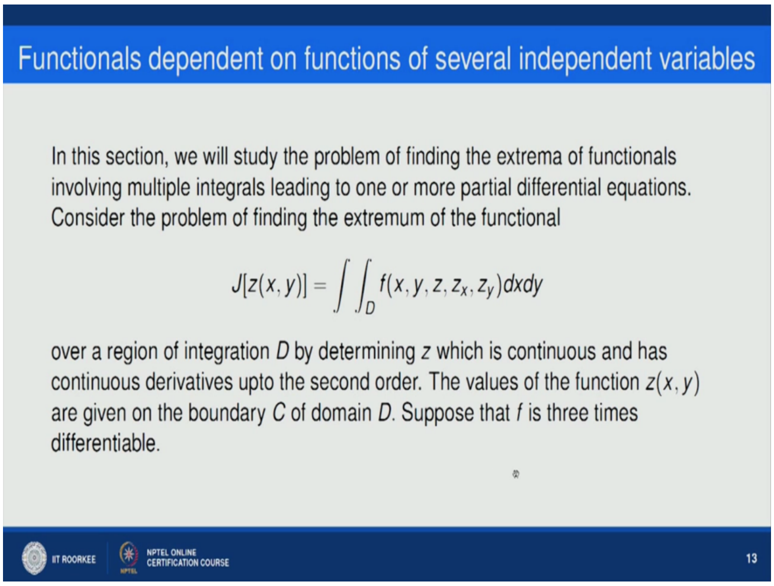
Integral Equations, Calculus of Variations and their Applications
Dr. D. N. Pandey
Department of Mathematics
Indian Institute of Technology Roorkee
Lecture 47
Functions of Several Independent Variables

Hello friends, I welcome to on this lecture and here,if you recall, we have discussed thefunctional and the curves, which minimizes the functional andnow in this lecture what we try to do here, weconsider the functional which depend on thesurfaces and here we want to find out theequationin terms ofsurfaces, which minimizes the functional. So herewe want to find out say extremal which are surfaces which minimizes the given function.

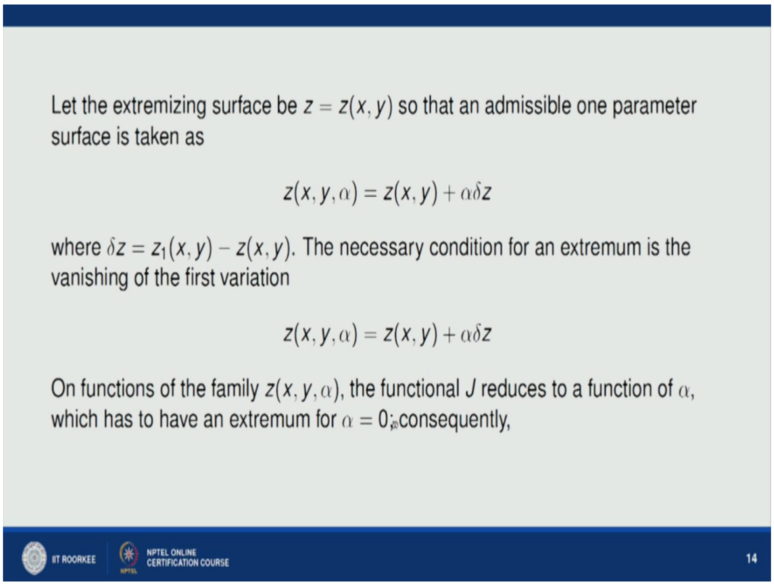
(Refer Slide Time: 1:00)



So let usdiscuss here. So in this lecture, we will study the problem of finding the extrema of functional involving multiple integrals leading to one or more partial differential equation and we can consider our functional as this, J of z of (x, y)which is double integral f(x, y, z, z of x, zy) over a region of integration D by determining z, which is continuous and has continuous derivative upto the second order. So here we want to minimize thisfunctional. So hereplease remember here the minimizer of this functional is your z of (x, y) which isnot a curve, but a surface right. So if so far we have discussed the cases where the minimizer of a functional is only a curve, but here we want tofind out say minimizer or say extremizer of a functional which is nothing but asurface in place of a curve.

So here this D represent thedomain on which the surfaces defined here andwe say that this domain D is bounded by the boundary C andthe value of z on the boundary C is given by some initial condition. In this case we can call this as boundary condition and here (we have) we assume that this ofintegrant f isat least 2 timesdifferentiable or we can without (lo) without any problem we simply assume that f is 3 times differentiable, but I think only 2 time differentiability is required here.

(Refer Slide Time: 2:45)



Soherewhat we want to know. We want to find out say Euler kind of conditions. So herewe proceed in a same way as we have proceeded for thecurves when the extremals are nothing but curves. So here also let usassume that the extremizing surface is z equal to z of (x, y). So asyou rememberwe have already seen that with respect to this extremizing surface we define a one parameter family of surfaces as a z of (x, y, alpha) equal to z of(x,y) plus alpha delta z where, this delta z is a variationin with respect to z here.

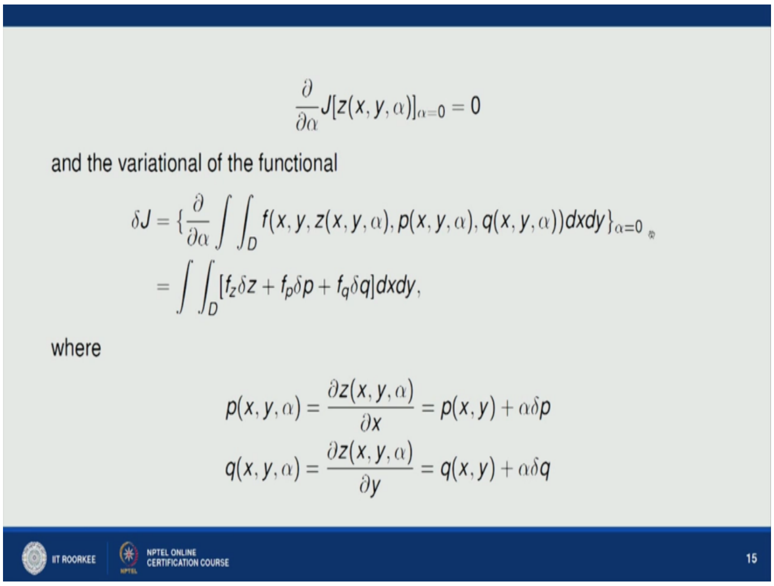
So let us take a z1(x, y) is another surfaces which is arbitrary, but fixed, arbitrary in the sense that you can take any surface having this same boundary condition as z as a boundary condition and fixed in the sense that you can take anysurfaces, but once you take this surface then it is fixed. So here delta z is nothing but z1(x, y) x, y minus z(x, y). Here we may also call thisz1(x,y) as a admissible surfacefor the extremalJ, right. So here z1 and z both satisfy this same boundary condition on the boundary of the domain (D) C.

So herewe want to **as** we have pointed out that thecondition that the for the extremal on the functional, the variation has to vanish andthatthat theorem is not depending on theform on the functional. So if it is true forthe functional for whichminimizer is or extremizer is your curve. So it is also true for this case also where the extremizer isnothing but surfaces. So here the necessary condition for an extremum is thevanishing of the first variations.

So first variation here is z of (x, y, alpha) equal to z of (x, y) plus alpha delta z. Here to be noted here that by suitablysuitable choice of alpha and z1, you canfind out any surfacehaving the same boundary condition,between same boundary condition on the boundary C by suitably choice of z1 and alpha. So z(x, y, alpha) is given by this we will represent any arbitrary surface having the same boundary condition and not only this if you look at your alpha equal to 0, then thisthis one parameter family is reduces to z of (x, y), which is the which is extremizer of the functional here.

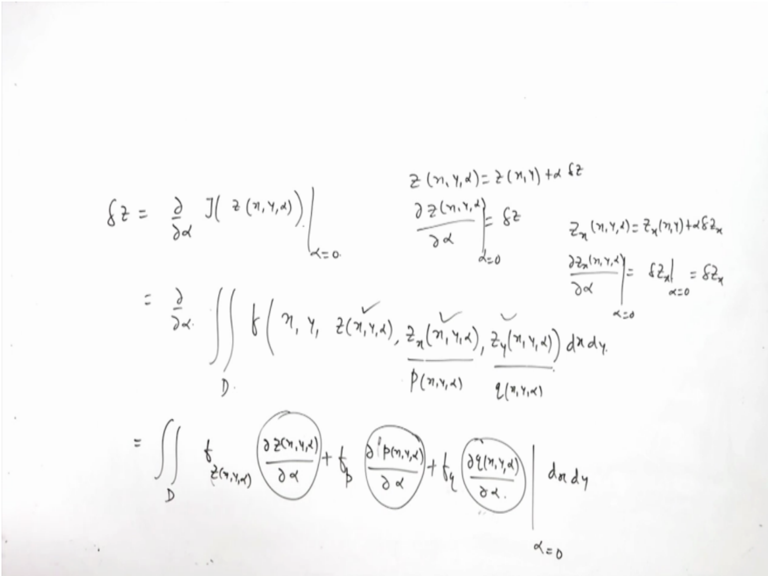
So here we can say that on function of the family z of (x, y, alpha) the functional z reduces to a fuction of alpha. So heredelta z is fixed z (())(6:05) by x isz(x, y) is already given as extremizing surface, delta z is fixed becausez1 is arbitrary, but once it is taken it is fixed. So here delta z is also fixed. So the only varying quantity is your alpha. So you can say that z(x, y, alpha) is nothing but a function of alpha and hence your functional is reduces to a function of a alpha.So here in the same way we can say thatthe extremum will occur at alpha equal to 0.

(Refer Slide Time: 6:39)



And hence,you can say thatdabba by dabbaalpha J of z(x, y)x, y, alpha is equal to 0 at alpha equal to 0. Soto find out this (we)this is this we denote as dabba J. So dabba J is equal todabba by dabba alphaand you write down thefunctional (z) J of z(x, y, alpha), so it means that define your J as for thisone parameter family of surfaces z of x, y, alpha. So here you can write J of this as f of x,y in place of z(x, y), now I am writing z(x, y, alpha) and here zxit can written asnotationally you can write it p(x, y, alpha) and zy is written asq of (x, y, alpha) and we want to find out say derivative of this with respect to alpha at alpha equal to 0.

(Refer Slide Time: 7:42)

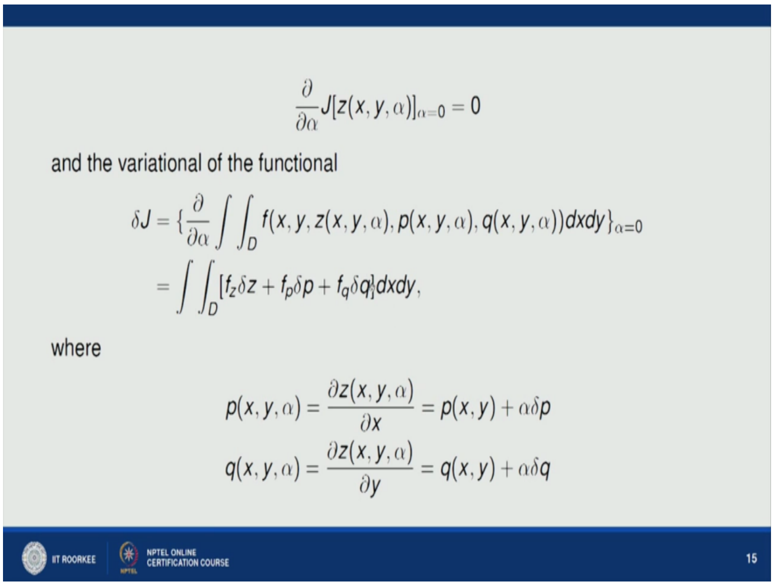


So let me write it here, you del here delta z is equal to dabba by dabba alpha and now J ofx j ofz of (x, y) here you can writez (x, y, alpha). So here you can write it at alpha equal to 0. So here you can write it dabba by dabba alpha and this isdouble derivative on D and here, you can write it f, so x, y is already z(x, y, alpha) and z of x. So herewhen you write a z of x(x, y, alpha) and z of y (x, y, alpha) and that is respect tothis integral is with respect todx and dy.

So herethis we denote as p(x, y, alpha) and this we denote as q of (x, y, alpha). So when you find out say,derivative with respect to alpha then you will get, this is as D andhere f(x, y). Now here this is what this is (independ) depending on alpha. Thisdepending on alpha. So here this, so by using theformula of derivative you can simply say that it isdabba by dabba alpha of this quantity is given as f with respect to z (x, y, alpha) and dabba of z (x,) z(x, y, alpha) dabbaz(x, y, alpha) dabba alpha plus f of p here and dabba p(x, y, alpha) upon dabba alpha at f of q and dabba q(x, y, alpha) dabba alpha right and here, this is dx and dy and at alpha equal to 0.

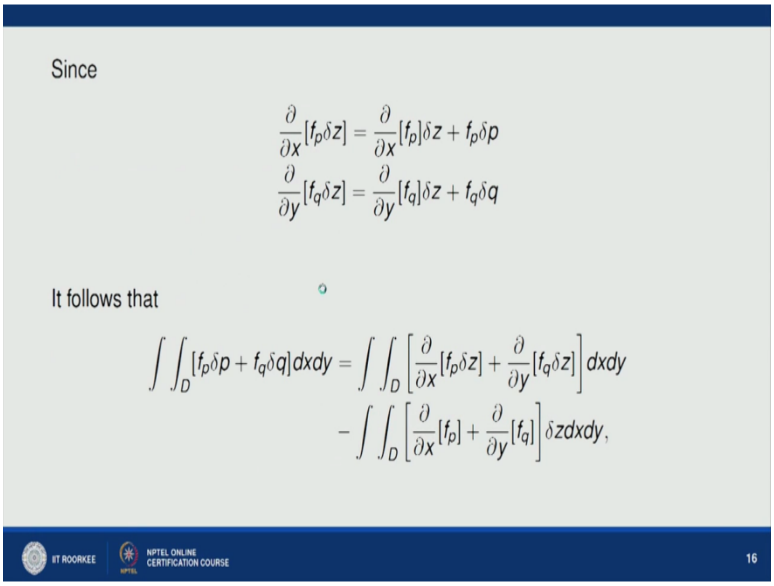
So hereif youwrite down what is z of (x, y, alpha). So x, y, alpha is equal to z x of y plus alpha delta z, right. Soalpha delta this is delta z. So when you find out say, derivative of this z(x, y, alpha) with respect to alpha and at put alpha equal to 0, you will getthat dabba z(x, y, alpha) upon dabba alpha. It is coming out to be delta z. So at alpha equal to 0, also it is coming out to be delta z. So this reduces to delta z and similarly when you find out say derivative of this with respect to dabba alpha, you will see what when you differentiate with respect to x you will get what, itzx (x, y, alpha) equal to zx (x, y) plus alphadeltazx and when you differentiate with respect to alpha, you will get dabba by dabba zx (x, y, alpha) dabba alpha is equal todelta z of x here. So at alpha equal to 0 also you will get the same thing, because it is not involving any alpha. So it is your delta z of x, so which we call this as delta p here.So similarly you can calculatethe derivative here at alpha equal to 0 and it is coming out to be delta q here.

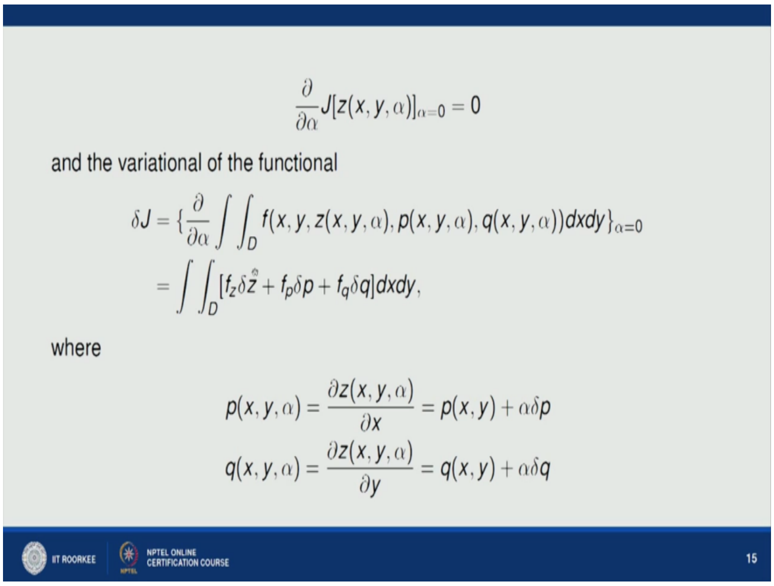
(Refer Slide Time: 11:38)



So let me write it here, so here we can write it,the variation of the functional you can write it fzdelta z plus fp delta p plus fq delta q dx dy, where delta pand fthis p(x, y, alpha) is related by this. So p(x, y, alpha) is equal to dabba z (x, y, alpha) with respect to dabba of x that we have already defined and it is coming out to be p(x, y) plus alpha delta p. Similarly you can define q(x, y, alpha), which is the derivative ofz(x, y, alpha) with respect to y and it is given as q(x,y) plus alpha delta q. So when you differentiate with respect to alpha youwill get here as delta p and delta q.

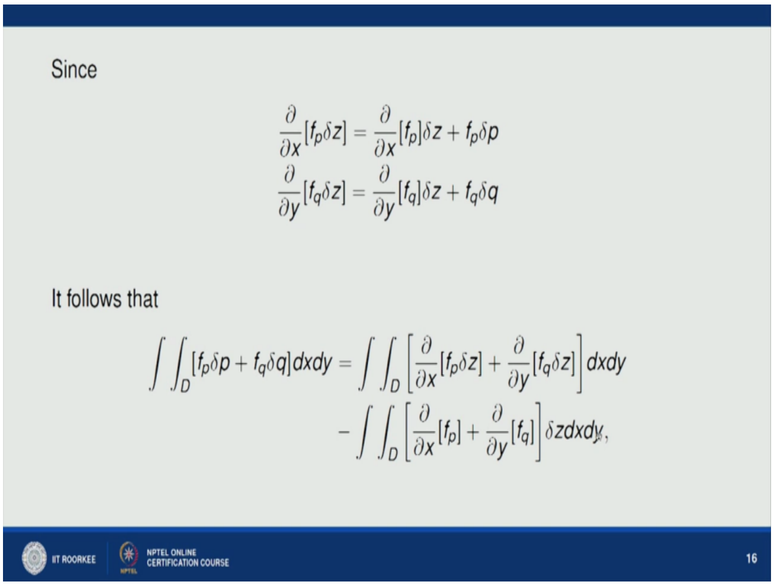
(Refer Slide Time: 12:27)





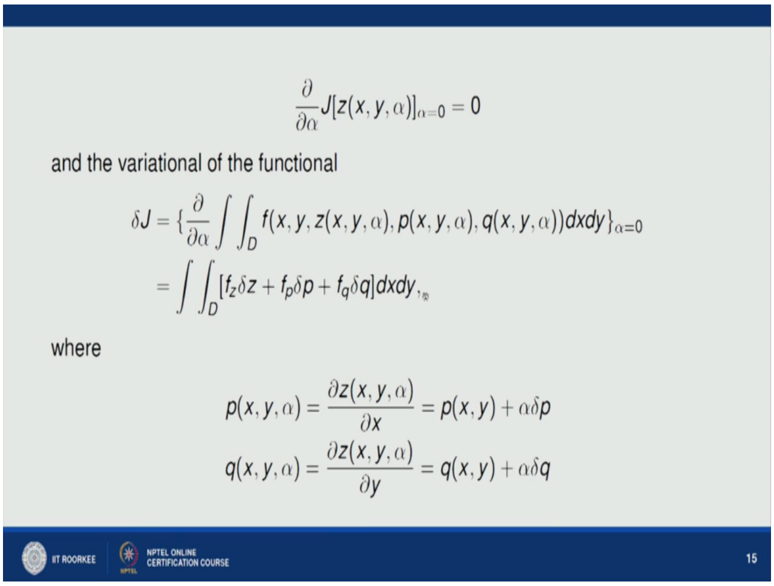
So now, so delta j is equal to 0 means thisdouble integralis going out to be 0 (())(12:33). Now the problem is that wedo not want this term delta p and delta q, because you want everything in terms ofz only. So what we try to do here is, then we try to do simplify these two terms.

(Refer Slide Time: 12:51)



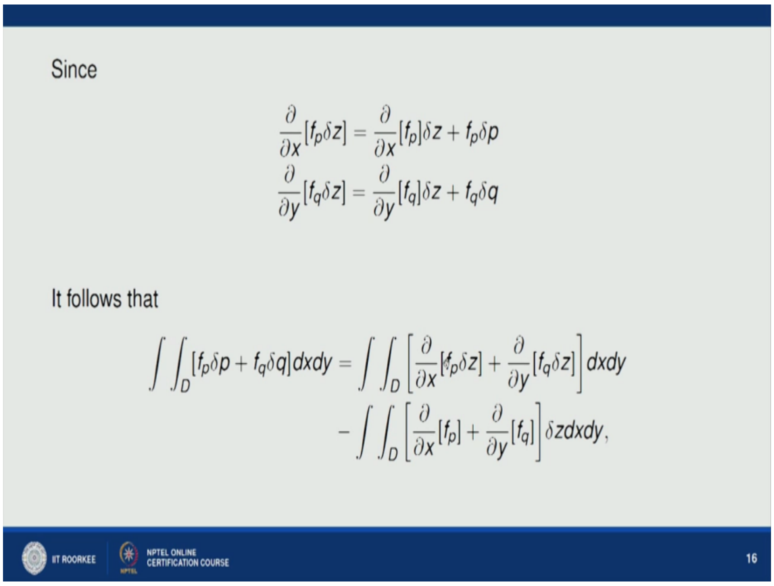
And forsimplifying these two terms, here we use this relation that dabba by dabba x of fp delta z is equal to dabba by dabba x of fp delta z plus fp dabba z byand delta z by dabba x.so here you can say using the chain rulething you can write it like this dabba by dabba x of fp delta z is equal to dabba by dabba x fp delta z plus fp delta p.Similarly, you can definedabba by dabba y of fq delta z, which is given by this. What we want isthis thing that integral of fp delta p plus fq delta q, which is here. So here you can say that the value of fp delta p plus fq delta q is equal to this summation minus this summation. So you can write it that integraldouble integral fp delta p plus fq delta q dx dy is given as double integral dabba by dabba x of fp delta z plus dabba by dabba y fq delta z that issummation of this minus summation of this, that is minus dabba by dabba x of fp plus dabba by dabba y of fq delta z dx dy.

(Refer Slide Time: 14:04)



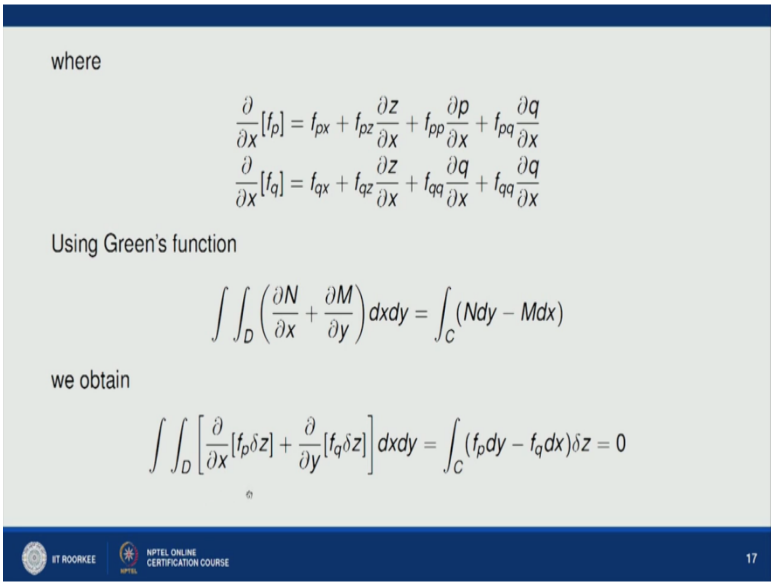
So herewe can simplify or we can say that in this integral which is coming out to be 0. We are trying to find out say, value of this integral in terms of delta z delta xdelta z dx and dy.

(Refer Slide Time: 14:16)



So herethis is coming out to be thisdifference of these two integral and forsimplifying these two integral,let usapplysaygreens theorem for uh the first integral.

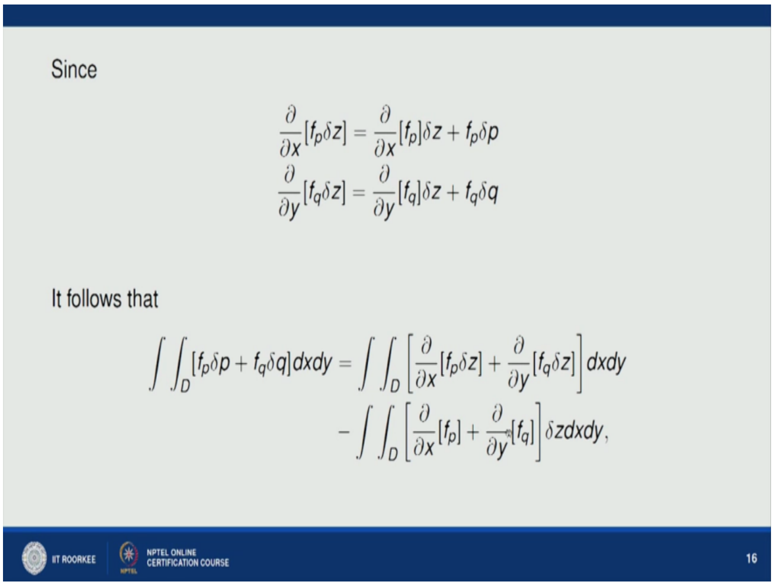
(Refer Slide Time: 14:31)



So if you remember the greenfunction theorem isgreens theorem isdouble derivative dabba N xplus dabba N by dabba x plus dabba M by dabba y dx dy is equal to C Ndy minus Mdx. So here D is the domain bounded by the curve C. So it means that herethe integral on the domain isreduce to the integral on the boundary of this. So here if you recallthen, here N is going to be fp delta z and M is going to be fq delta z. So using thisgreens theorem, you can write itN is fp delta z. So here fp delta z dy minus fq delta z d of x.

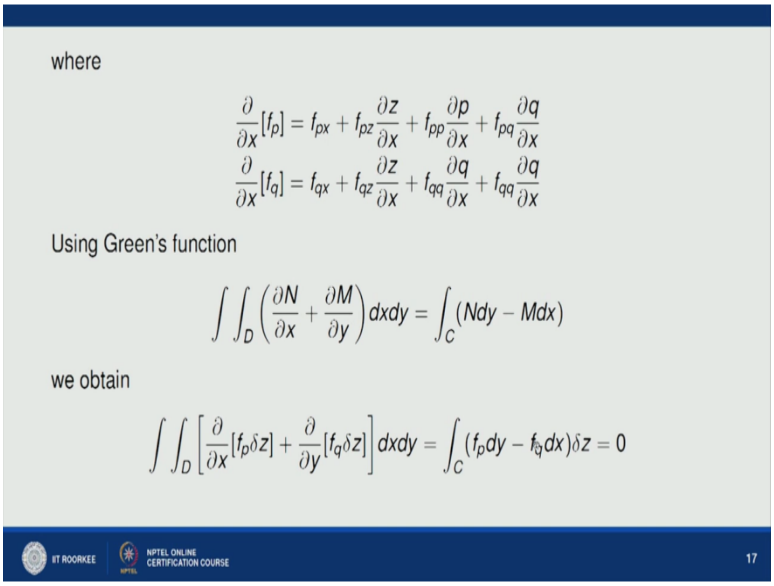
So now this is coming out to be 0. Why becausedelta z is what? Delta z is the difference ofz1(x, y) minusz1(x, y) and x z of (x, y) and both z1(x, y) and z(x, y) satisfy the same boundary condition on the boundary C. So it means that the delta z, which is a difference between z1 and zis satisfying the(z) delta z has to be 0 on the boundary. So using this fact you can say that this integral is coming out to be 0, which is nothing butthis double integral.

(Refer Slide Time: 15:55)



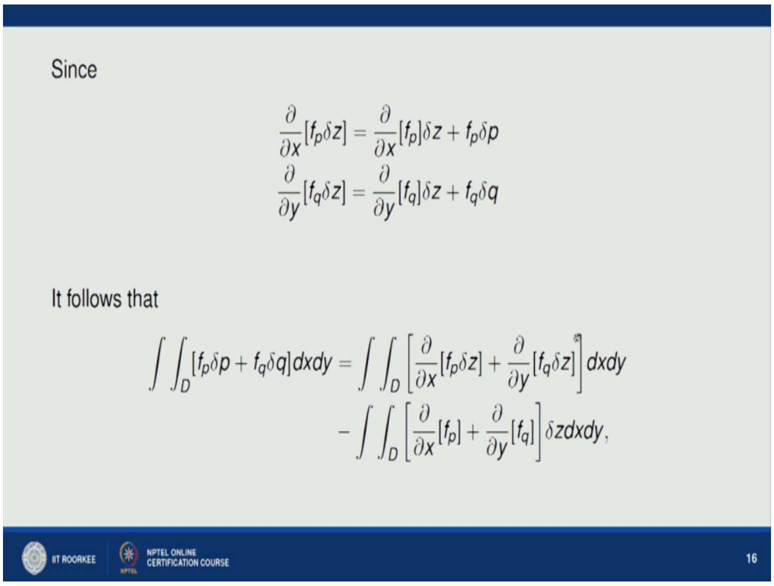
So it means that the first term in this case in this is coming out to be 0, I cannot apply the same theorem for the second integral.

(Refer Slide Time: 16:08)



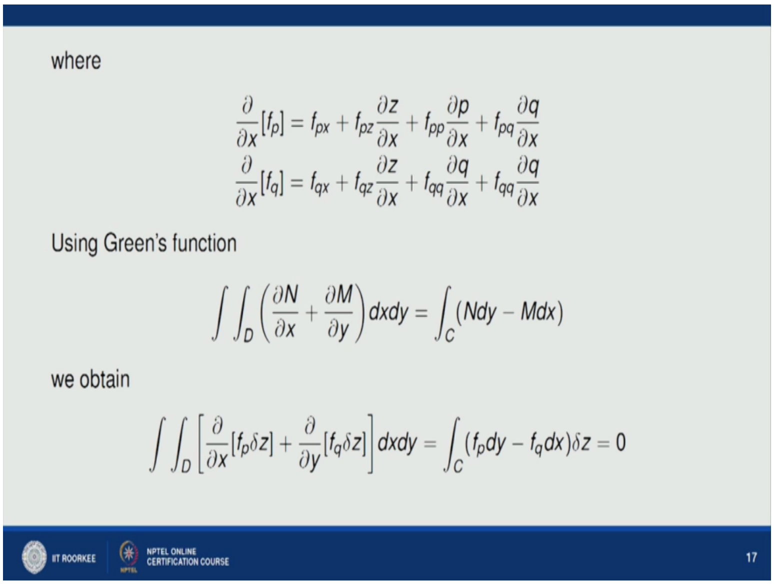
Because here this will reduce toyourfp dy minus fq dx, butdelta z is notthere.

(Refer Slide Time: 16:13)



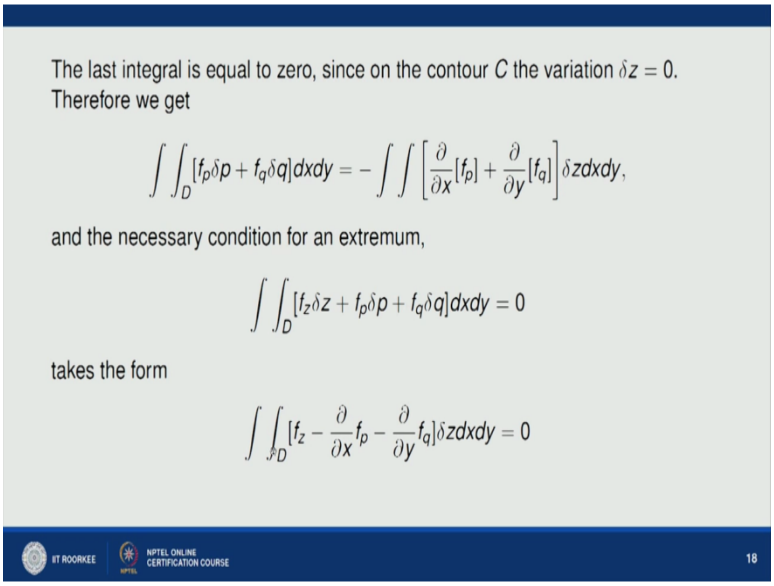
So here wemay not applythese thing.

(Refer Slide Time: 16:23)



So herewe simply say that first integral is coming out to be 0.

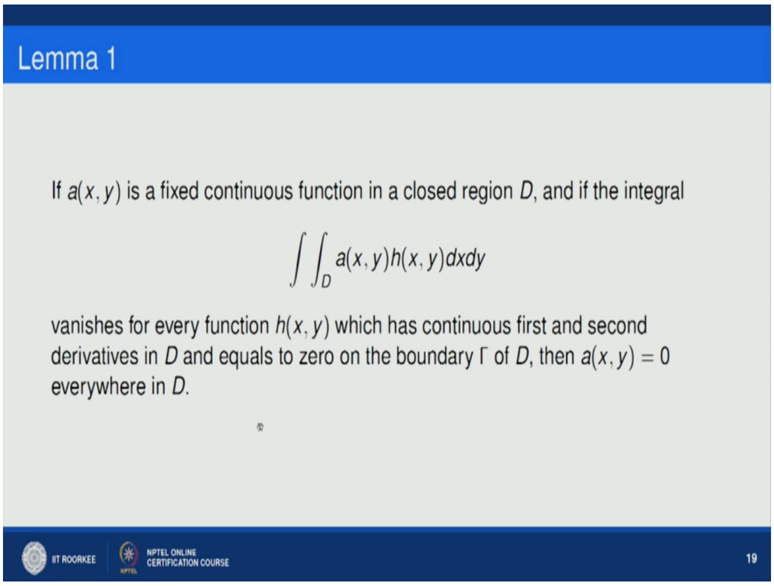
Refer Slide Time: 16:26)



And using this we cansimplify ourintegral double integral fp delta p plus fq delta q dx dy is equal to integral on D dabba by dabba x of fp plus dabba by dabba y fq delta z dx dy. So herethe equation that total variation is 0 is reduce to this integral that double integral fz delta z plus fp delta p plus fq delta q dx dyis now reduced tothis form. So here fp delta p thishere we are utilizing this equality. So using this equality, you can write that thesecond and third term can be replaced by this. So here you can say that double integral fz minus dabba by dabba x of fp minus dabba by dabba y fq delta z dx dy is equal to 0.

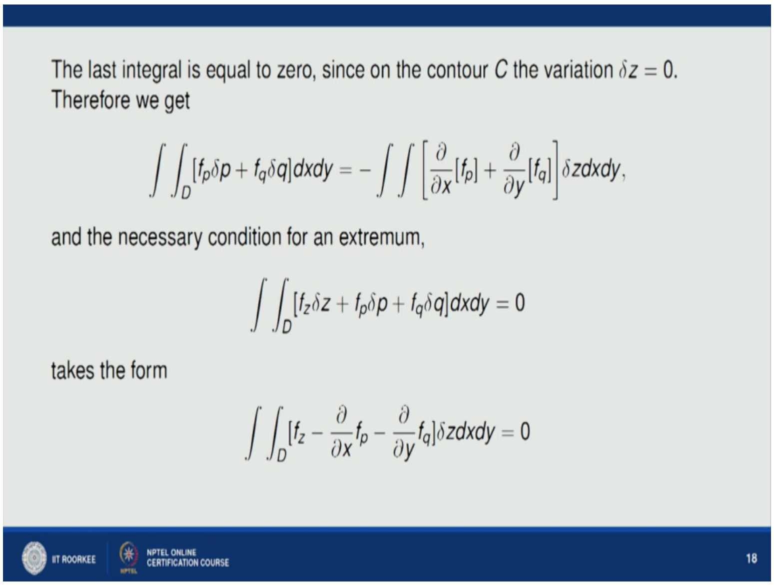
So here if you look at, sincewe have assumed that f isthrice differentiable or you can say f issecond derivative of f is continuous. So you can say that that this is a continuous function. Now look at this part delta z, so delta z is basically acontinuous function in fact, it is alsodifferentiable function. So here we can say that delta z isequal to 0 on the boundary andhaving continuoushaving first and second order derivativecontinuous on theon the domain D. So here to get some information about quantity we use the generalization oflama (()) (18:03), which we have discussed forin the case of curves.

(Refer Slide Time: 18:04)



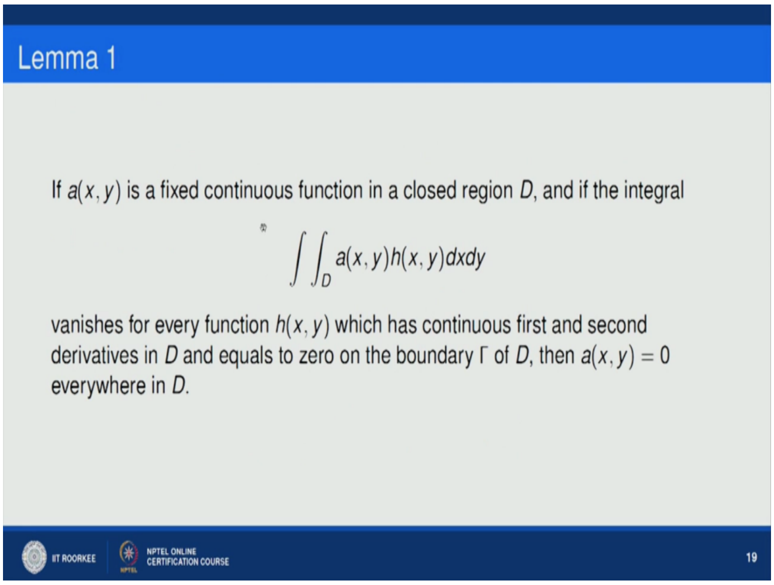
So here let us consider what is this lama. It says that if a(x, y) is a fixed continuous function in a closed region D and if this integral, this is a one integral a(x, y) h(x, y) dx dy is equal to 0 means vanishes for every function h(x, y), which has continuous first and second order derivative in domain D and equals to 0 on the boundary, then a(x, y) is equal to 0 everywhere in D.if you remember if you recall here h(x,y) is your delta z.

(Refer Slide Time: 18:39)



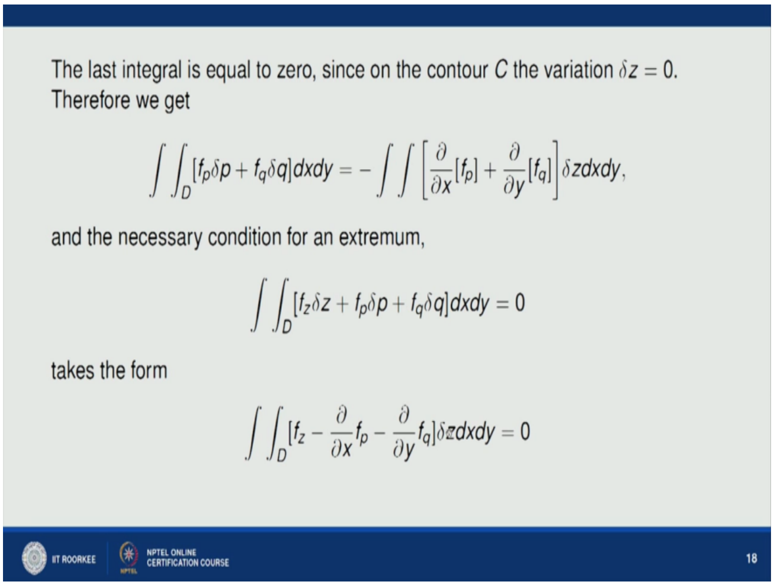
So here delta z isvanishing on the boundary, because it isyour, it is a difference of two admissible curve z1(x, y) and the extremal surface is at (x, y) who satisfy the same boundary condition. So it delta z is going to vanish on the boundary and it is arbitrary in the sense, because z1 is arbitrary surfacein the neighborhood ofz.

(Refer Slide Time: 19:02)



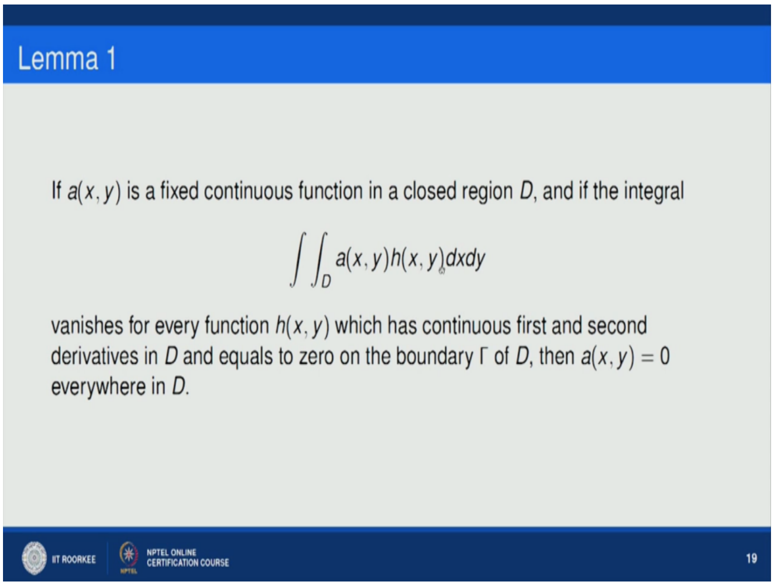
So you can say that if wetake this lama as true.

(Refer Slide Time: 19:06)



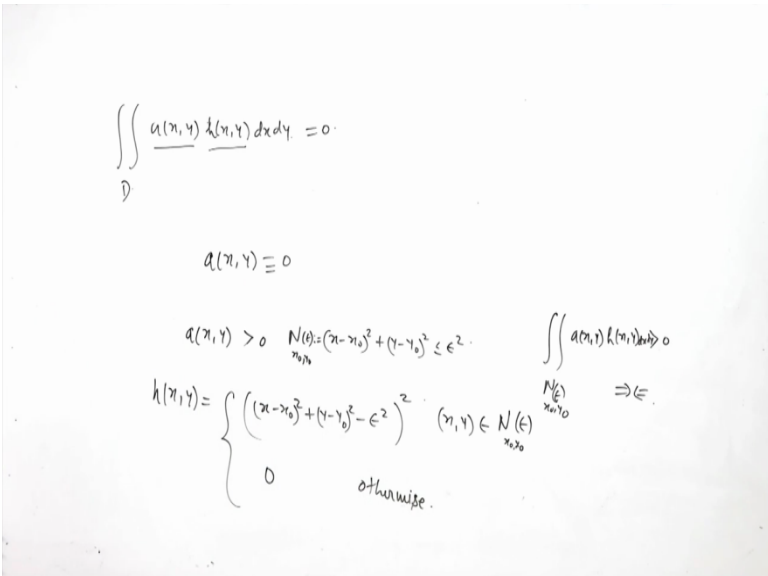
Then we can say that the a(x, y) which is this quantity has to be 0 on this.

(Refer Slide Time: 19:12)



Sofor thisto prove this we may generalize the concept produce in previous lectures.

(Refer Slide Time: 19:24)



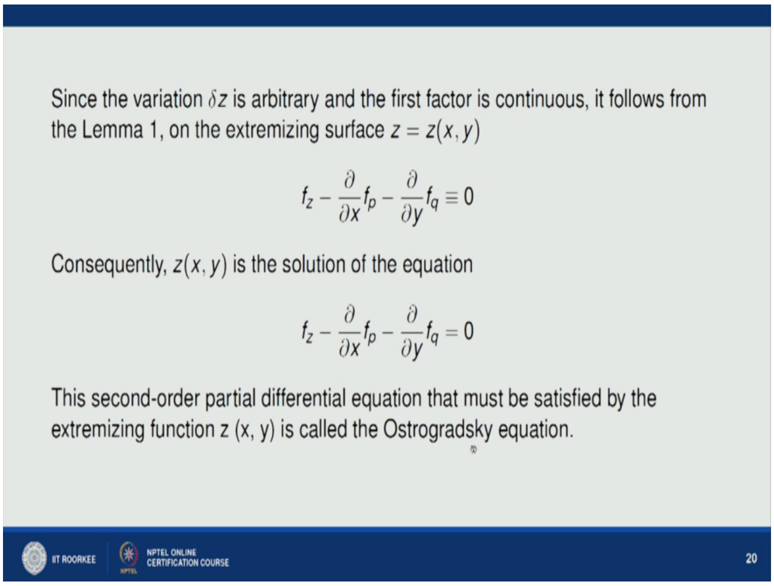
So here our claim (())(19:25) is that if a(x, y) is a fixed continuous function and h is thefunction, (which) whose first and second order derivative iscontinuous and(on a) on the domain d and it vanishes on the boundary of the domain D, then this a(x, y) is coming out to be(())(19:44) equal to 0 function. So claim is at a(x, y) is (())(19:48) equal to 0. So heresuppose it is not true that a(x, y) isnon-zero function. So it is takingsay non-zero value.

So let us assume without loss of the generality (())(19:59) that a(x, y) is say positive on let us see strictly positive on some kind of a small space. So we can write itdomain is as x minus x not whole square plus y minus y not whole square less than or equal to say epsilon square. So here what we can take here, we can takeclose ball around x not y not with the radius epsilon and we say that your a(x, y) is taking the positive value on this small domain, inand epsilon is small enough such that this domain is lying inside your D.

Now we can choseour h(x, y) in a way such that, you can write it h(x, y) as x minus x not whole square plus y minus y not whole square minus epsilon square and whole square. So by taking this, you can take h(x, y) as this when (x, y) lying inside your domain, let us call this as domain as x not y not with radius epsilon. So you can define in domain like this.So when (x, y) belongs to this domain and epsilon x not y not then your h(x, y) is this otherwise 0.

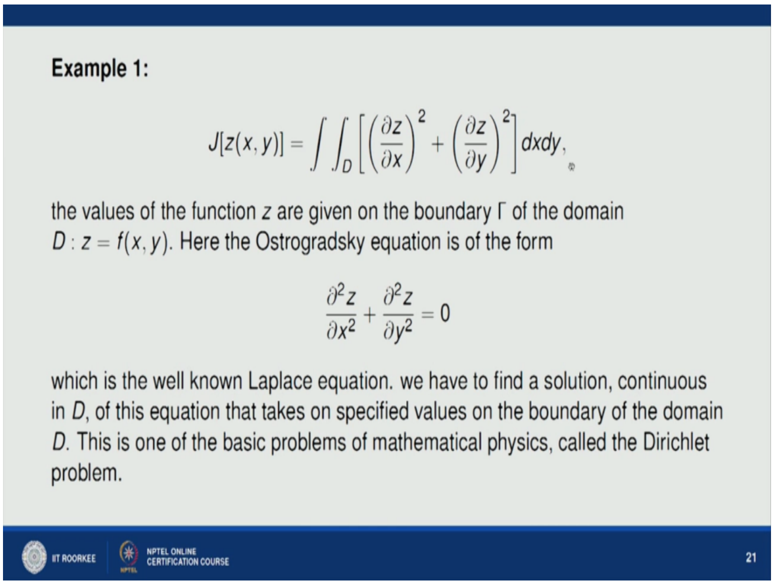
So in this case if we take h(x, y) like this then h(x, y) will be 0 on the boundary and it satisfy thecondition that the first and second order derivative iscontinuous.So herethe h(x, y) satisfy all the assumption on the lama and also that the h(x, y) and product a(x, y) both product is going to be positive on this small neighborhoodaround x not y not with the radius epsilon and in this case, if you find out the integral here, then integral on this N epsilonn epsilon x not y not is going to be positive, dx dy is going to be positive. So this is a quantradiction here, because uh we have assumed that thisintegral is nothing but 0. So it means thatthe assumption that a(x, y) ispositive in some domain is not correct and hence, your a(x, y) has to be 0.

(Refer Slide Time: 22:41)



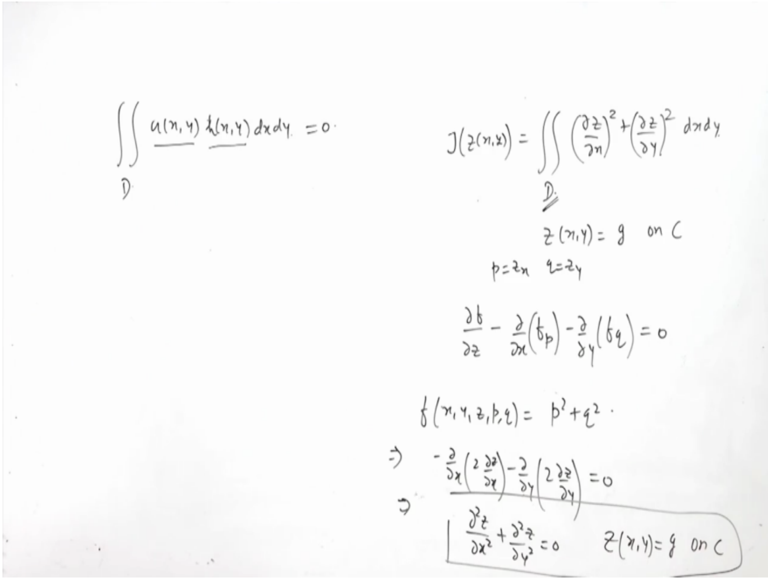
So usingthis lama, you can say that since the variation delta z is arbitrary and the first factor is continuous. It follows from thislama that the on the extremizing surface z equal to z(x, y). This equation which issecond order partial differential equation ishas to be 0. So fz minus dabba by dabba x fp minus dabba by dabba y fq is (())(22:57)equal to 0 andwith the, so it means that the extremizing surface will be a solution of this equation along with the boundary conditionwhich z equal to z(x, y) takes on the boundary C. So tis second order partial differential equation that must be satisfy by the extremizing function z(x, y) is call the Ostrogradsky equation.

(Refer Slide Time: 23:28)



So let us discussexample based on this. So first example is the meanminimize the extremal,J of z(x, y), which is given by this double integraldabba z by dabba x whole square plus dabba z by dabba y whole square dx dy.

(Refer Slide Time: 23:45)

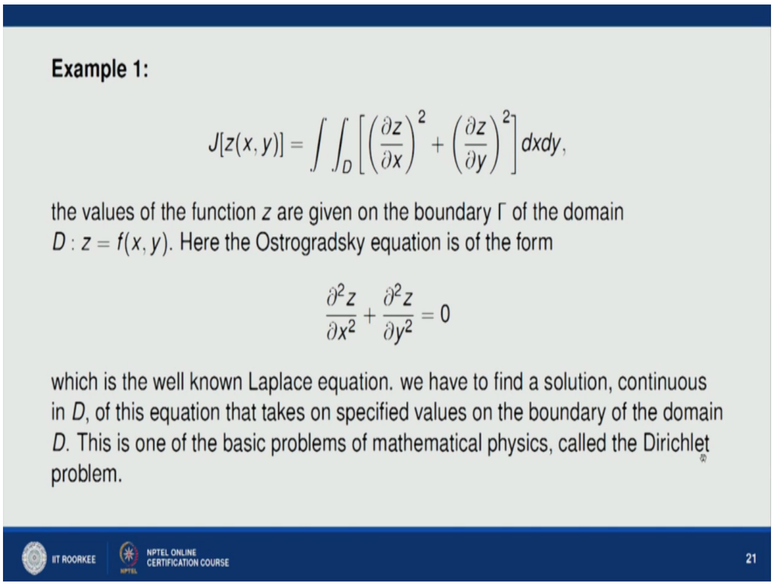


So herea functional is this J of z of (x, y) is equal to double integral D dabba z by dabba x whole square plus dabba z by dabba y whole square dx dy with the condition that z(x, y)take some value say somevalue say g on say C which is a boundary of domain D. So when you find out sayI willEuler (())(24:17)Ostrogradsky equation for this. It is coming out to be, so here you can write it here as f by dabba z minusyou can write it here as(dabba by D) dabba by dabba x of f of p minus dabba by dabba y of f of q, where p and q is basically zx. So here p is yourzx and q is zy or here you can say that your is equal to 0.

So here f ofx, y, z, p and qis given by, if you look at this is nothing but p square plus q square. Sothis is (Euler) Ostrogradsky equation and you can see that, there is no component of z here. So you can simply find out that minusdabba by dabba x, it is nothing butwhen you differentiate, it is 2p, so dabba by dabba xminus dabba by dabba y and 2 of (dabba by) dabba z by dabba y is equal to 0, when you simplify this you 2 (you can) 2 minus 2 you can take it out and you can say, it is nothing but dabba2 z by dabba x square plus dabba2 z by dabba y square isequal to 0.

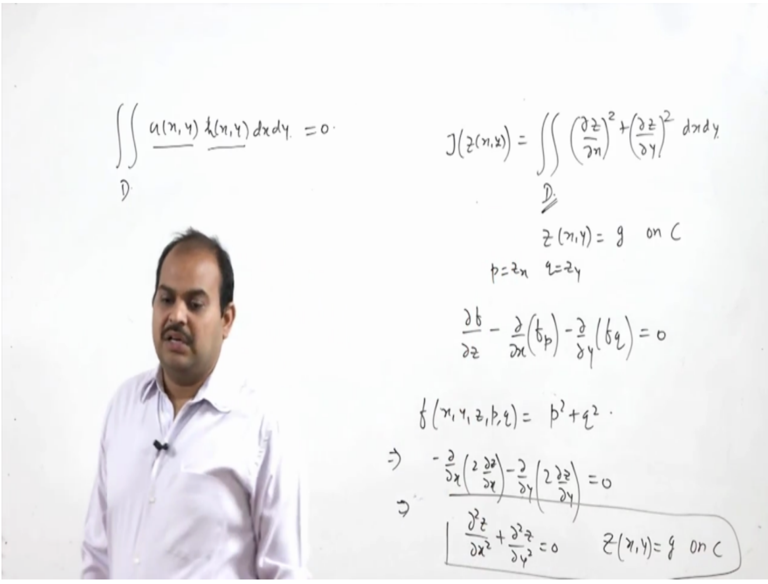
So extremalsurface is must satisfy this equation and if you recall this equation is nothing but Laplace equation. Soextremalsurface is the solution of the Laplace equation whenzsatisfy certaincondition on someboundary of the domain D. Soyou can simply say that if you recall this is nothing but yourDirichlet problemfor the Laplace equation.

(Refer Slide Time: 26:21)



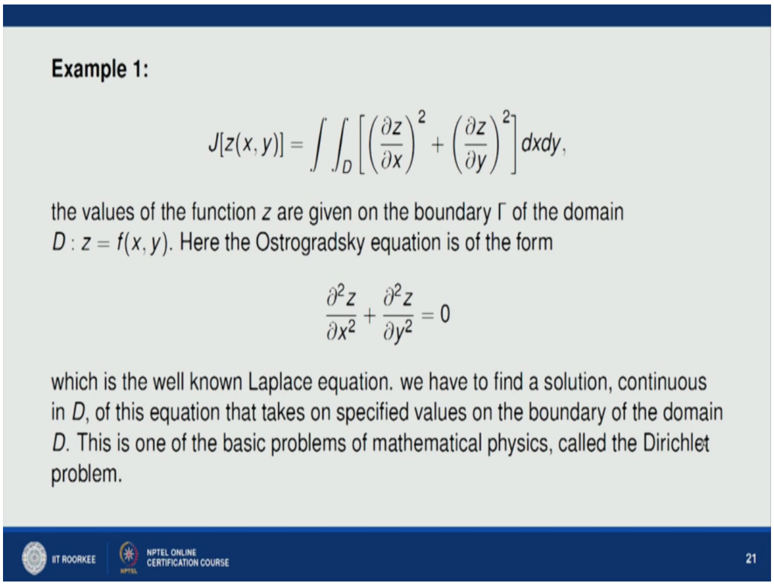
So herewe can say that theEuler Ostrogradsky equation for this functional is coming out to be the Laplace equation. So we have to (find it)find a solution, continuous in D, of this equation that takes on specified values on the boundary, we havesaygiven that z(x, y) is equal to (()) (26:36) on the boundary and this is one of the basic problem of mathematical physics and which is known as Dirichlet problem.

(Refer Slide Time: 26:49)



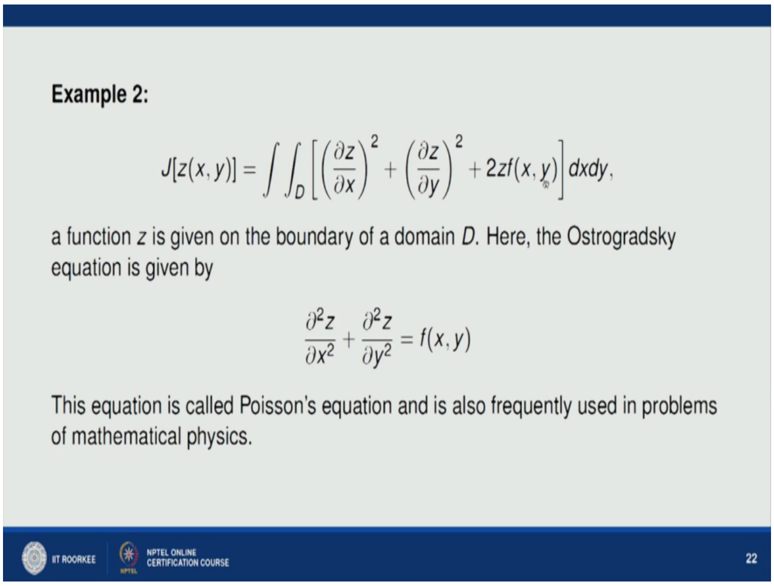
Here,one thing to be noted here that what we started with is to minimizing the variationalminimizing the functional given in this form and it is coming out to be that the minimizer of functional is coming out to be the solution of theDirichlet problem. Soand solution of the Dirichlet problemis thetwice differentiable function, buttwice differentiable function, which satisfy this equation, butthe solution of this is coming out to be the minimizer of this variational. Sothis observation is very very importantand it say, it help us togive a new direction to a new field, which is known assaythe way ofhow to define a solution in a generalized sense (())(27:41) and this willa rise tosay to define a generalize solution in terms of (())(27:47).

(Refer Slide Time: 28:02)



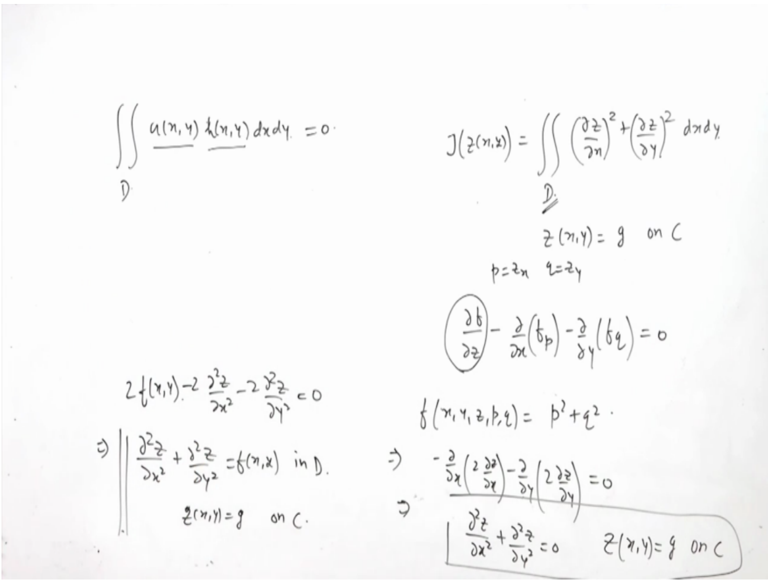
So right now we are no discussing about that problem, so here we simply say that if we have a function like this then the minimizer also extremizer is given bysay this Laplace equation with the condition that z(x, y) satisfy some specified values on the boundary and hence we can say that the extremizer of tis problem is nothing but the solution of the Dirichlet problem.

(Refer Slide Time: 28:13)



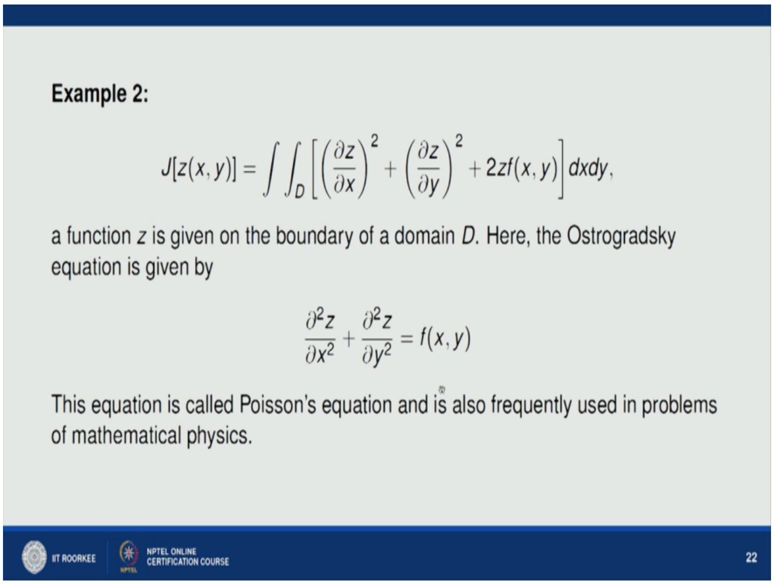
Nowmoving on to the next example, here we have a functionalsay double integral dabba z by dabba x (square)whole square plus dabba z by dabba y whole square plus 2q z f(x,y) dx dy. Now in a same way, we can find out say Euler Ostrogradsky equation asdabba2 z by dabba x square plus dabba2 z by dabba y square equal to f of (x, y). So here if you look at what is a difference here, in between this first example and second example. Insee second example we have added this term 2z f of (x, y).

(Refer Slide Time: 28:49)



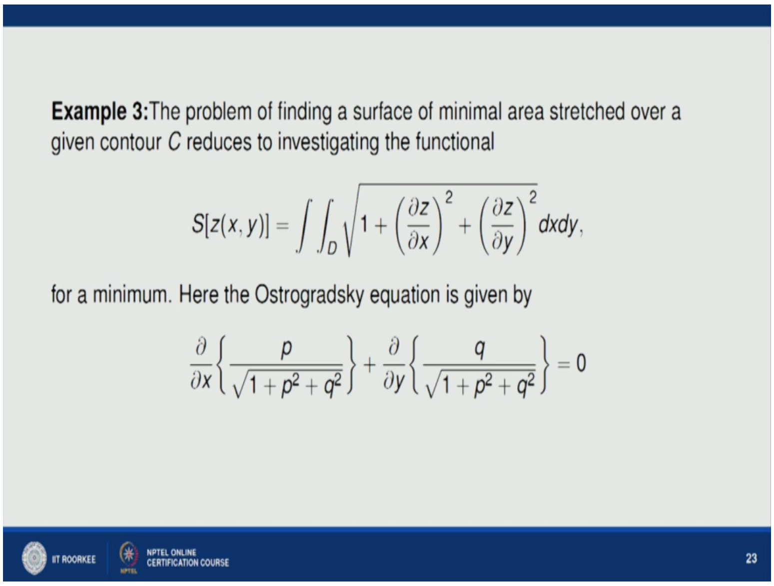
So if youuse the Euler Ostrogradsky equation then this term is also present there. So when you simplifythe contribution with respect to this is going to be 2 of f of (x, y), (())(29:02) it is same\, so you can get that insecond example we have this (exam)this thing minus 2 dabba2 z by dabba x square minus 2 dabba2 z by dabba y square is equal to 0. So you can simplify tis as dabba2 z by dabba x square plus dabba2 z by dabba y square is equal to f of (x, y) and here your z of (x, y) is equal to g or your curve C. Is it okay?So this is in D and this is on this is. So if you look at this isa non-homogeneousDirichlet problem.

(Refer Slide Time: 29:59)



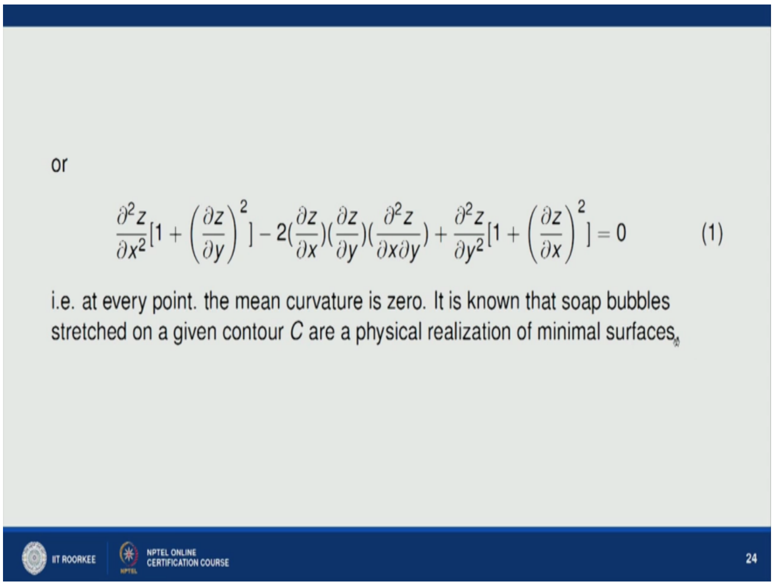
Sothe solution, so or we can say the extremizer of the functionalgiven here, z equal to J of z of (x,y) is equal to this. The extremizer of this must satisfy the poisson equation and this is also one of thevery very (())(30:04) is more steady problem in the physics mathematical physic and you can say that the solution on this extremizersolution on thisthe extremizer of this functional is nothing but the solution of the poisson equation.

(Refer Slide Time: 30:17)



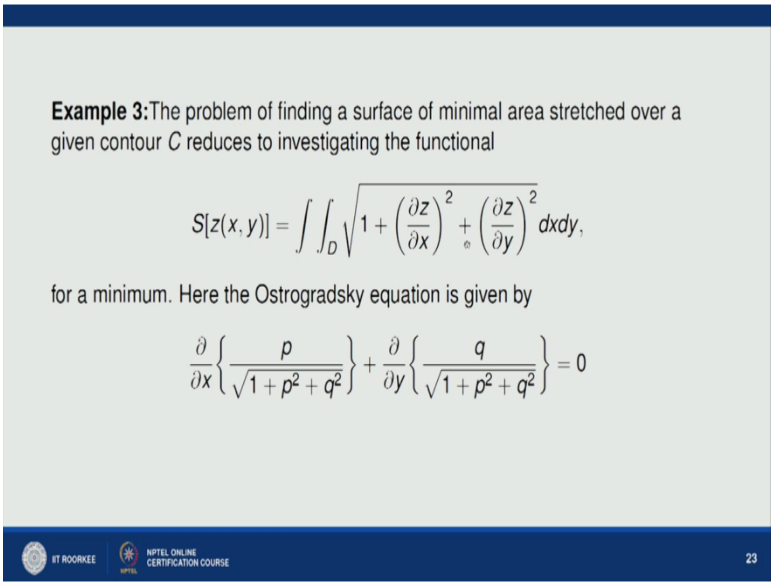
So in example 3we want to find out say surface of minimal area stretched over a give contour C. So here we have contour C and we want to find out a surface which gives the minimal area and it is a generalization of the problemhaving the minimum surface area in one dimensional.so here your functional is coming out to be S z of (x, y), which is given as double (deriva)double integral under root 1 plus zx whole square plus zy whole square dx dy and ina same way you can find out say Ostrogradsky equation if you remember here integrant is not involvingfunction z. So here you can simplifythe Euler Ostrogradsky equation, which is given by dabba by dabba x of p upon under root 1 plus p square plus q square plus dabba by dabba y of q upon 1 plus t square plus q square and it isgiven as0.

(Refer Slide Time: 31:21)



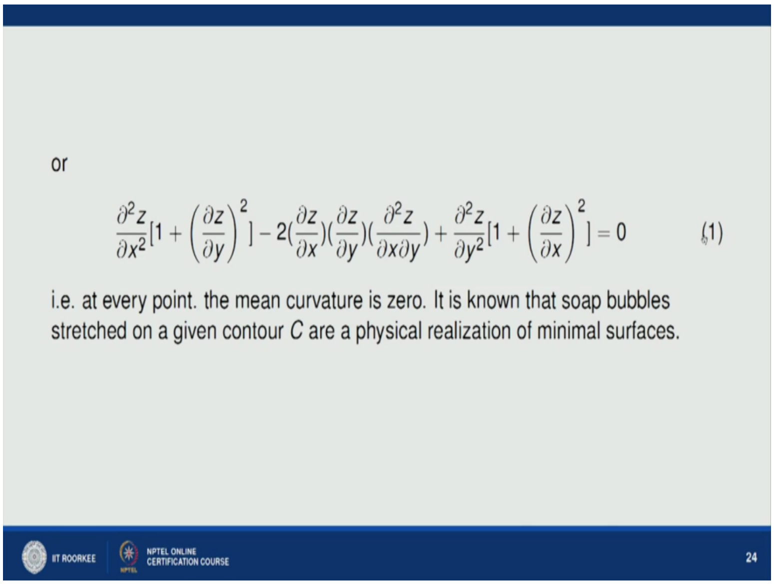
And when you simplify, it is coming out to be that z xx 1 plus zy whole square minus 2dabba zx zy zxy plus zyywithin bracket 1 plus zx square equal to 0. So here mathematicalphysicthissurface, the solution of this differential equation represent a surface whosemean curvature is 0. So it means that at this isthe surface at which every point, the mean curvature is 0 andwe know thatthat the soap bubbles stretched on a given contour C are surfaces havingmean curvature as 0 and these kind of surfaces are known as minimal surfaces.

(Refer Slide Time: 32:17)



So here we can say that the solution of this problemswhich minimizes this functional are known as minimal surfaces.

(Refer Slide Time: 32:27)



And it satisfy thepartial differential equation given by this one. So here we stop our discussion and in next classwe will see in the continuation of this and in next problem we will see,how to definehow to takesay example based on this, which is known as iso-parametric problem some more example based onthis Euler's equation we will discuss that is iso-parametric problem and we also discussthe functional derivative and invariance ofEuler's equation in nextlecture. Thank you very much for listening us, we will meet in a next lecture.