Integral Equations, Calculus of Variations and their Applications
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Lecture 51
Invariance of Eulers Equation and Isoperimetric Problem-1

Hello friends welcome to today's lecture. Here in today's lecture we will continue with the previous lecture. In previous lecture we have defined the concept of variational derivative and as an application we have shown the invariance of the Eulers Equation. So in today's class we will discuss some example of Invariance of Eulers Equation and we try to discuss the problem which is known as Isoperimetric Problem.

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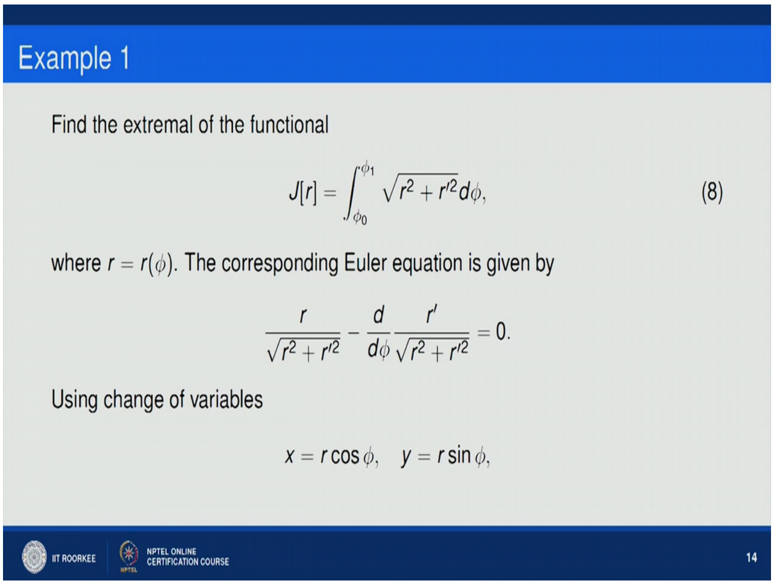

So here we start, so here if you recall we have discussed the concept of variational derivative and then we have discussed the Invariance of Eulers Equation and now we are moving to the first example based on invariance of Eulers Equation.

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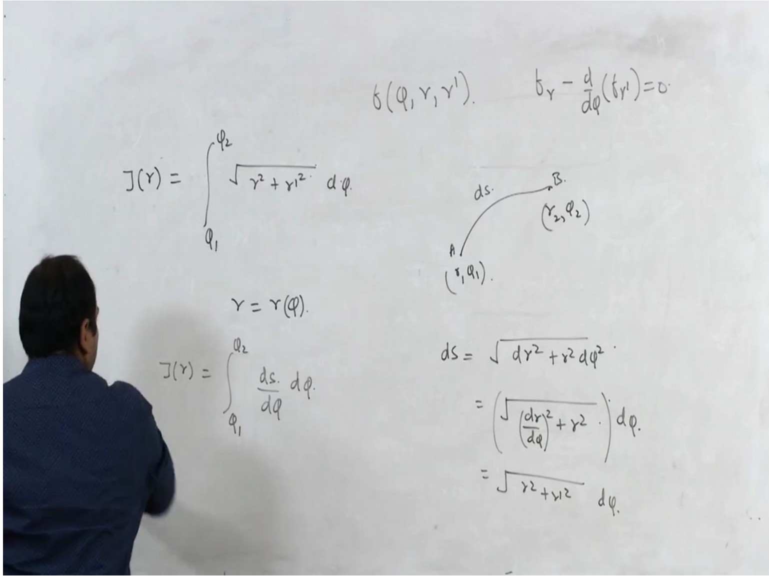

So the example is given by this, so here we have a J of r which is given as some phi 1 to say phi 2 under root of r square plus r dash square d of phi and here we want to minimize we want to extremes the functional given here. So here your r is given as r of phi where so here this actually represent the distance between say r1 and phi1 be one point and r2, phi2 is another point and this represent the arc element with joining these two points let us call this as point A and B and this simply represent the length and element ds here.

So here we want to we can consider here that in polar coordinate system we want to find out say the curve which minimizes the distance between these two point A and B. So for that we are trying to find out so we are extremizing this functional also, so here your ds is given as say r square sorry ds is equal to dr square under root r square d phi square. So which we can represent as say dr d phi whole square plus r square which is nothing but d of phi so which is nothing but r square plus r dash square d of phi.

So basically this represent what J of r is equal to say phi 1 to phi 2so ds here ds by d phi and d phi. So basically we want to extremizing this functional given in here. So here we want to find out say the minimum distance between A and B along the curve r here r equal to r of phi here, right. So if you look at with the help of Invariance of Eulers Equation in this particular form your this f is given as function of phi, r, and r dash and in this case your Eulers Equation is written as f of r minus d by d phi of f of r dash equal to 0.

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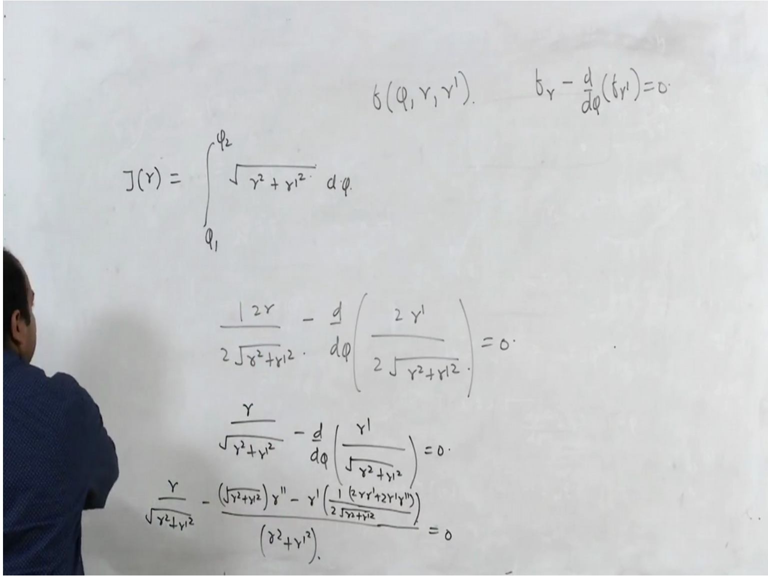

So if you want to find out the Eulers Equation will be what? Eulers Equation will be here you can write it 1 upon 2 under root r square plus r dash square and here we have 2 of r minus d of d of phi and this is what here we can write it 1 upon 2 under root r square plus r dash square and when you differentiate this you will get 2 of r dash and that is equal to 0. So here we can simplify this and we can get this is r upon under root r square plus r dash square minus d by d phi upon r dash of r dash upon under root r square plus r dash square that is equal to 0.

So to find out say extremizing curve for this functional we need to solve this particular equation which is r upon under root r square plus r dash square minus to find out say derivative of this we use the formula for derivative of f by g, so here we have r square plus r dash square and here we have under root r square plus r dash square and derivative of r dash that is r double dash minus r dash and then derivative of this that is 1 upon 2 under root r square plus r dash square and then we find out say derivative of this here. So that is it is 2r r dash plus 2r dash r double dash. So that we are writing it here, okay so that should equal to 0.

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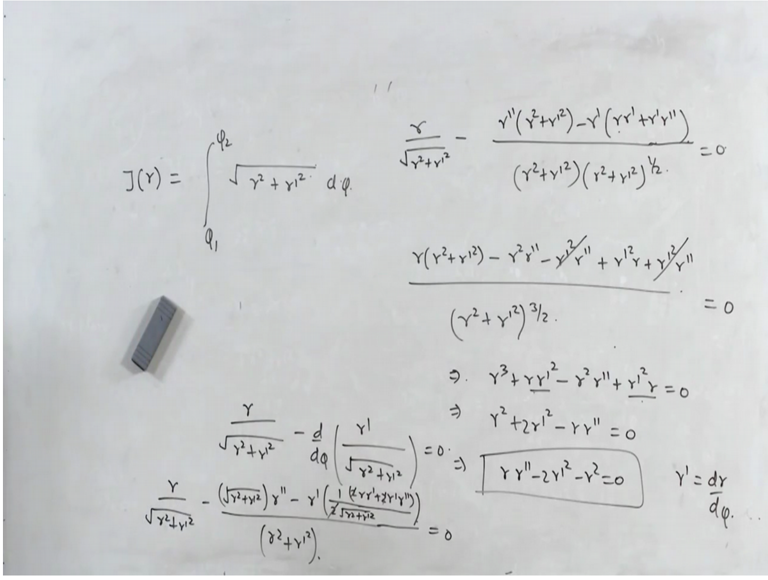

So to simplify this let us simplify this here so here we can simplify this as r upon under root r square plus r dash square minus if you take the this under root r square plus r dash square multiply by that and simplify you will get r double dash multiply by r square plus r dash square minus r dash and if you simplify these two these two will be cancelled out here we will get what r dash into rr dash plus r dash r double dash and this will be and in denominator we have r square plus r dash square into r square plus r dash square square root of this that is 1 by 2 and this is equal to 0.

Then we can take this thing in denominator and you can say that it is r into r square plus r dash square minus you simplify this here when you simplify you will get r square r double dash minus r dash square r double dash plus r dash square r then here we will get plus r dash square r double dash equal to 0. So we can say that this will be satisfied if the numerator is equal to 0. Now if you look at is there anything which is canceling out, so here if you look at r dash square r double dash this will be cancelled out here so we have this thing.

So this represent r cube plus rr dash square minus r square r double dash plus r dash square r has to be 0. So dividing by say r dash throughout we can say that r square plus r dash square minus so here this is we divide by r here so it is r square this and this term are same so we can add them and we can divide by r, so 2r dash square minus rr double dash is equal to 0. So if you simplify little bit further take it on the other side we have rr double dash minus 2r dash square minus r square is equal to 0.

So the Eulers Equation in this particular for this particular function is given by this which is quite difficult to solve here, here r dash is your dr by d phi. So what we try to do here we try to use the take the help of Invariance of Eulers Equation and try to find out a new coordinate system where this equation is solvable. So for that what we try to do here we use the change of variable that is x equal to r cos phi and y equal to r sin phi.

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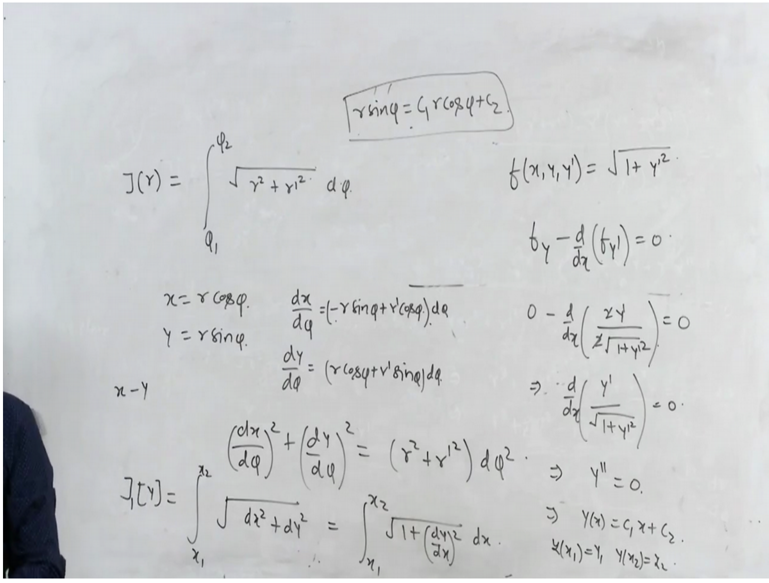

So what we try to do here in place of polar coordinate system we want to find out in a we want to consider the same problem in a rectangular coordinate system. So here we take x equal to r cos of phi and y equal to r of sin of phi. So to find out the equalant this thing so you simplify now dx by d phi. So which is given by say minus r sin phi plus your r dash you can write it here as cos of phi d phi so d phi and here similarly you can find out dy by d phi so that is r cos of phi plus r dash and here you can get this is what is r r cos of phi plus r dash sin of phi d phi, right.

So it means that new coordinate system that is x-y coordinate system we have J1 y is equal to the corresponding let us say phi 1 is given by x1 so and phi 2 is corresponding to say x2 and this under root r square plus under root r square plus r dash square is given by under root say dx square plus dy square and so this can be written as x1 to x2 and this is 1 plus dy by dx whole square and d of x here.

So it means that this function is now reduced to J1 y and your integrand is under root 1 plus dy by dx whole square d of x. So here your f of x, y, y dash is given by under root 1 plus y dash square and which is which is quite easy to solve in fact we have already solve here. So here your Eulers Equation is given by f of y, f of y minus d by dx of f of y dash is equal to 0. So there is a no y term here so will be 0 minus d by dx of f of y dash that is 1 upon 2 1 plus y dash square 2y dash here is equal to 0 when you can solve this you will have d by dx of y dash upon under root 1 plus y dash square equal to 0.

And when you solve this in fact this we have already solved somewhere then this is coming out to be solving y double dash is equal to 0 which is nothing but y of x is equal to c 1 x plus c 2 here where c 1 and c 2 are integration constant that you can find out with the condition that your y of x 1 is equal to y 1 and y of x 2 is equal to some y 2. So with the help of these two condition you can find out c 1 and c 2 and you can say that the extremal for this functional is a straight line.

So here we can say that that the extremal of this functional is given by say r sin phi is equal to c 1 r cos of phi plus c 2. So it means that here though it is the Eulers Equation if we consider here is quite difficult to solve but if we use the change of coordinate system and the property that is change of Invariance of Eulers Equation we have converted this functional into a new functional which is quite easy to solve and with the help of Eulers Equation in new coordinate system it is quite easy to solve and here our extremal curve is coming out to be straight line satisfying these two condition.

And if you consider the original functional then the extremal curve for this functional is given by r sin phi equal to c 1 r cos phi plus c 2 which is quite difficult to solve in the in a polar coordinate system (())(13:25).

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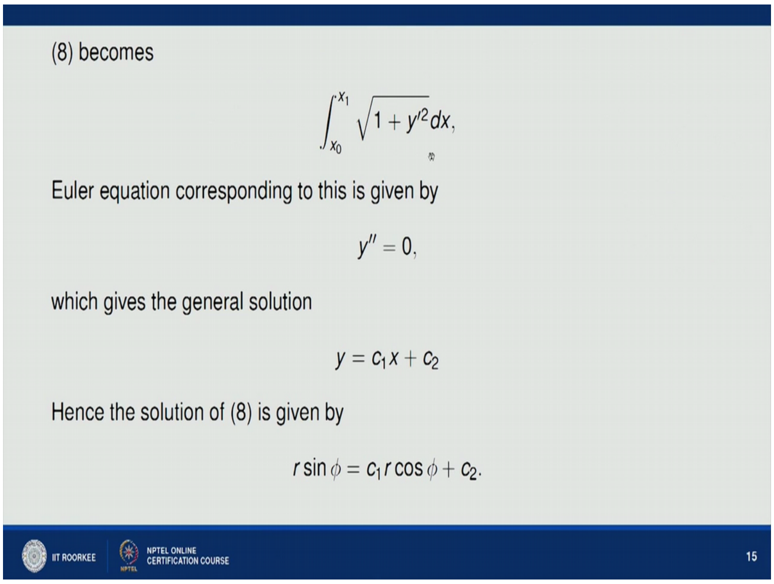

So here what we have discussed here so this functional is reduced to this x not to x 1 under root 1 plus y dash square d of x and the corresponding Eulers Equation given by this and which have a solution by equal to c 1 x plus c 2 this this is just now we have proved.

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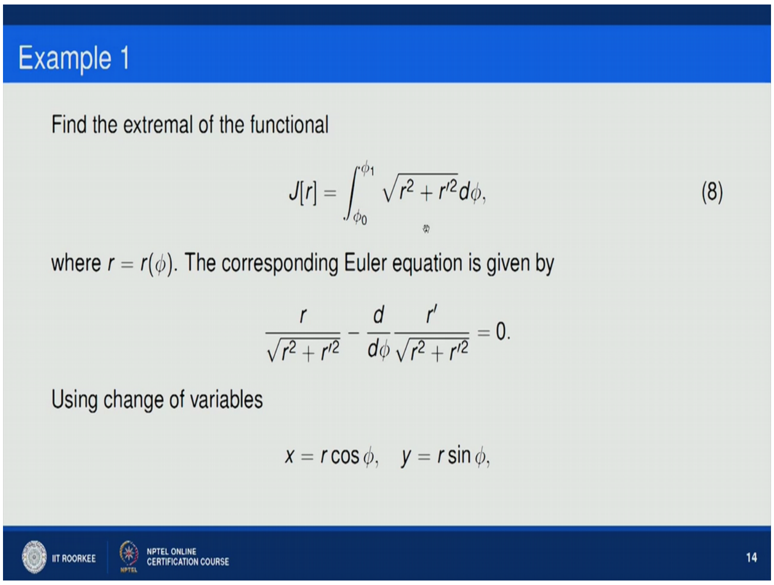

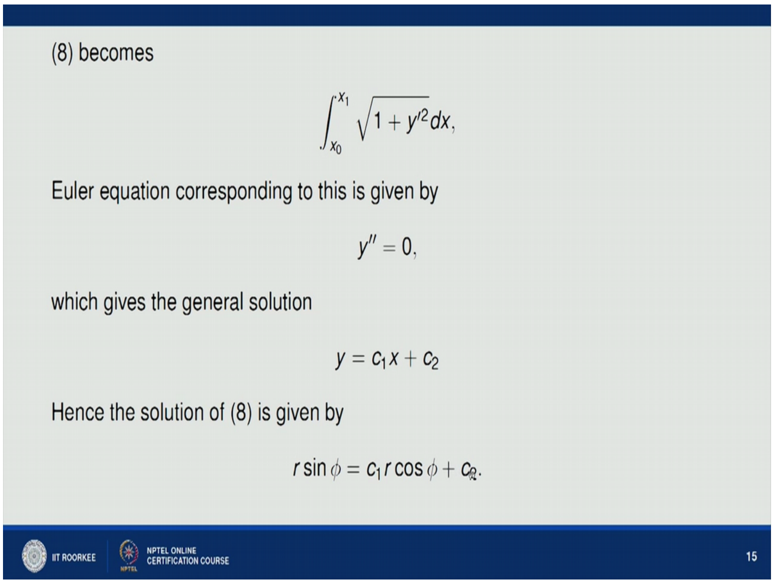

And extremal extremal further function which is given here is given by r sin phi equal to c 1 r cos phi plus c 2.

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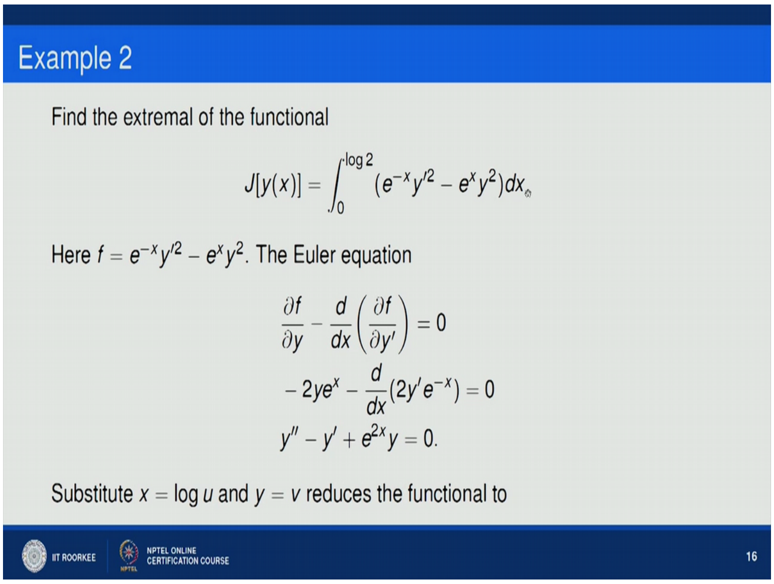

Now let us move to second example and and the example here is to find out the extremal of J of y of x which is 0 to log of 2 e to the power minus x y dash square minus e to power x y square d of x which is given in terms of a rectangular coordinate system x, y. So here your if you look at the integrand is e to power minus x y dash square minus e to power x y square. So if you want to find out the extremal so we we know that extremal must satisfy the Eulers Equation which is given by f of y minus d by dx of f of y dash equal to 0.

So if you calculate say fy, fy is nothing but minus 2 e to power x y here and f of y dash will be 2 of y dash e to power minus x and when you simplify you will get this equation f double dash minus y dash plus e to power 2x into y equal to 0. So it means that if y is a extremal of this functional then it must be a solution of this second order differential equation.

Now if you look at this second order differential equation y double dash minus y dash plus e to power 2 x y equal to 0 here we do not know any general method to solve this particular problem though we can solve this particular problem by changing the independent variable by saying that e to power x equal to a new independent variable that that is t then we can solve this we can convert this problem into a second order differential equation with a constant coefficient in terms of independent variable t and we can solve it.

But we can avoid that and we can say that this can be simplified further if we take x as log of u and y equal to v in fact we can say that e to power x as a new variable that is u and y as simply v.

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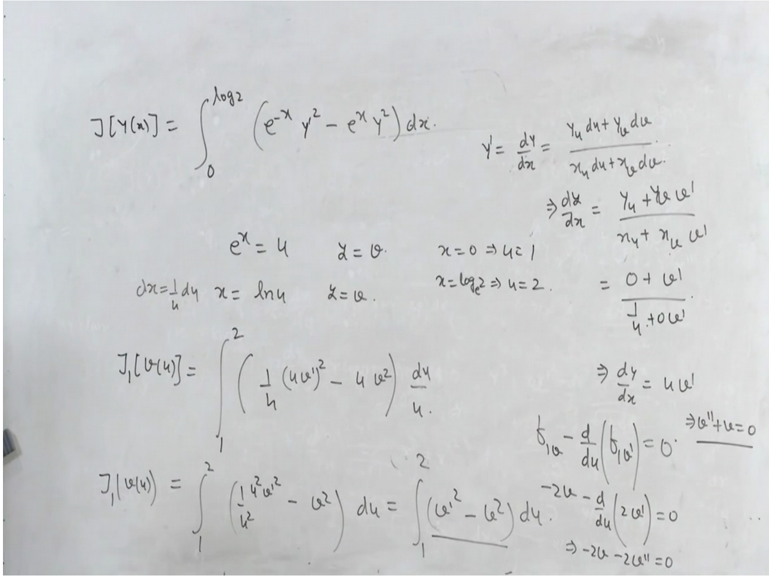

So here we have written the function given in this form. Now let us assume that e to power x is equal to u and here y is given as v. So here I can write x as lnu and y as v then we try to find out say a this functional in terms of u and v. So it means that we want to find out J v1 of u, so here let us find out say limit of u here so limit of u will be what when x equal to 0 then u is given as 1 and when x is equal to log of 2 so it means that when x equal to 0 then this represent u equal to 1, when x equal to log of 2 base e then this represent this will give you u equal to 2. So limit for u will be 1 to 2 then e to power minus x will be say 1 upon u.

Now y dash square will be what? So we have to calculate what is y dash. So y dash will be dy by d of x, so dy is given as say yu du plus yv dv divided by xu du plus xv dv. So everything is given here so y of u will be 0 here. So we can write this as we can further simplify this as dy by dx as this is Yu plus Yv v dash xu plus xv v dash. So here y of u will be simply 0 plus y of v y of v will be 1, so this is v dash and x of u will be say 1 upon u and x of v is 0 into v dash. So here we will give we will have dy by dx is equal to so this will go up and it is uv dash. So dy by dx is given as uv dash.

So 1 upon u and here we have uv dash square of this minus this is e to power x as u and y square as v square and d of x is given by here you can find out dx as 1 upon u d of u so it is d of u upon u. So you can simplify it further it is 1 to 2 and when you take when you multiply u inside then it will be what so you can say that 1 upon u square and this is u square v square u square v dash square minus this will cancel out here and we have v square and this is du. So this is nothing but 1 to 2, this is v dash square minus v square and d of u.

So it means that this functional is now reduced to new functional which is given as J1 v of u and here your integrand is v dash square minus v square. So now if we want to find out say extremal of this functional then we can apply your Eulers Equation in new coordinate system and here your you can say that it is what f 1 of v minus d of du f 1 of v dash equal to 0 so f 1 v will be so minus 2 of v minus d of d of u and f 1 v dash will be 2 of v dash is equal to 0.

And when you simplify this this is what minus 2 of v and this is minus 2 of v double dash is equal to 0 and which is nothing but so here we have v double dash plus v equal to 0 which is a very simple second order differential equation with constant coefficient and the solution is given as simply C1 cos of u plus C2 sin of v.

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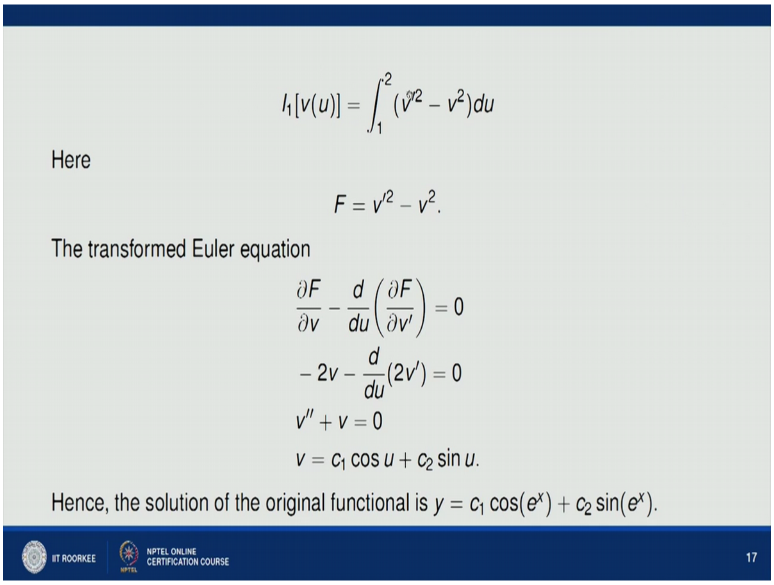

So let us look here in the slides, so here we have just seen that this functional this J of y of x can be written as I 1 v of u that is 1 to 2 v dash square minus v square d of u, here your integrand is v dash square minus v square and then we can find out the Eulers Equation in new coordinate system as F of v minus d by du of F of v dash equal to 0 it is coming out to be v double dash plus v equal to 0 and the solution is given as v equal to c 1 cos of u plus c 2 sin of u. And we convert into the original functional then our extremal is given by y equal to c 1 cos of e to power x plus c 2 sin of e to power x.

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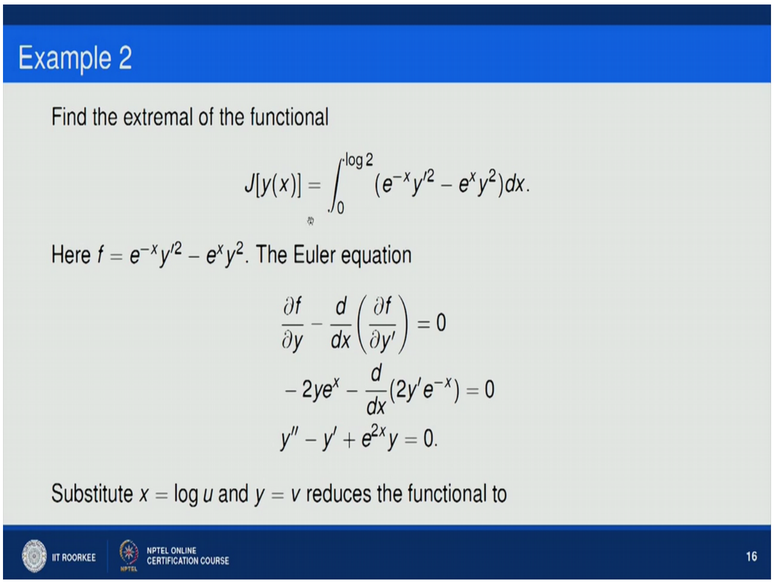

So here if we start with the original coordinate system that is x-y coordinate system here then Eulers Equation is this which is little bit difficult to solve it here but if we change the coordinate system.

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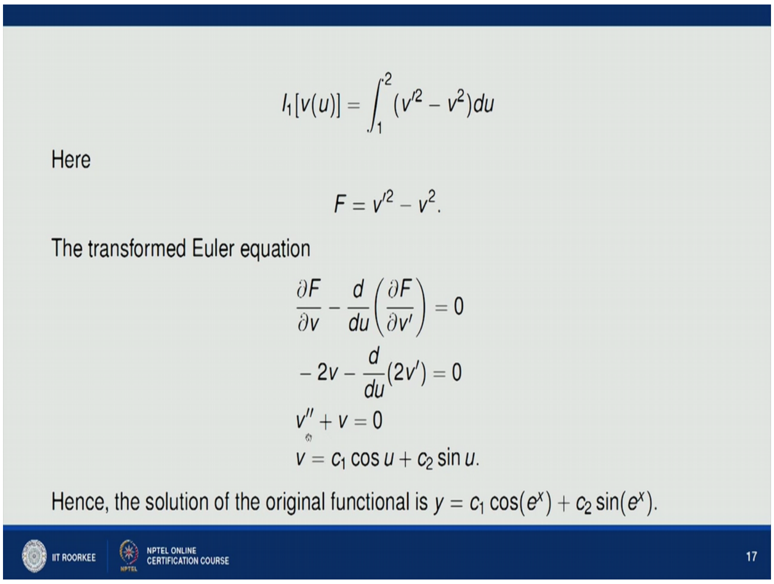

Then we have seen that it is very easy to obtain the extremal curve in new coordinate system. So that is the very useful application of Invariance of Eulers Equation.

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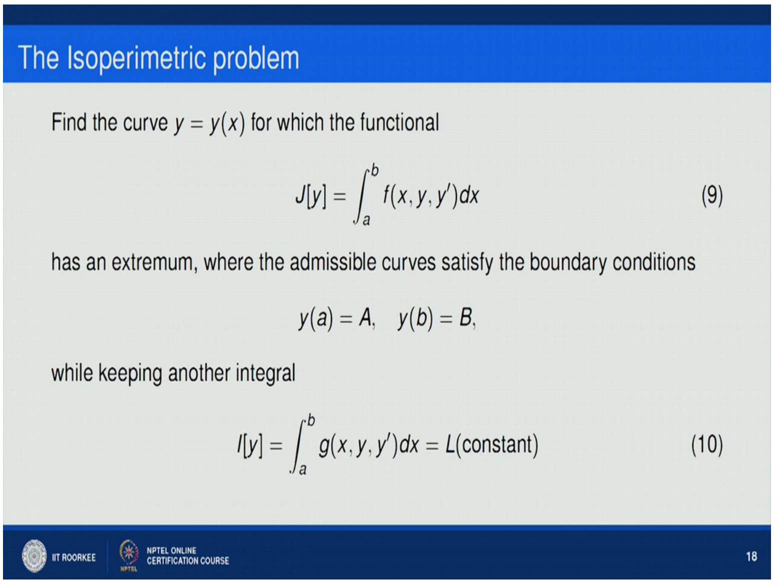

Now we will move to next problem that is Isoperimetric Problem. So what is Isoperimetric Problem? This is a very old problem, so in Isoperimetric Problem we have to find out that among all the close curves with the given length say some given length say let us say L we need to find out the curve which inclusis the maximum area. So it means that we have to find out a curve with the fixed length means parameter is fixed then we want to find out say the curve which have a maximum area and answer is also known from the very beginning and it is coming out to be a circle.

But circle but how to prove it we will try to prove it here a new problem which is known as Isoperimetric Problem here what we try to do here we want to find out the curve y equal to y of x for which we extremis this functional which is given as J of y between equal to A to B f of x, y, y dash dx with the boundary condition y of a equal to A, y of b equal to B, provided that the other condition which is given in terms of integral here that I of y a to b g of x, y, y dash dx remains as a constant. So it means that the problem is what we need to extremis the functional this provided that this value remain constant.

So among all the function for which it is true we want to find out the curve which extremises this function.

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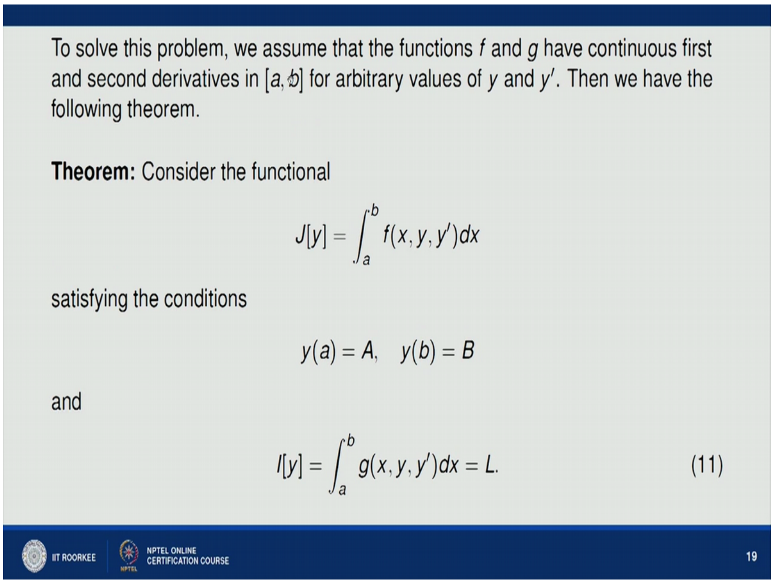

So to solve this problem we assume here that the function f and g have continuous first and second derivative in entire interval a, b for arbitrary values of y and y dash. Then we have the following theorem.

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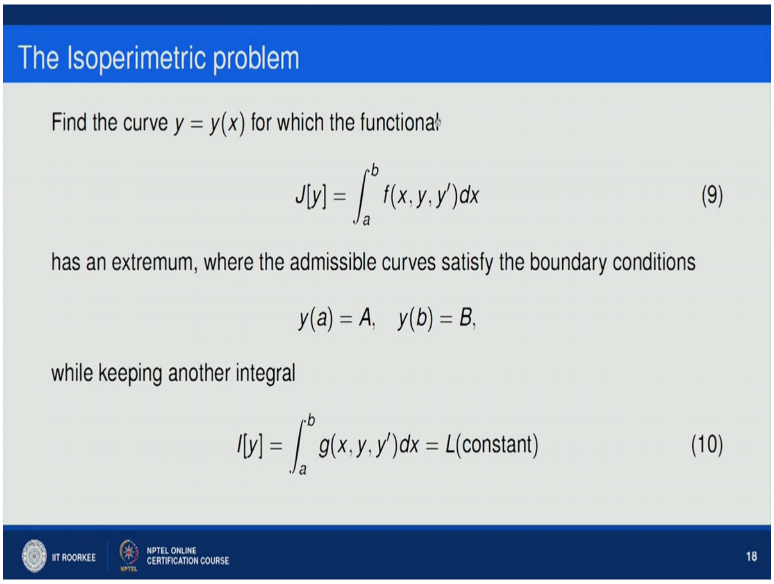

So to find out the extremal of this problem to extremis a functional given a another functional take a constant value.

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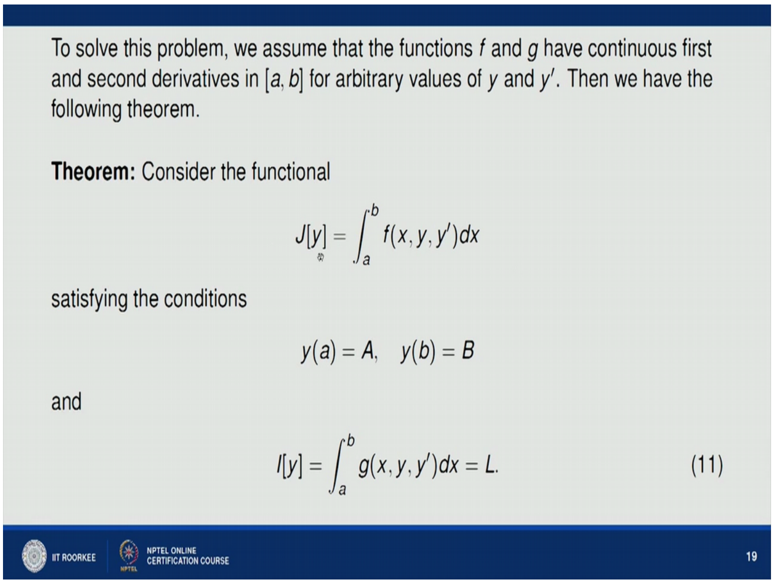

We have the following theorem which says that consider the functional J of y which is given as a to b f of x y, y dash d of x. Satisfying the condition y of a equal to A, y of b equal to capital B and the integral functional I y which is given as a to b g x, y, y dash dx take the value L.

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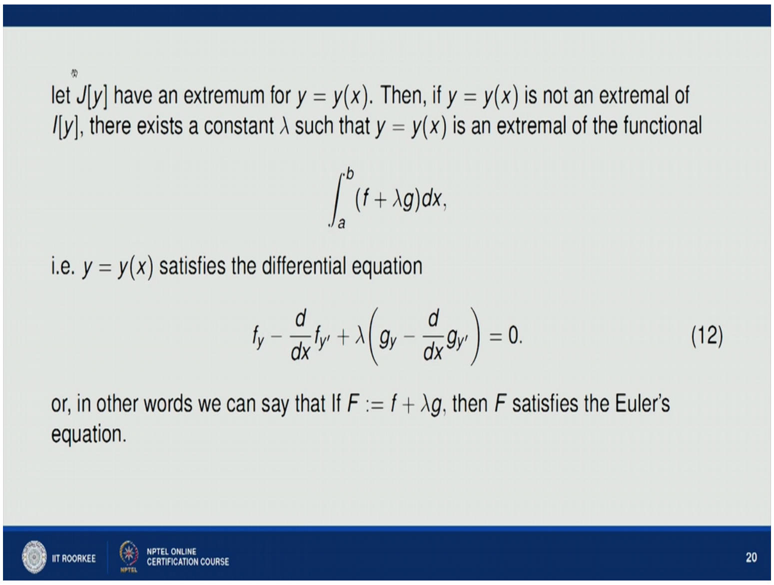

And we want to say that if J y have an extremum for y equal to y of x and y equal to y x is not an extremal of the other functional that is I of y then there exist a constant lambda such that y equal to y of x is an extremal of the functional a to b f plus lambda g d of x. It means that this y equal to y x is extremizing this functional it means that this functional lambda g this integrand f plus lambda g satisfy the Eulers Equation which is given as f of y minus d by dx f of y dash plus lambda g y minus d by dx of g y dash equal to 0.

Or in other word we can say that if f is denoted by f plus lambda g then f satisfy the Eulers Equation at a point of x in this interval a to b. So that is what is a theorem we want to prove this theorem. So to prove this theorem we use the concept of variational derivative.

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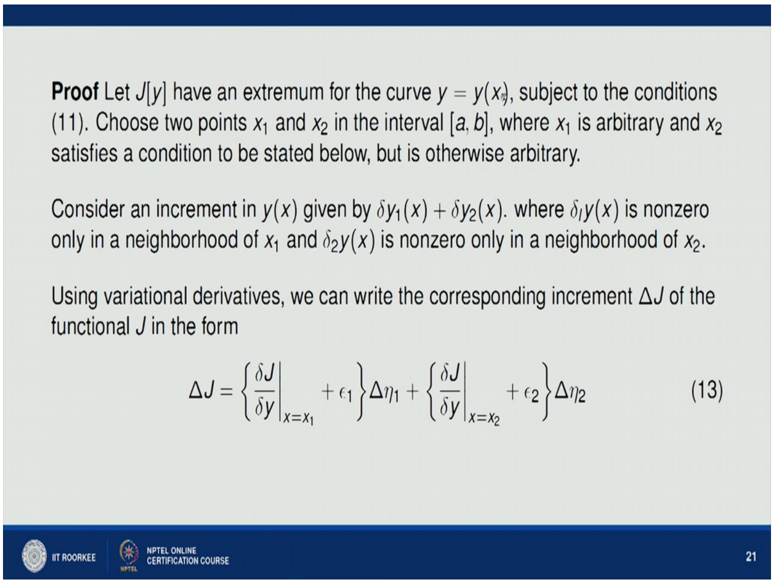

So here let us consider the proof here, so let J y have an extremum for the curve y equal to y of x, subject to the condition that the extremal the other functional have a constant value that is f. Now to find out that let us choose two point x1 and x2 in the interval a, b where x 1 is arbitrary and x 2 is also arbitrary but satisfying certain condition which we will discuss later on the idea is that when whenever we try to find out say that y of y equal to y equal to y of x is extremum curve we try to find out we perturb the curve y of x and try to see the functional value for those perturbed curves.

And here we cannot do arbitrary perturbation as we have done while proving the Eulers Equation while finding the Eulers Equation. Here if we change a little bit this y of x then it may happen that the value of this functional will also change so it means that if we take in place of y as some other curves say y 1 it may happen that this I of y will also change.

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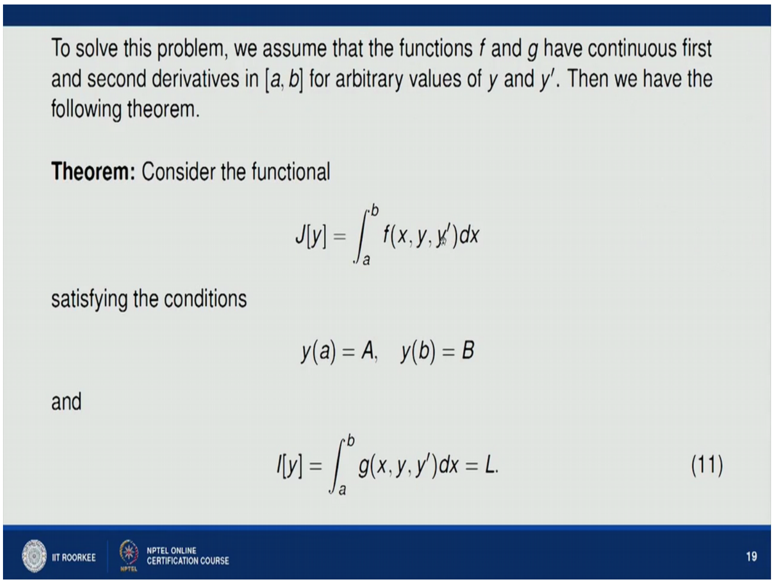

So here we have to keep this thing in mind that we can consider only those perturbation of y perturbation means derivation from y such that this remain constant.

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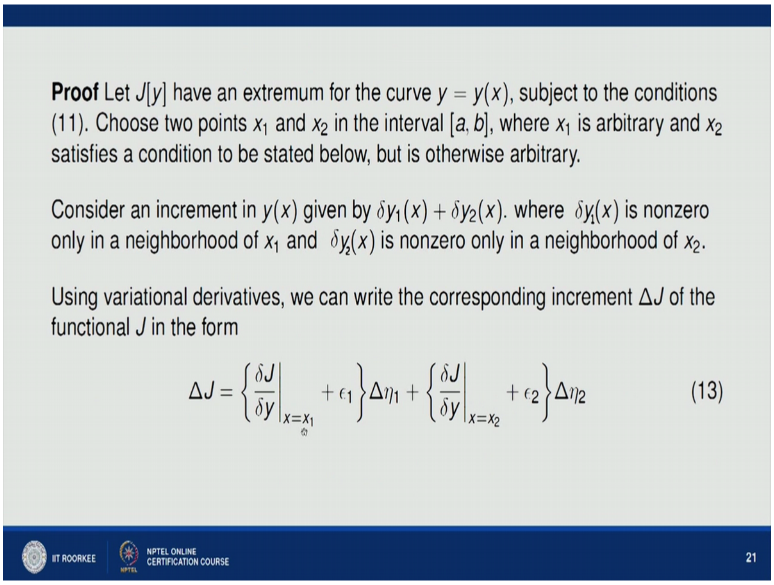

So here what we try to do here let us consider an increment in y of x given by delta y 1 x plus delta y 2 x where delta y 1 delta 1 y x is nonzero only in neighborhood of x 1 and delta 2 y x is nonzero only in neighborhood of x 2. So if we change delta 1 y of x in the neighborhood of x 1 then it may happen that your value of the I will also change, to balance that we consider another point x 2 where we where we take another variation in y of x which is given as delta 2 y of x such that your value remain constant.

So using variational derivative let us see how it is how we can write in terms of mathematics mathematical symbols. So using variational derivative we can write the corresponding increment delta J of function J in this form. So here delta J is given as delta J by delta y given at x equal to x 1 plus epsilon 1 delta eta 1 plus delta J upon delta y given at x equal to x 2 plus epsilon 2 delta by 2.

So here if you remember they are only change in the neighborhood of x 1 and in the neighborhood of x 2. So here delta J is given in terms of variational derivative at the point x equal to x 1 and the point x equal to x of 2.

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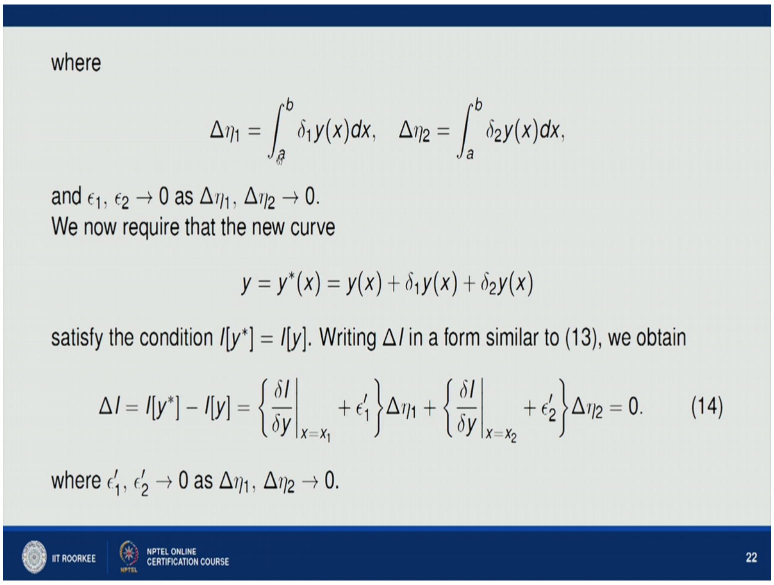

So here delta eta 1 is the area of the curve between a to b delta y 1 delta 1 y of x dx and delta eta 2 is the area in area of delta 2 y of x between a to b dx and as epsilon 1 and epsilon 2 tending to 0 as delta eta 1 and delta eta 2 tending to 0 so as this delta eta 1 and delta eta 2 tending to 0 then this epsilon 1 and epsilon 2 will also tend to 0. So now we have a perturb curve which is given as y of x plus delta 1 y of x plus delta 2 y of x and then for this we try to find out say that I of y will remain invariance means remain constant so I of y will be given as I of y star.

So for this we find out say increment in delta I in terms of variational derivative which is given as delta I is equal to I of y star minus I of y since both are same it has to be 0. Now this can be written as in terms of variational derivative as delta I by delta y at x equal to x 1 plus some epsilon 1 dash into delta eta 1 plus delta I by delta y evaluated at x equal to x 2 plus epsilon 2 dash into delta eta 2.

So here delta eta 1 is given by this area and delta eta 2 given by this area and epsilon 1 dash and epsilon 2 dash will tend to 0 as delta eta 1 and delta eta 2 tend to 0. So here this will give us a condition that how this delta eta 2 is given in terms of delta eta 1. So from this last equation 14 we try to find out delta eta 2 in terms of delta eta 1.

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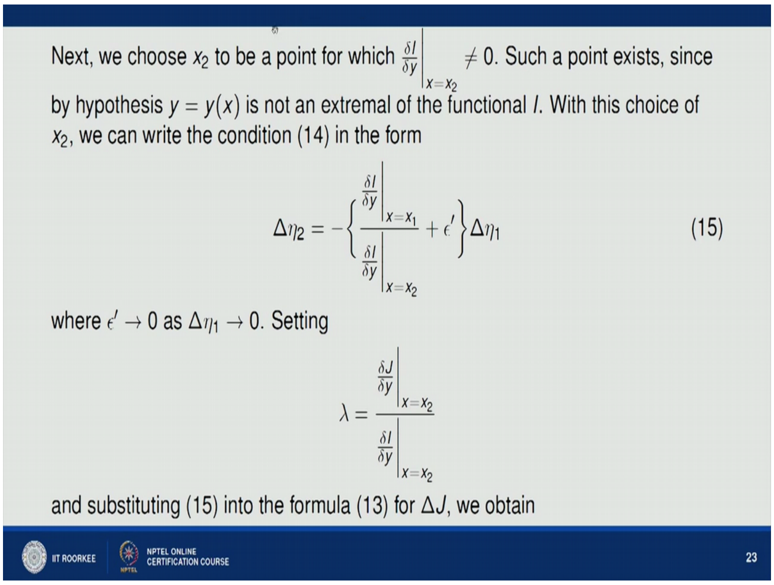

For this we assume that we choose our x 2 to be a point for which delta I by delta y at the point x equal to x 2 is nonzero. But how it can happen this will happen because by hypothesis we have already assumed that this y equal to y of x which is an extremal of the functional J but it is not an extremal of the functional I. So with this choice we can say that delta eta 2 can be written as minus of delta I by delta y given at x equal to x 1 divided by delta I by delta y given at x equal to x 2 plus epsilon dash into delta eta 1. Here as delta eta 1 is tending to 0 your epsilon dash is also tending to 0.

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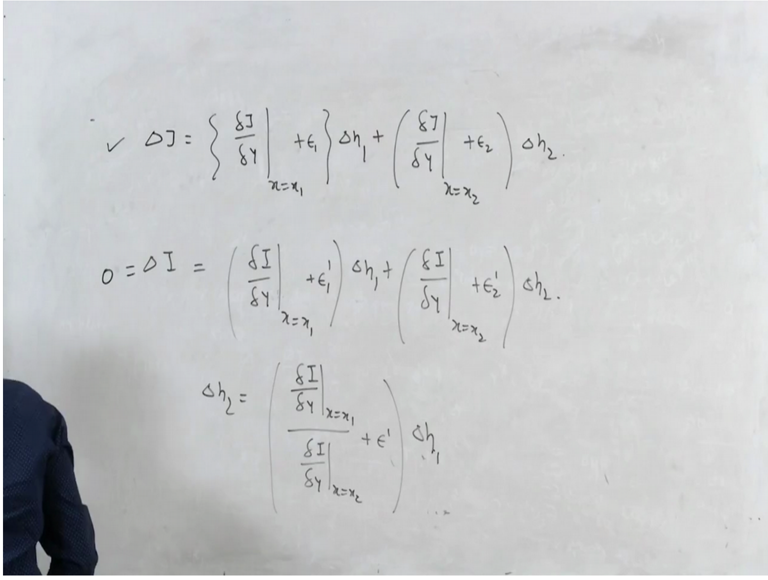

Let me write it here then we can understand in a more better way so we know that delta eta 2 given by this ratio delta I by delta y given at x equal to x 1 divided by delta I by delta y x equal to x 2 plus epsilon dash into delta eta 1. This we have obtained from the relation that delta I is equal to 0. So when delta I is equal to 0 means this coordinate is equal to 0 and since we have assume that this y is not an extremal of nonzero at any point between a to b.

So it means that delta I by delta y at the point x equal to x 2 is nonzero. So using this value here we can find out delta eta 2 in terms of delta eta 1 and it is given by this thing.

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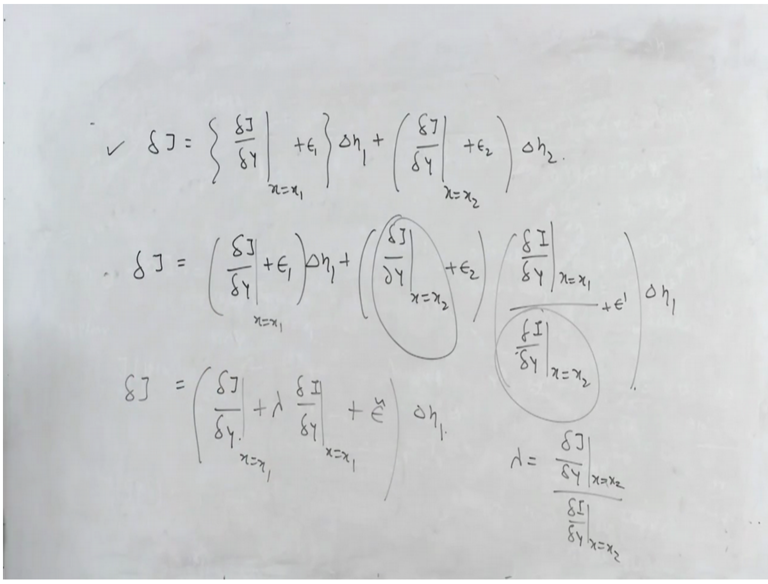

Now if you use delta eta 2 here in the functional in this value delta J is equal to this thing then your delta J can be written as let me write it here so here we can say that delta J is given as in fact this is not this is delta J, so here delta J is equal to you can say delta J upon delta y plus epsilon 1 into delta eta 1 plus here you have delta J delta y at x equal to x 2 plus epsilon 2 and here delta eta 2 I am writing it like this delta I by delta y at x equal to x 1 divided by delta I upon delta y given at x equal to x 2 plus epsilon dash into delta eta 1.

So when you simplify you can write here this when you simplify you will get delta J by delta y and this quantity we are assuming here this quantity divided by this quantity as lambda. So here we are assuming lambda as delta J upon delta y evaluated at x equal to x 2 divided by delta I by delta y evaluated at x equal to x 2 as lambda. So if we assume this then here we have lambda delta I upon delta y evaluated at x equal to x 1 this is already x equal to x 1 this is at x equal to x 1 plus some epsilon tilde into delta eta 1.

So here your delta J is given in this particular form. So please if you look at here we have remove the component x equal to x 2 and delta J is given by this where epsilon tilde is something which is tending to 0 as delta eta 1 is tending to 0. So here using this we can write delta J as this. So now as delta eta 1 is tending to 0 your epsilon is tending to 0.

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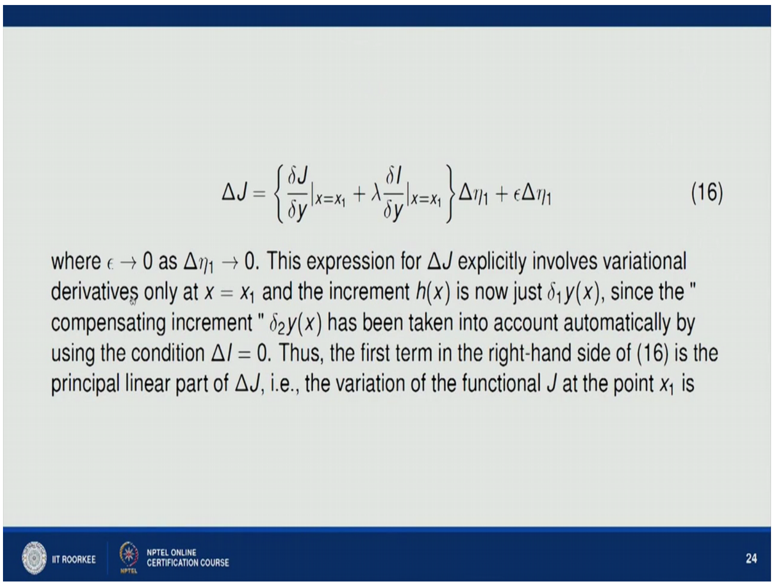

So this expression for delta J explicitly involves variational derivative only at x equal to x 1 and the increment h x is now just delta 1 y x, since the compensating increment delta 2 y x has been taken into account automatically by using the condition delta I equal to 0 that we have taken to find out delta 2 delta eta 2 in terms of delta eta 1. So it means that the first term in the right hand side of this equation 16 is the principal linear part of delta J, so delta J is given by this.

Now here this term is tending to 0 as delta eta 1 is tending to 0, so it means that this will give you the variation of the functional J.

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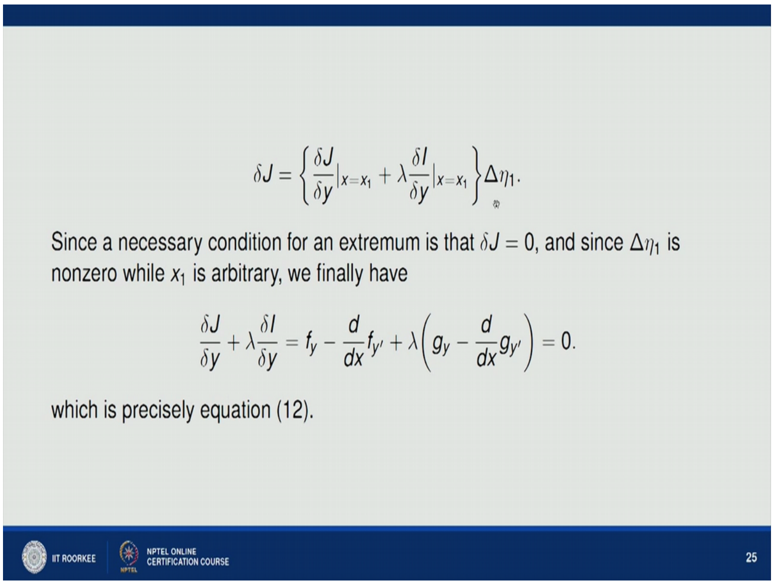

And we already know (delta). So it means delta J is equal to this quantity. Now we know that necessary condition for an extremum is that your variation is equal to 0. Now here delta eta 1 is nonzero while x1 is arbitrary we can say that this is this is equal to 0 for all the point of all the point between a to b. So it means that delta J by delta y plus lambda delta I by delta y is equal to 0 for all x. Now what is delta J by delta y? It is y minus d by dx of f y dash and delta I by delta y is g y minus d by dx of g y dash.

So this gives you a condition which is the preciously the condition which we want that this we have that f is the extremal extremal of the functional J provided that the extremal I will have a constant value we have the condition imposed on y is this. So here we have discussed the the general Isoperimetric Problem which is this that we want to extremis the functional J y here provided the other functional keeping the constant value.

So we have proved the following theorem which says that if you want to extremis the functional J y equal to a to b f of x, y, y dash dx and y is the extremal of this functional provided that this value of this functional is constant. We have just shown that that y equal to y of x is an extremal of the functional a to b f plus lambda g dx it means that if we denote this integrand as capital F then this capital F satisfy the Eulers Equation.

So this gives you an idea to extremize a functional such that your extremal curve satisfy the other extremal keeping as a constant value. So in next class we will discuss some example based on this. So here we stop with this note only and meet in next class, thank you very much for listening us, thank you.