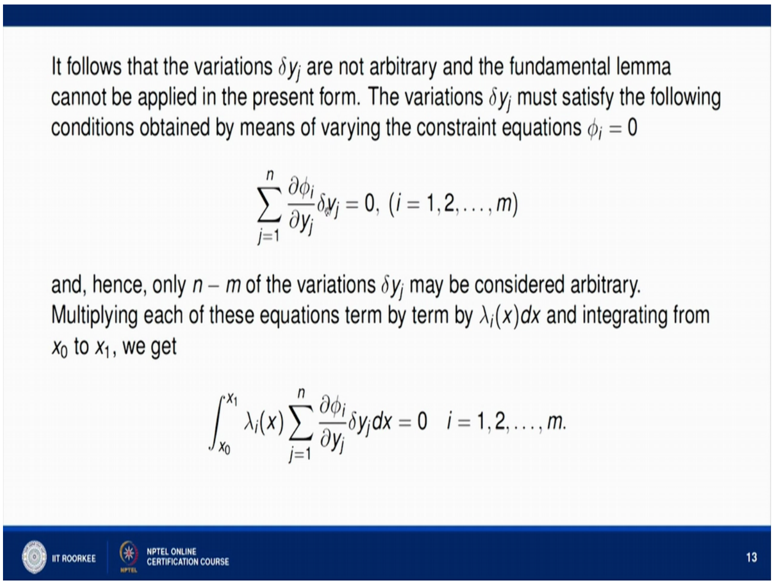

So here adding term wise all these m equations, which are satisfied by the permissible variation delta y j with the equation x not to x 1, j equal to 1 to n F of y j minus d by dx F of y j dash delta y j dx equal to 0. So this we obtain by taking delta v equal to 0. Now we have this equation which we multiply by lambda j and we are getting this we have new integral that is x not to x 1, j equal to 1 to n deba by F by deba y j plus lambda i x deba phi by deba y j minus d by dx of deba F by deba y j dash delta y j dx equal to 0 here. So here we introduce a new notation that is F bar which is written as F plus summation i equal to m lambda i x phi of i.

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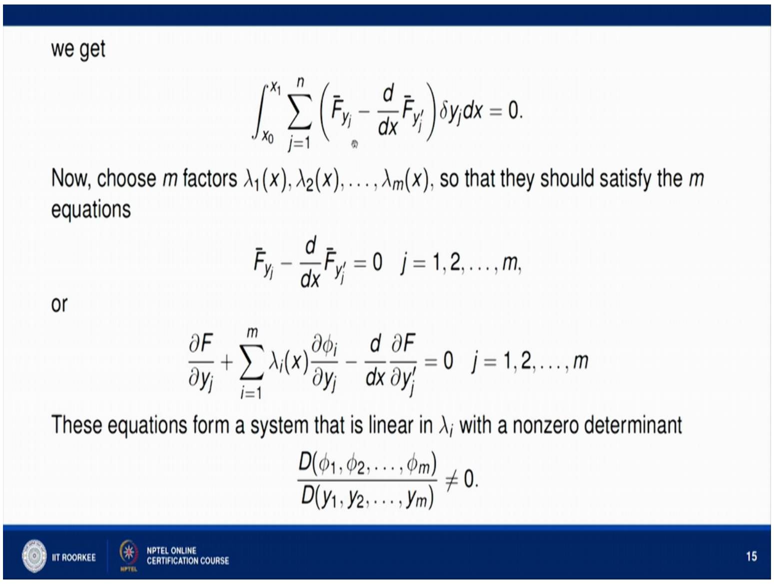

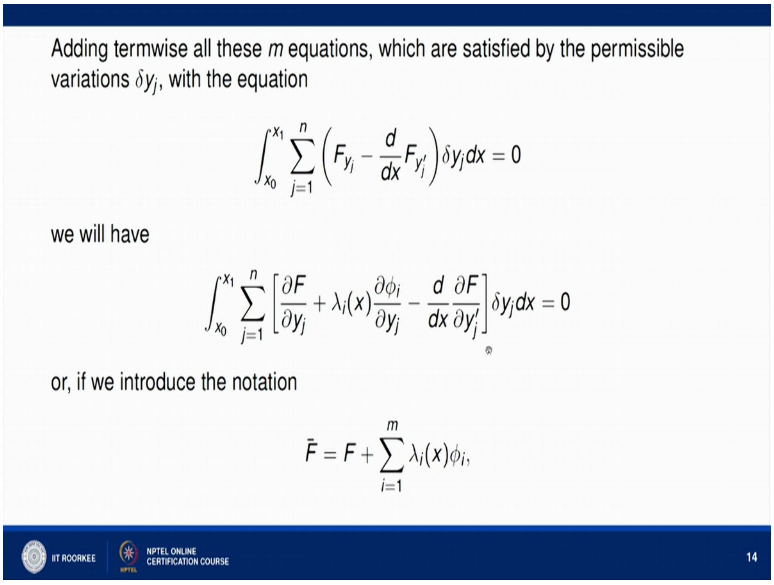

So if we take this new notation that is F bar we can rewrite this condition into this form x not to x 1, j equal to 1 to n F bar y j minus d by dx of F bar y j dash delta y j dx equal to 0. So here in this this can be rewritten in terms of F bar as follows. So here we have incorporated our constraint as well as the condition that delta v equal to 0 into this integral that x not to x 1, j is equal to 1 to n F bar y j minus d by dx F bar y j bar y j dash delta y j dx equal to 0.

Now still we cannot apply the fundamental lemma because delta y j are not arbitrary so for this we have taken the factors lambda 1 to lambda m completely arbitrary but now let us assign some values of lambda 1 to lambda m so that this equation so j so this equation F bar y j minus d by dx F bar y j dash is equal to 0 for j equal to 1 to m. So we try to find out lambda 1 to lambda m such that these m equations is true and what is this if you simplify this if you look at this is nothing but deba F by deba y j plus summation i equal to 1 to m lambda i x deba phi i by deba y j minus d by dx deba F by deba y j dash equal to 0.

If you look at this linear equation system of linear equation in terms of lambda i and we can find out the solution here solution of this system of linear equation in terms of lambda i because the coefficient matrix if you look at deba phi i by deba y j this has a nonzero determinant because we have assumed that these phi i are independent constraint. So keeping this thing in mind we can always find out such factors lambda 1 to lambda m such that these equations are true for j equal to 1 to m. So if we have this equations true for j equal to 1 to m putting it into this equation.

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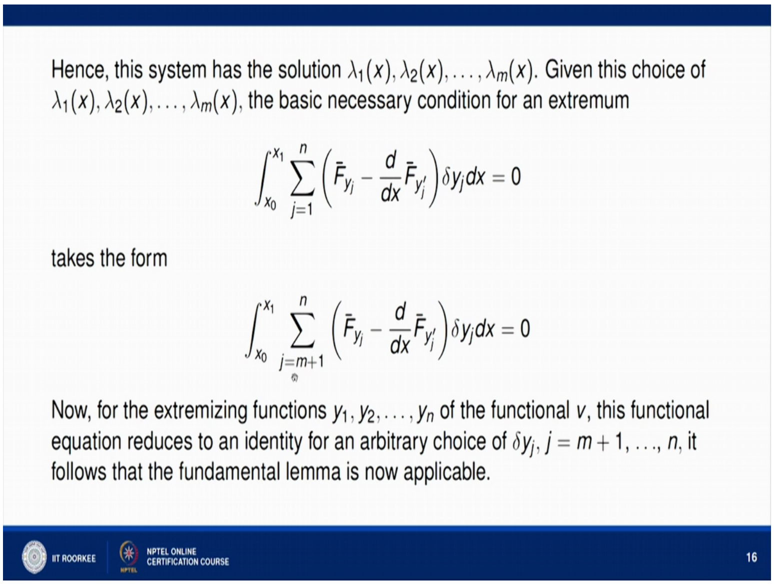

We can have that we have this x not to x 1. Now summation is reduced to j equal to m plus 1 to n initially it is from 1 to n now it is reduced to m plus 1 to n F bar y j minus d by dx F bar y j dash delta y j dx equal to 0. So now this delta y j are arbitrary in some sense so we can say that for the extremizing function y 1 to y n of the function v, this functional equation reduce to an identity for an arbitrary choice of delta y j, j equal to m plus 1 to n. So now we try to apply the fundamental lemma, so what how we can apply by taking say all the variation as 0 except one.

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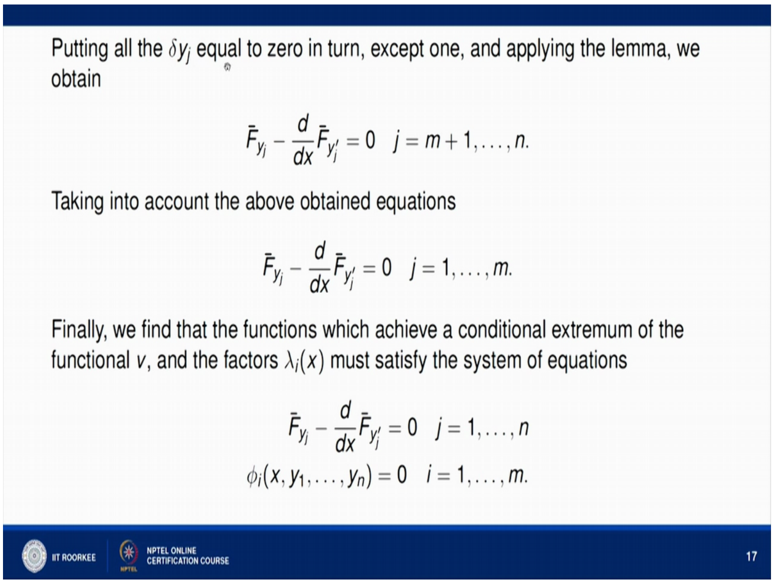

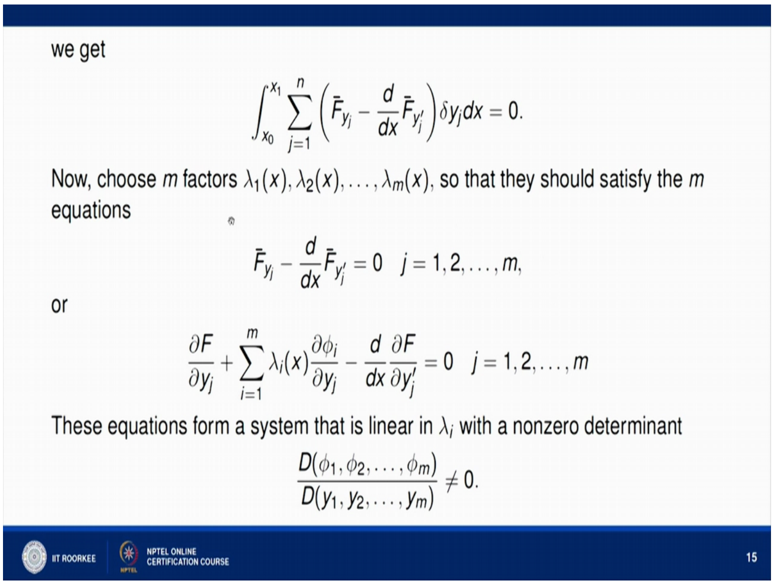

So let us say that putting all the delta y j equal to 0 in turn, except one, and applying the fundamental lemma we obtain this F bar y j minus d by dx of F bar y j dash equal to 0 for each j equal to m plus 1 to n and if we so here we have n minus m equation and if we remember we have these m equation which we have obtained by choosing the values of these factors lambda 1 to lambda m.

So this along with these n minus m equation we can rewrite a set of equation as this that taking into account the above obtained equation we have F bar y j minus d by dx F bar y j dash equal to 0 for each j equal to 1 to m, so 1 to m is earlier one and this m plus 1 to n we have this equation F bar y j minus d by dx of F bar y j dash equal to 0 for each j equal to now

1 to n. So 1 to m is obtained by choosing lambda 1 to lambda m and this m plus 1 to n obtained by applying the fundamental lemma.

So we have now n such equation given here and m set of equation given by the constraint. So we have total n plus m equation and this n plus m equation we can find out the function y 1 to y n and the function lambda 1 to lambda m.

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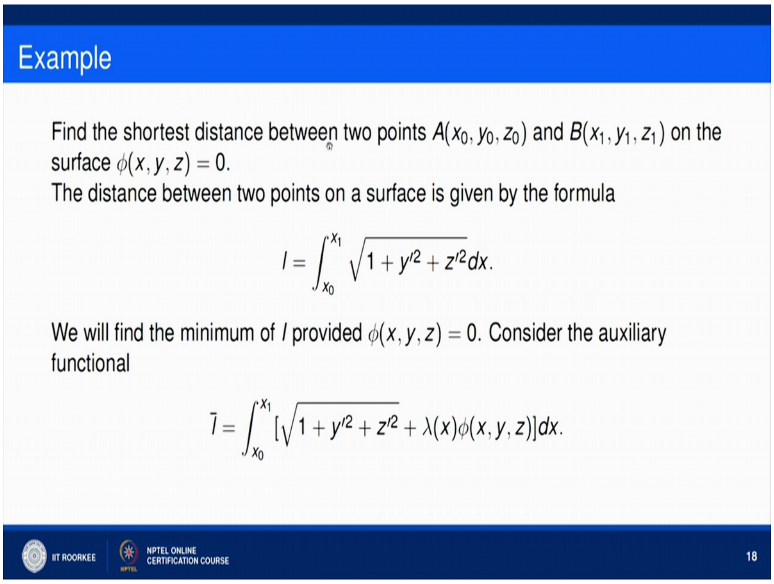

So let us consider one example here and the example is that find the shortest distance between points A and B where the coordinates of A as x not, y not, z not and coordinates of B are x 1, y 1, z 1 on the surface phi of x, comma y, comma z equal to 0. So here shortest distance so the formula of distance is given here I as x not to x 1 under root 1 plus y dash square plus z dash square d of x.

And the condition that these two point lie on phi of x, comma y, comma z equal to 0 and we are measuring the distance only on this surface. So this is the condition we have to consider and now this we can solve by the theorem given just now. So here consider the auxiliary function I bar equal to x not to x 1 under root 1 plus y dash square plus z dash square plus lambda x phi of x, comma y, comma z dx. So here we have only one subsidiary equation or one condition so we consider only one function lambda x like this. Now here if we assume this as a new functional F bar then we can write down the Eulers Equation for this F bar.

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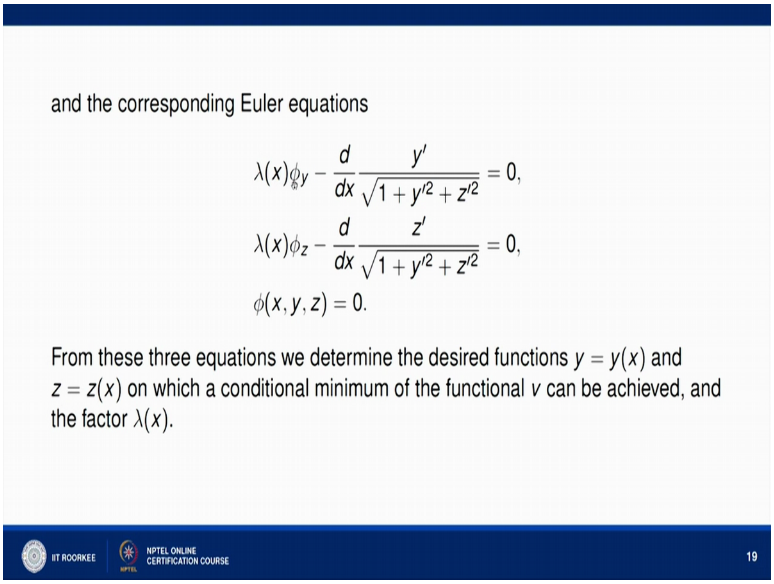

And we can say that the corresponding Eulers Equation are reduced to lambda x phi y minus d by dx of phi dash upon under root 1 plus y dash square plus z dash square equal to 0. Similarly corresponding to z lambda x phi z minus d by dx z dash upon under root 1 plus y dash square plus z dash square equal to 0 and the condition that is phi of x, comma y, comma z equal to 0. So from these three equation we can find out the function y equal to y of x and z equal to z of x on which the condition minimum of the functional can be achieved and we can also find out the function lambda x from these three equation.

So here we have applied our theorem to find out the extremum of this functional this variational problem I equal to x not to x 1 under root 1 plus y dash square plus z dash square d of x provided the condition phi of x, comma y, comma z equal to 0.

So here we stop and in next class we will discuss when the condition is not a finite equation but it is a differential equation in place of this finite equation. So in next class we will discuss the similar kind of problem but the conditions are replaced by differential equation rather than this finite equation and consider and we will also consider certain application of this said theory, okay so thank you for listening us we will meet in next lecture, thank you.