Integral Equations, Calculus of Variations and their Applications
By Dr. P.N. Agrawal
Department of Mathematics
Indian Institute of Technology Roorkee
Lecture 49
Variational problems of general type

Hello friends, I welcome you to the lecture on variational problems of general type, so far we consider the variational problems which involved a process which were found by integrating a certain differential expression,

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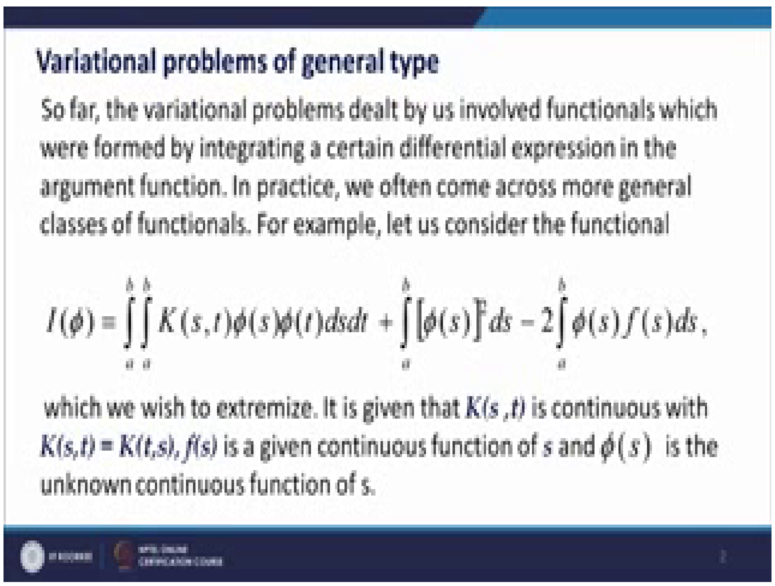

If you recall we had considered such type of functionals, now in practice we often come across more than the (cases) classes of (function) functionals, for example let us consider the functional i phi equal to integral a to b.

Integral a to b k(s, t) into phi s phi t, dsdt plus integral over a to b, phi s whole square into ds minus two times integral a to b, phi s into f (s) ds which we wish to estremize.

It is given to us that the function k(s, t) is a continuous function with the property of symmetry that is k(s, t) is equal to k(t, s) and f(s) is the given continuous function of s, phi s is the unknown continuous function of s.

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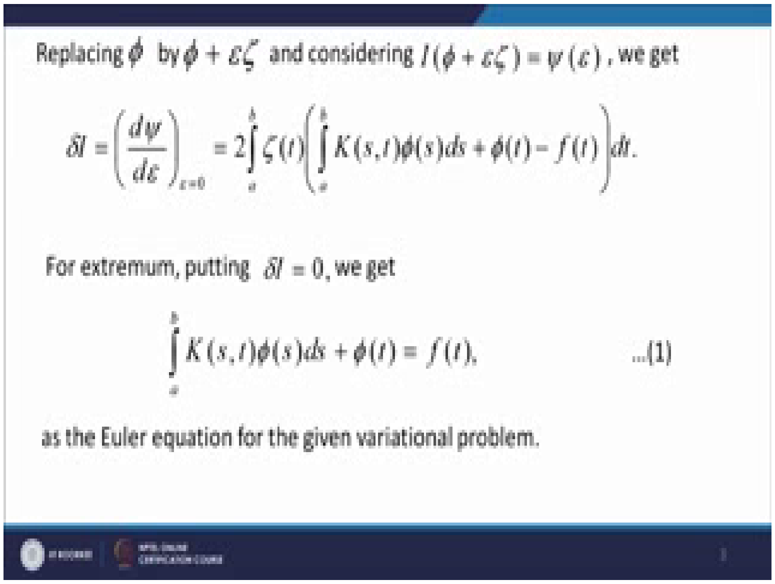

Now in order to find the variation of this functional, we replace phi by phi plus epsilon zeta

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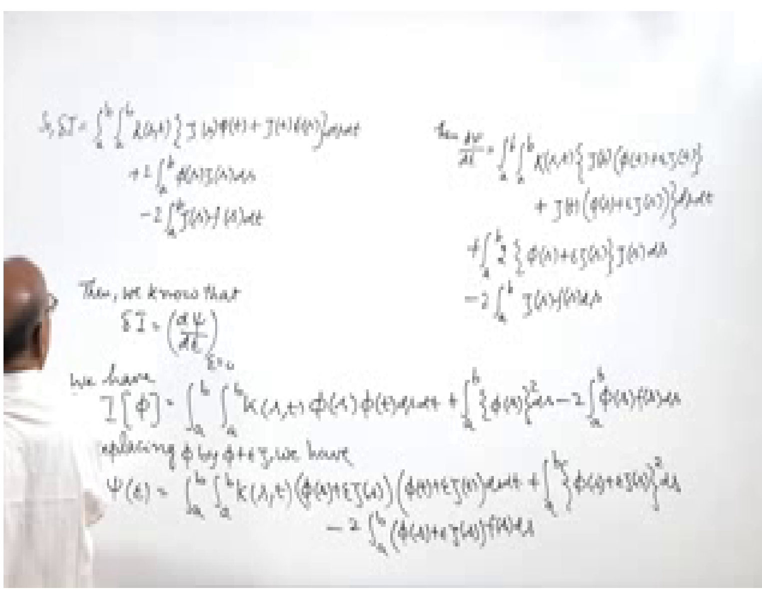

And let us consider i be replaced by i phi be replaced by i phi plus epsilon zeta and let us assume that I phi plus epsilon zeta is equal to psi epsilon, (is equal to) psi is a function of epsilon, then we know that delta I is nothing but d psi over d epsilon at epsilon is equal to zero, so let us apply this formula (al lal) let us see how we get the value of delta epsilon I.

We have I phi equal to integral over a to b, integral over (a) a to b k(s,t) phi(s) phi(t) d (s)dt plus integral over a to b phi (s) whole square d(s) and then we have minus two times integral over a to b , so in this functional let us replace phi by phi plus epsilon times zeta ok, so replacing we have psi epsilon is equal to integral a to b, integral a to b k(s,t) phi(s), phi(s) (mea) phi we are replacing by phi plus epsilon zeta.

So phi(s) will become phi(s) plus epsilon zeta s, phi (t) will be, phi(t) plus epsilon zeta(t) into d(s) dt and here we shall have phi(s) will become phi(s) plus epsilon zeta(s) minus two times integral a to b phi(s) will become into f(s) ds, now in order to determine delta e, let us differentiate psi with respect to epsilon, so then d psi over d epsilon will be k(s,t) when we differentiate (th th th) this with respect to epsilon we have here product of two functionals.

Which depend on epsilon, so we apply the formula of derivative of a product of two functionals, so first we differentiate this and when we differentiate this with respect to epsilon we get zeta (s), into phi (t) plus epsilon zeta (t) plus, now we differentiate ok second function with respect to epsilon, so zeta t into phi(s) plus epsilon zeta(s) d(s) dt and then we come to this one second, so plus integral over a to b, two times phi(s) plus epsilon zeta(s).

And then here when we differentiate with respect to epsilon, we get zeta(s), now when we differentiate here we get minus two times integral a to b, this will be when we differentiated this (with respect) with respect to epsilon we get zeta(s) into f(s)ds, now delta e is equal to d psi by d epsilon and when epsilon is zero, so (d) delta e is equal to, let us put epsilon equal to zero, so what we get integral over a to b, a to b then we get k(s,t) zeta(s) into epsilon is zero.

So phi(t) plus in the second term zeta(t) into phi(s), d(dt), and when we get two times integral a to b when we put the epsilon zero phi(s) into zeta(s)ds minus two times integral a to b zeta(s) into f(s) ds, so this is delta e, when we put d psi over d epsilon when we find d psi over d epsilon at (psi) d epsilon equal to zero, we get delta e and now this can be put further in this form , this can be written as now (you can) we can see here that this is k(s, t) zeta(s) phi(t).

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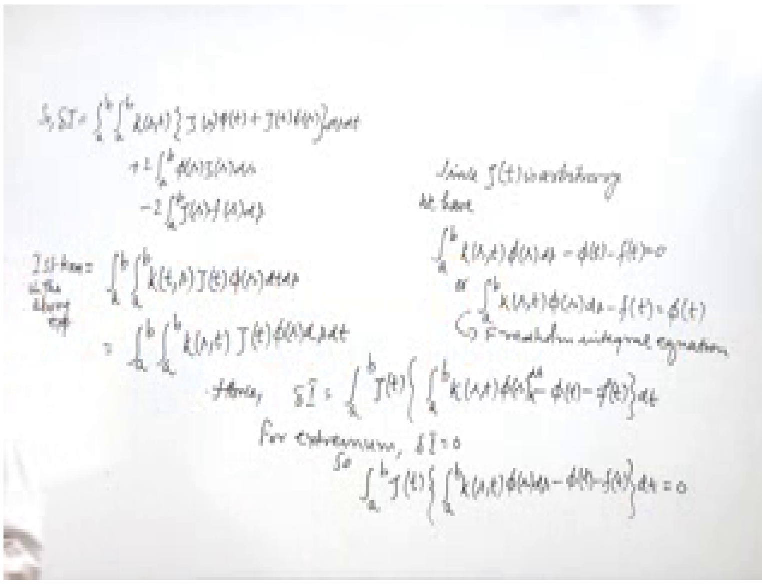

Here we have k(s, t) zeta(t) into phi(s) in the first (expre) integral if you interchange s and t the limits will remain the same, we will have integral a to b, integral a to b here we shall get k(t, s) and then here we will get zeta(t) into phi(s) and we will get dt ds when we interchange t and s, now we are assuming that k(s, t) and k(t, s) are same this is first integral, first term in the above expression, so this is nothing but integral a to b integral a to b.

k(s, t) is equal to k(t, s) so k we can write k(s, t) into zeta(t) into phi(s) also we can write dsdt, now we can write it so thus we can write delta i s , so the first term in the above expression can be written like this and hence delta e is equal to integral over a to b zeta(t) times integral over a to b, k(s, t) into phi(s) minus point t into sorry minus f(t) dt and for maximum delta e is equal to zero, (so) hence so for a maximum.

So we get integral a to b zeta(t) times integral a to b k(s, t) into minus phi(s), here it will be ds also, now this I am ready to write because this is the definite integral, here when we change the integral (they be) variables (s, t) what do we find k(s, t) as k(s, t) becomes k(t, s) but k(s, t) and k(t, s) (as) k(t, s) are same so we get k(s, t) and then zeta(t) phi(s) into ds dt, here also we get zeta(t) into phi(s) ds dt so this becomes double.

Therefore we get when we combine first term with the second term we get twice integral a to b integral a to b, k(s,t) zeta (t) into phi(s) ds dt and here these are (intra) definite integrals we can change the (variable of) variable of integration from s to t, so this will become two times a to t, k to b phi(t) zeta (t) dt and this is minus two times integral a to b zeta(t) f(s) this was ds, so zeta(t) f(t) dt, so that quantity we have written in this form.

Now for a maximum delta e is equal to zero, so we get this since delta zeta(t) is added to the, zeta(t) is arbitrary we have the condition that integral a to b k(s,t) into phi(s) ds minus phi(t) minus f(t) equal to zero or integral a to b k(s,t) into phi(s) ds minus f(t) is equal to phi(t), which is a Fredholm integral equation of (sec) second kind, now so we notice that we get this Fredholm integral equation which is the either equation for the given variational problem.

Because this equation we are getting by putting delta equal to zero, so this Fredholm integral equation, so what we want to emphasize is that the either equation for the given variational problem is nothing but the Fredholm integral equation where phi is the unknown function.

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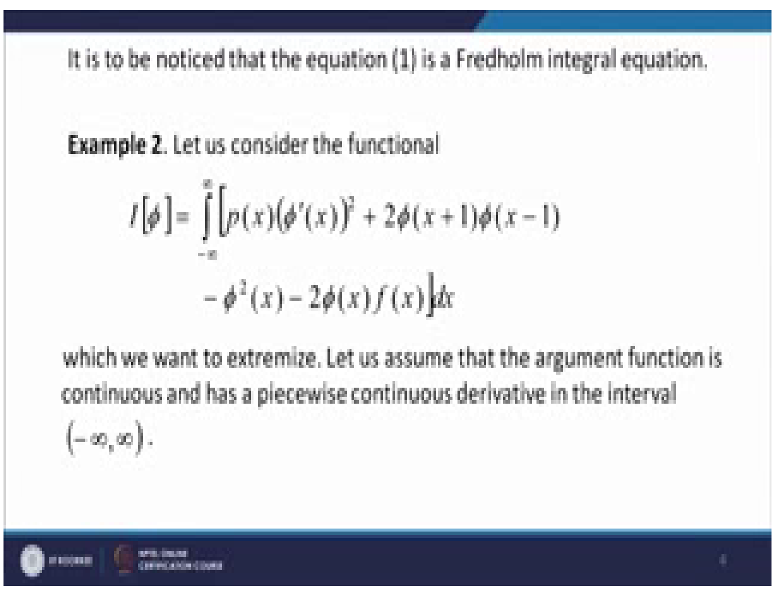

Let us know go to the second example, let us consider this functional i phi equal to integral over minus infinity to infinity p (x) into five dash x whole square plus two phi (x) plus one .

Into phi x minus one phi square x minus two phi(x) into f(x) dx which we raise to extremize, let us assume that the argument function is continuous and has a piecewise continuous derivative in the interval minus infinity to infinity,

So now see how we get the reification for this version version problem so again we will, what we will do is, we will replace phi by phi plus epsilon zeta and then we shall call i phi plus epsilon zeta as psi epsilon.

We will differentiate psi epsilon with respect to epsilon and then put epsilon equal to zero, epsilon equal to zero to get the delta e and then for an (15:45actimum b o) delta e is equal to zero, so when we put zeta delta e equal to zero and you decide that zeta(t) is an arbitrary function, we shall get the i s equation for the variational problem.

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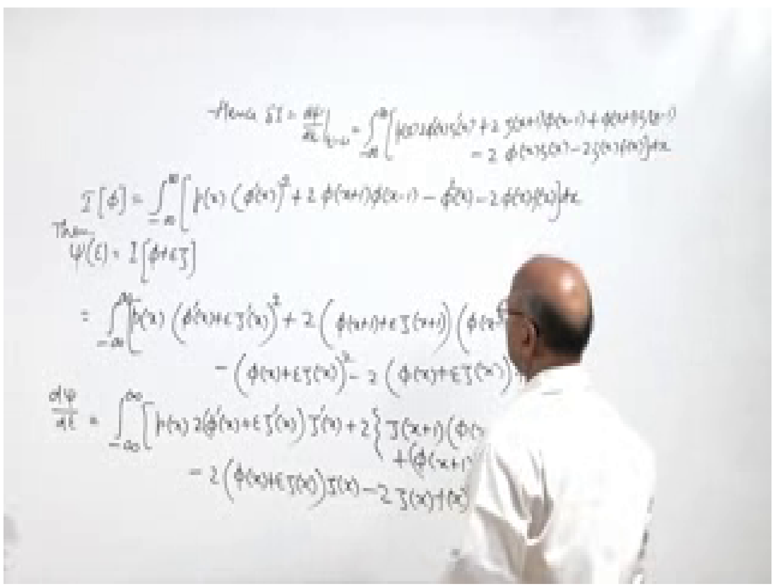

So let us see are we are given that, i phi equal to integral over minus infinity to infinity d(x) into phi dash (x) whole square. Plus two times phi(x plus1) phi (x minus 1) and then we have minus phi square x, minus two times phi(x) into f(x) ds

So let us put phi plus epsilon zeta in place of phi and call it as dull psi epsilon, psi epsilon equal to i phi plus phi dash (x) will be phi dash (x) plus epsilon zeta dash(x) whole square plus two times phi (x plus 1) will be phi (x plus 1) plus epsilon zeta (x plus 1) then phi(x minus 1) will be phi(x minus 1) plus epsilon zeta (x minus 1).

Minus phi square x will be phi(x) plus epsilon zeta x whole square and then minus two times phi(x) plus epsilon zeta x into f(x) dx, so let us differentiate psi (dis) , psi with respect to epsilon, so d psi over d epsilon will be equal to minus infinity to infinity px derivative here will be two times phi dash x plus epsilon, zeta dash x into zeta dash x plus here what we will get that the product of two functions of epsilon we have so two times phi.

When we differentiate with respect to epsilon we get zeta(x plus one) into phi(x minus one) plus epsilon zeta (x minus one) and then we differentiate the second one, so phi (x plus one) plus epsilon zeta (x plus one) into zeta (x minus 1) and then we come here so minus two times phi (x) plus epsilon zeta x into zeta x and then we come here so we get what, derivative with respect to epsilon will give you zeta(x) here.

So to minus two zeta x f(x) into dx, now let us put epsilon equal to zero here in order to get delta i, so then delta i equal to, this equal to what we will put, we will put epsilon equal to zero here, so we get minus infinity to infinity d(x) then here we get two phi dash x zeta dash x, here we will get two times zeta x plus one into phi x minus one and here we shall have, and phi (x plus 1) zeta (x minus 1) then we have minus two times epsilon equal to zero phi (x) zeta (x). And here we get minus (j) two zeta (x) f(x) dx.

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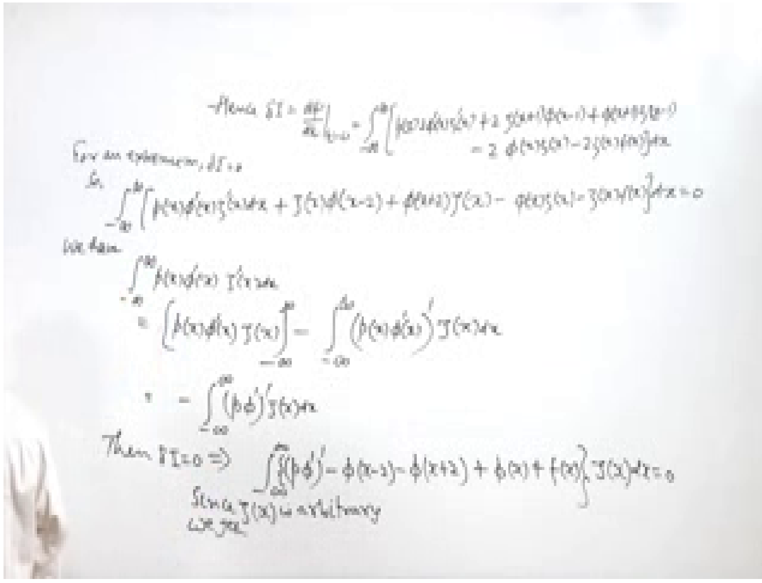

As we know for an extimum delta e equal to zero, so what we will get minus infinity to infinity we can write it as d(x) we can divide the whole equation, this is equal to zero and after that we can divide the whole equation by two so p(x) phi dash(x) into zeta dash (x) dx plus now integral over minus infinity to infinity zeta(x plus 1) phi (x minus 1).

We can put as replacing (x plus 1) by x, what we will get the limits determine minus infinity to infinity, so we will get zeta (x) phi in place of x we are putting (x minus 1) so (x minus two) plus in the second one we are putting(x minus 1) in place of x, so (phi) no in place of (x minus 1) we are putting x, so x is replaced by (x minus) (x plus 1) so phi (x plus 2) zeta (x) and then here we get minus two phi(x) zeta(x) sorry not 2, 2 is divided phi(x) zeta (x).

And then we get minus zeta(x) f(x) dx equal to zero, now we do the following , let us look at the first integral here, integrating by powers p(x) into phi dash(x) into zeta dash (x) mean we write this is equal to integration by powers, so this I call as one function, this as second function, so (minus) sorry so we have integration with first function into integral of zeta dash (x) which is zeta (x), derivative of first function so p phi dash x.

p(x) into phi dash (x) is dash, derivative of the first function into zeta(x) dx, this is equal to minus integral minus infinity to infinity p phi dash dash into zeta(x) dx, where we assume that p(x) tends to zero as x tends plus minus infinity, so this quantity goes to zero as x goes to plus minus infinity, now (let let) putting this value here ok, so so then delta e equal to zero, we imply that first term will be replaced by this ok.

So what we will have minus infinity to infinity p phi dash dash ok, we will take zeta(x) coming from all the terms so minus phi(x) minus 2 minus phi (x) plus 2 ok, plus phi(x) plus f(x) into zeta(x) we get this when we substitute this value here ok, multiply the whole equation by minus 1 we get this,

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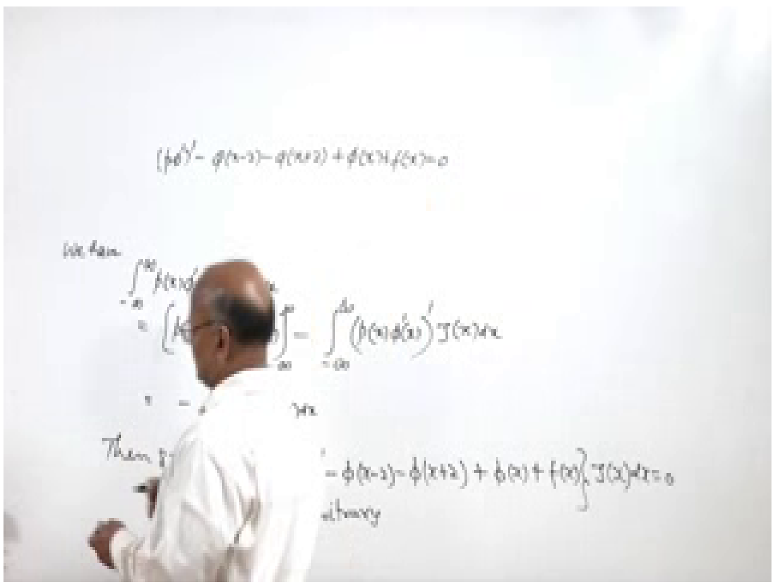

Now see zeta(x) is again, since zeta(x) is arbitrary we get p phi dash dash minus phi(x) minus 2. This equation which is the Euler equation for this problem and we noticed that this equation is nothing but the (differential) difference equation for the argument function phi(x).

So such variational problems are often encountered in practice where we in the first case we have seen that we get the Euler equation as the Fredholm integral equation of the (ff) second kind and in the second case we get a differential difference equation for the unknown function phi(x),

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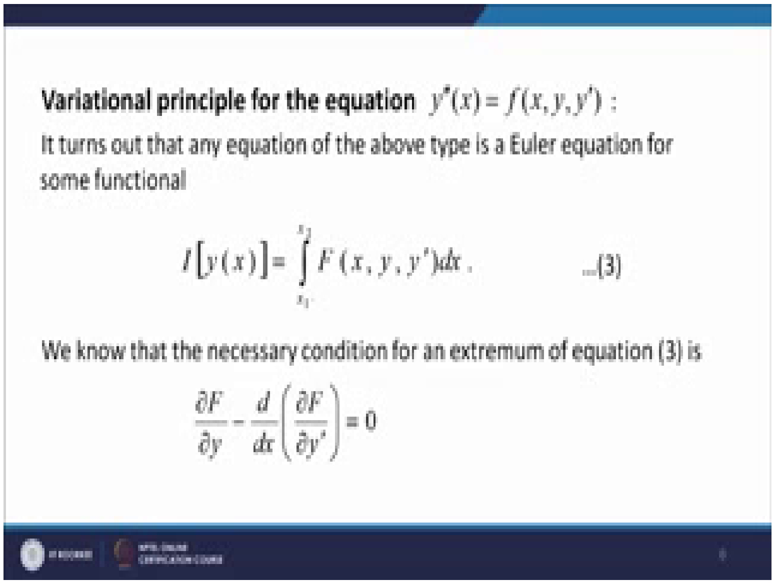

Now let us also consider, next we consider variational principle for the equation by double dash x equal to f(x) y by dash ok, so let's consider the variational principle for the equation by double dash equal to f(x) by by dash,

What we (see) noticed here is that any equation of this type is a either equation for some functional i by x equal to integral x one to x two f(x) by by dash d(x) how we will show this.

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Let us (let us notice) let us remember that the nursery condition for an extimum of the equation i by x equal to integral x one to x two f(x) by y dash is delta four delta y minus d over d(x) of,

This we know already that if we have a functional like this then in nursery condition for an extimum of this equation is this,

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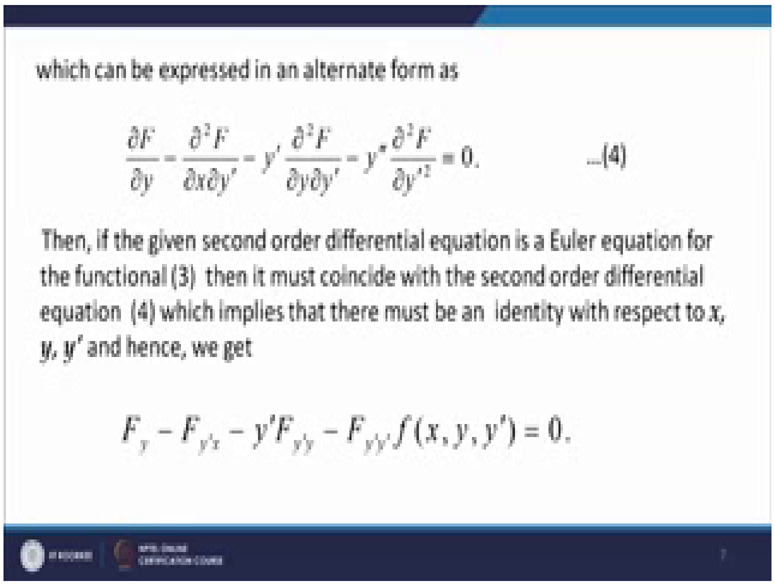

Now if this (will) either equation can be pretend in a alternate form as delta f over delta y minus delta square f over delta x delta y dash minus y dash into delta square f by delta y delta y dash minus y double dash into delta square f over delta y dash square equal to zero.

So if the given second order differential equation is a either equation for the functional given by three then it must coincide with the second order differential equation which is the equation number four, which implies that there must be an identity with respect to x y n y dash.

So what we will get, the equation four will then become f y minus f y dash x minus y dash times f y dash y minus f y dash y dash and then this , y double dash we shall replace by f(x,y,y dash) this is given to us y double dash x equal to f(x) by y dash,

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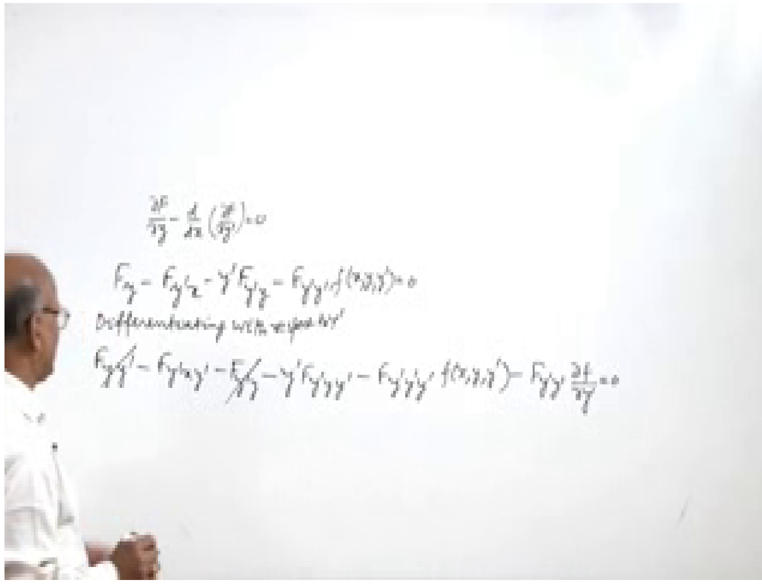

So this will be replaced here and then what we will do is let us differentiate this equation with respect to y dash, so we have f y minus f y dash x minus f y dash times f y dash y and then we have f y y y dash y dash into f(x) y y dash equal to zero F y minus f y dash (x) minus y dash times y dash f y dash y minus f y dash y dash f(x) y y dash equal to zero,

Now this equation we differentiate with respect to y dash, so what we will get, now differentiating with respect to y dash, what we get f y y dash minus f y dash x y dash minus derivative with respect to y dash will give you the derivative of y dash with respect to y dash is one.

So f y dash y minus y minus y dash times f y dash y y dash and then we get f y dash y dash y dash f(x) y y dash minus f y dash y dash into delta f over delta y dash equal to zero, now f y y dash is same as f y dash y assuming the continuity of second order with partial derivative here, so this will cancel with this and then what we get ,

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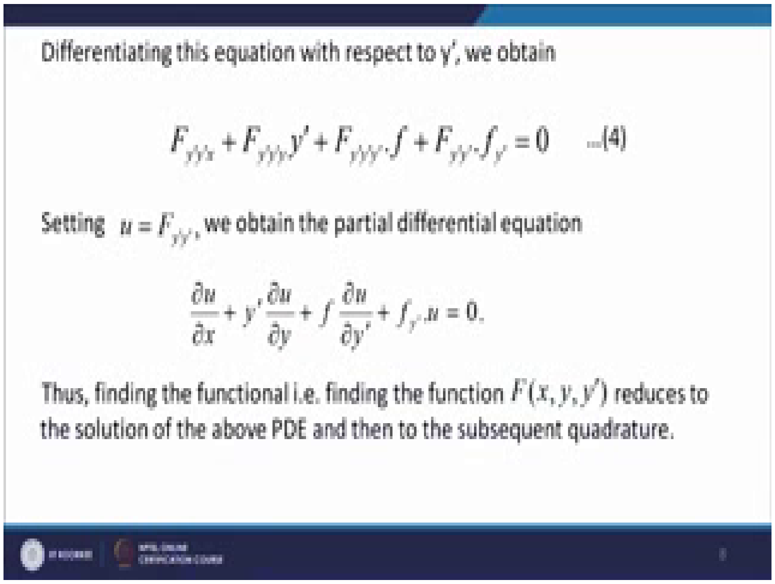

We will get the equation as f y minus f y dash x minus y dash f y dash y minus f y dash y dash. Into f(x) y y dash (with respe) equal to zero with respect to y dash, we arrive at the equation this which is nothing but f y dash y dash (x) plus f y dash y dash y into y dash plus f y dash y dash y dash into f plus f y dash y dash into f y dash equal to zero.

Now let us set f y dash y dash to be equal to u then this equation will give you partial derivative of u, first term will give partial derivative of u with respect to x.

Plus y dash times partial derivative of u with respect to y plus f times partial derivative of u with respect to y dash plus f y dash into u equal to zero, so thus in order to find the functional that is the function f(x) by y dash we have to solve the above PDE for the function u and then we have to do the subsequent quadrature because once u is known the functional f can be found from the equation u equal to f y dash y dash.

With that I would like to conclude this lecture thank you very much for your attention.