Integral Equations, Calculus of Variations and their Applications
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Lecture 45
Brachistochrone problem and Euler's equation-1

Hello friends, welcome to my lecture on Brachistochrone problem and Euler's equations.

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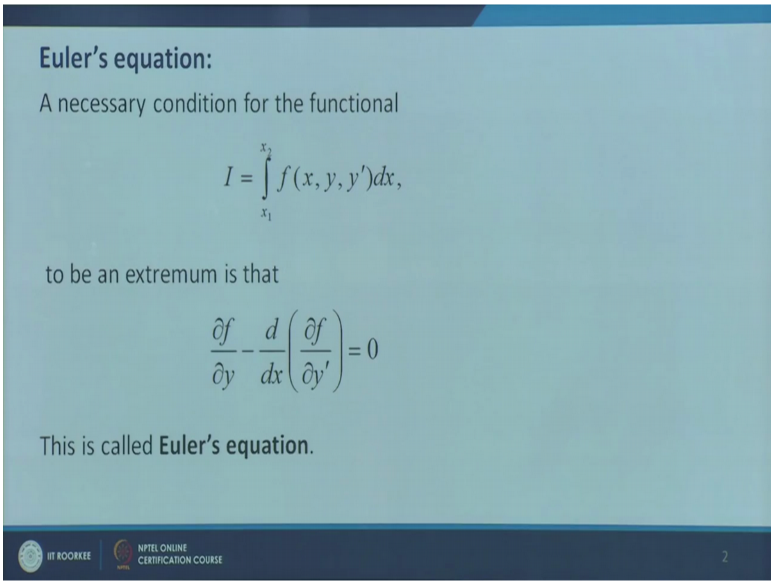

We know that a necessary condition for the functional I equal to integral x1 to x2 f(x, y, y dash) dx to be an extremum is that delta f over delta y minus d over dx delta f over delta y dash is equals to 0 and we call this equation as the Euler's equation.

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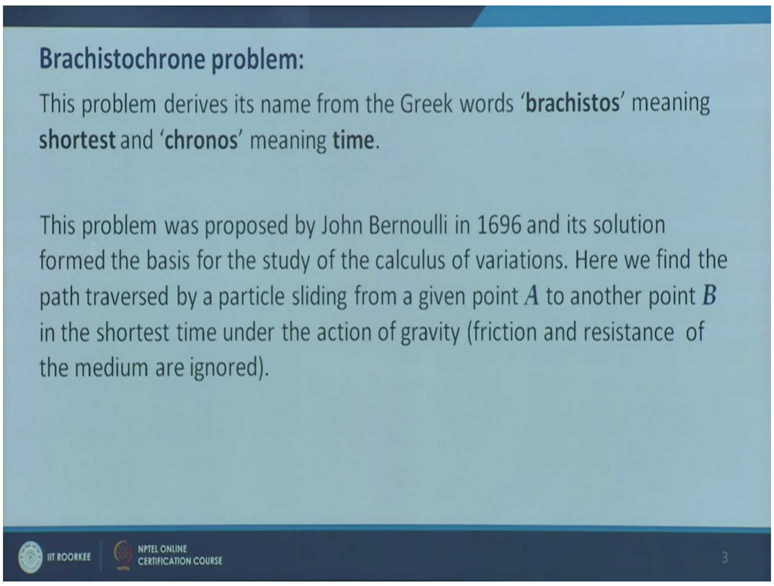

(The) this problem which we are going to discuss now this brachistochrone problem derives its name from the greek words brachistos and brachistos means shortest and chromos means time. So brachistochrone problem actually discusses the shortest time that a (())(1:04) will take and sliding down from 1 point to the other. This problem was proposed by John Bernoulli in 1696 and its solution formed the basis of the study of calculus of variations. Now here we shall find the path traversed by a particle which slides from a given point A to another point B in the shortest time under the action of gravity. We shall assume that the friction and resistance of the of the medium are ignored.

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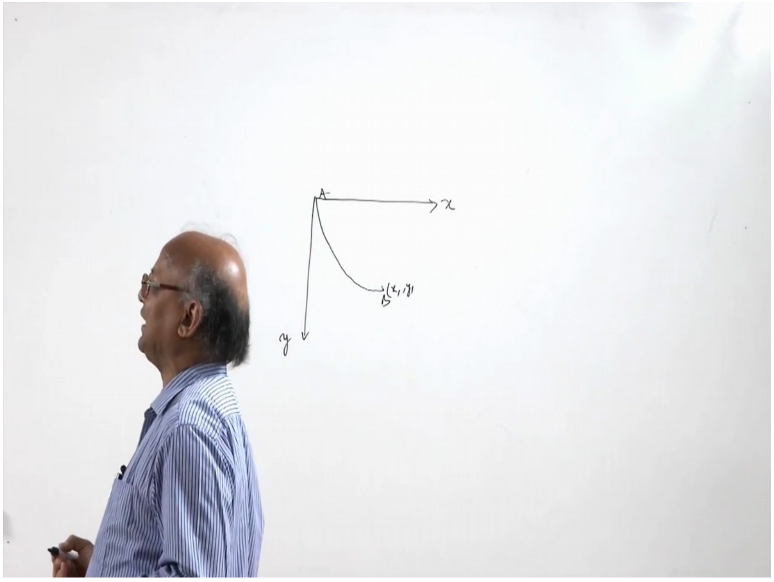

So let us say let say the point A be at the origin, the y axis is pointed downwards and this our x-axis. So let A be the origin, y be measured downwards and B be the point at (x1, y1). Let say this is the point (x1, y1). The particle slides down from the point A to the point B in time let us say t.

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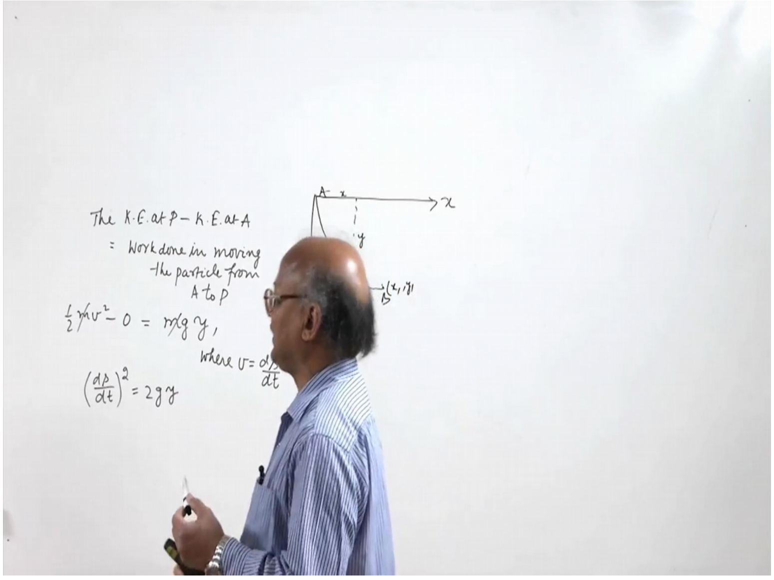

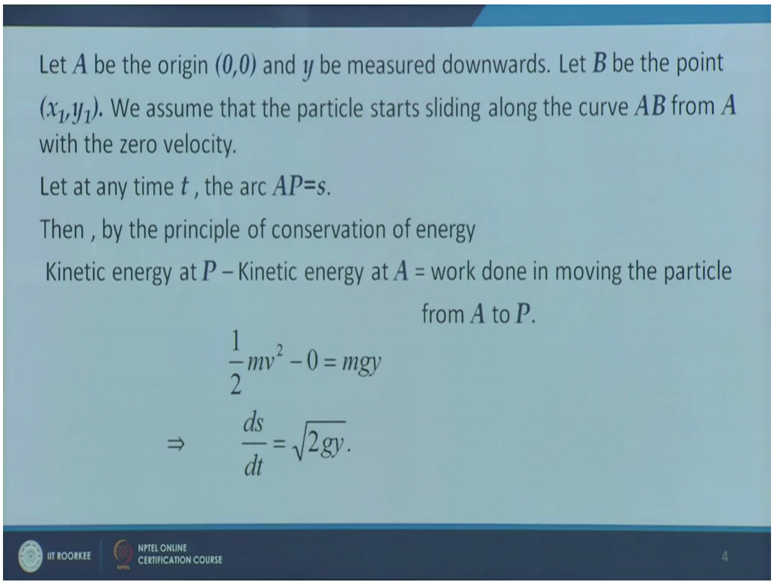

So then we assume that the particle starts sliding from the point A along the curve AB with the 0 velocity and let us say take any arbitrary point P on its path, so the coordinates of P are let us say x, y then y is this vertical height and this is x. So by the principle of conservation of energy the kinetic energy at p will be equal to the minus kinetic energy at A will be equals to work done in the moving the particle from A to P.

Now if V is the velocity of the particle then the kinetic energy at the point P let us say, the velocity of the particle be V. So then half mV square minus, it starts at the point A with 0 velocity. So the kinetic energy there at the point A is 0. So minus 0 equal to mg y by the height vertical height of the point P. Now this B is equal to ds by dt. So what do we have? We can cancel m and then we have ds by dt whole square equal to 2gy or we can say ds by dt is equal to under root 2gy.

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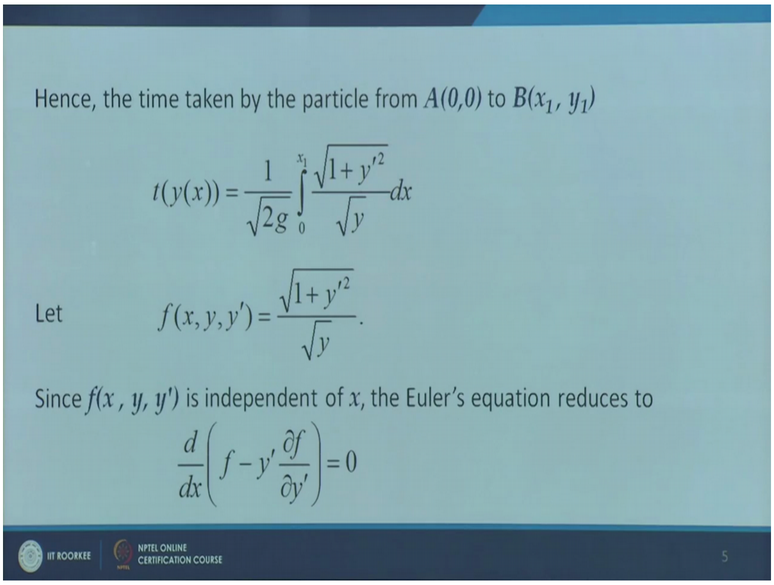

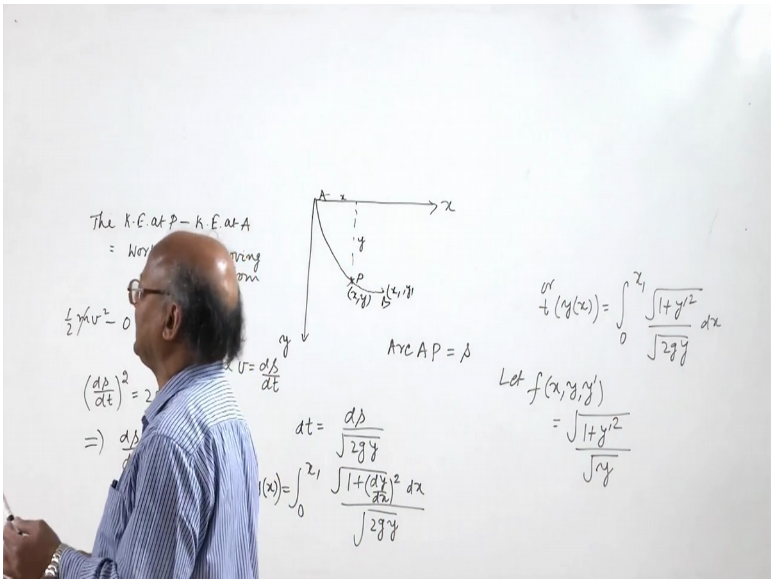

Now let us say t is the time taken by the particle from sliding from A to the point B whose co-ordinates we have taken as (x1, y1) then t will be a function of y okay, y will depend on X and we shall have from there if you can if you take the square root ds by dt will be equal to under root 2gy, where (())(4:29) we have taken a positive sign here, because as t increases s also increases, the arc AP is s.

Now subtract the variables, so we have dt equal to ds upon under root 2gy when we integrate from 0 to from A equal to 0 to the point P, we shall have 0 to t is y(x) will be equal to 0 to x1 and then ds is equal to under root 1 plus dy by dx whole square into dx divided by under root

2gy or we can write it as. So we can write t y(x) equal to one over root 2g is a constant we can take it outside the integral. So one over root 2g, integral 0 to x1 under root 1 plus y dash square over root y and let us now assume that f(x, y, y dash) be equal to under root 1 plus y dash square over root y. So let us say, 1 over root 2g is a constant so it does not play any role in determining the extremum value of this integral. So we have taken it outside.

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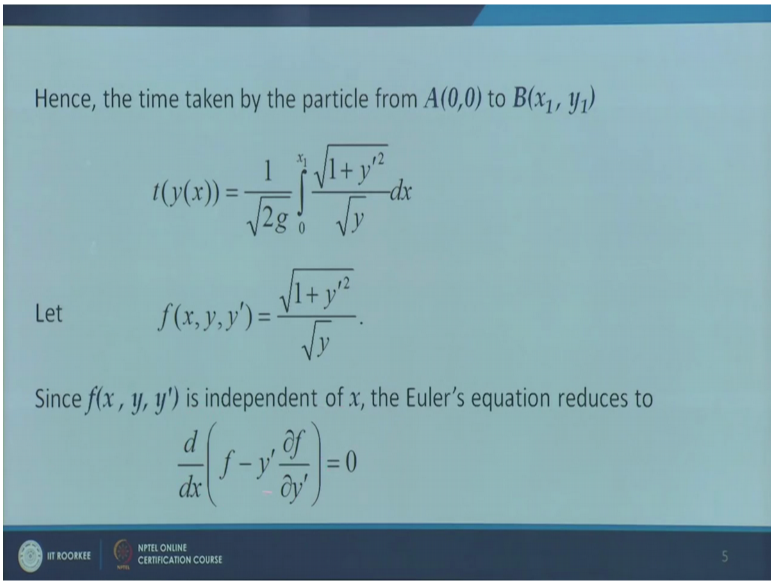

And now let us recall that if the function f(x, y, y dash) is independent of x then the Euler's equation is given by this alternate form. The Euler’s equation give is reduces to d over dx f minus y dash delta f over delta y dash is equal to 0. Here you can see that f(x, y, y dash) depends only on y and y dash it does not depend on x. So the Euler’s equation reduces to this.

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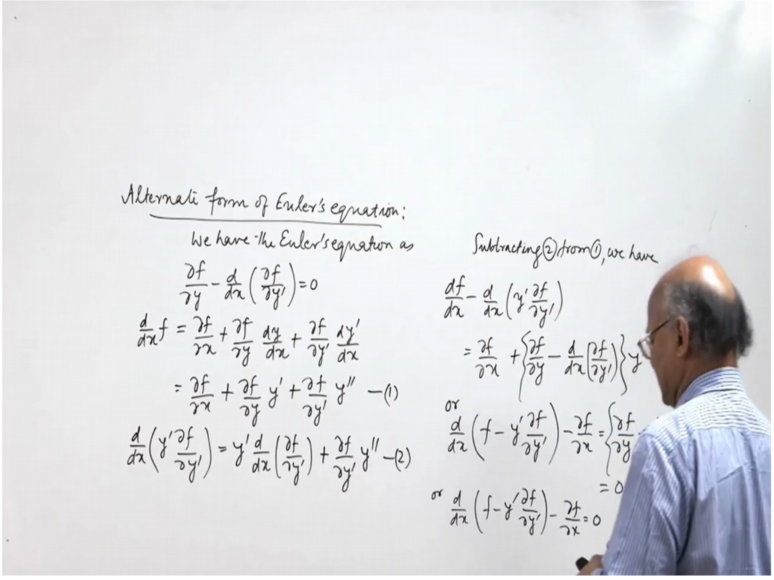

And the proof of this is a we alternate form of with an easily do this alternate form of we have the Euler's equation as. This is the Euler's equation. Now let us find its alternate form. let us find first d over dx of f. Now f is the function of x, y and y dash. So I can write it as delta f by delta x delta f by delta y into dy by dx plus delta f by delta y dash dy dash by dx or I can write it as like this and then let us find d over dx of y dash into partial derivative of f with respect to y dash then what we will get y dash times d over dx of plus into y double dash. Let us call this as equation 1 and this as equation 2. So when you subtract the equation 2 from 1. So subtracting 2 from 1 we have, df by dx minus d over dx of y dash delta f by delta y dash.

Now what will happen when we subtract 2 from 1? This term this term delta f by delta y dash y double dash will cancel with delta f over delta y dash y double dash and what we will get delta f by delta x plus delta f by delta y minus d over dx of delta f by delta y dash into y dash or we can say I can write it as d over dx of f minus y dash delta f by delta y dash that is the left hand side minus delta f by delta x equal to delta f by delta y minus d over dx of delta f by delta y dash into y dash.

Now from the Euler’s equation we know that this is 0 delta f by delta y minus d over dx delta f over delta y dash is equal to 0. So right hand side is equal to 0 and thus we get (the) or I can say d over dx of f minus y dash delta f by delta y dash minus delta f by delta x equal to 0. So this is another form of the Euler's equation and in case f is independent of x, it reduces to d over dx f minus y dash delta f over delta y dash equal to 0 like here.

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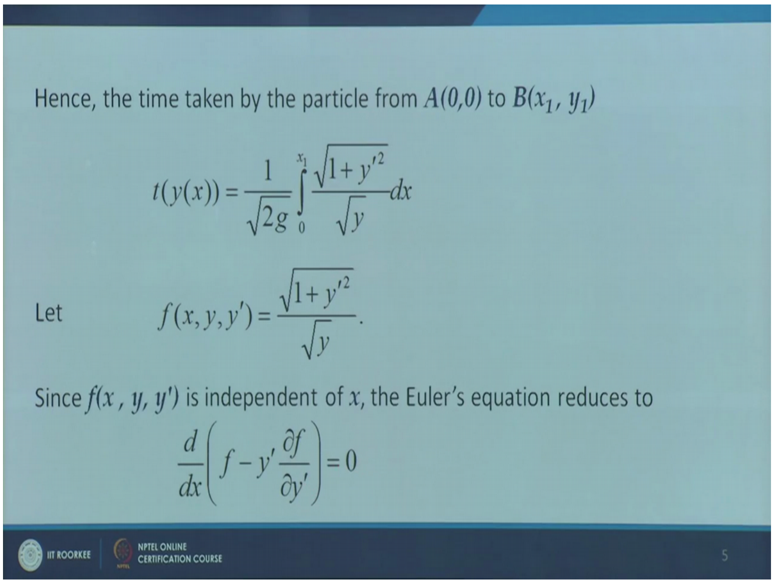

Here, f is a function of y and y dash only. It does not depend on x. So the Euler's equation this one reduces to this equation. Now from here when we integrate with respect to x we will have f minus y dash delta f over delta y dash is equal to some constant.

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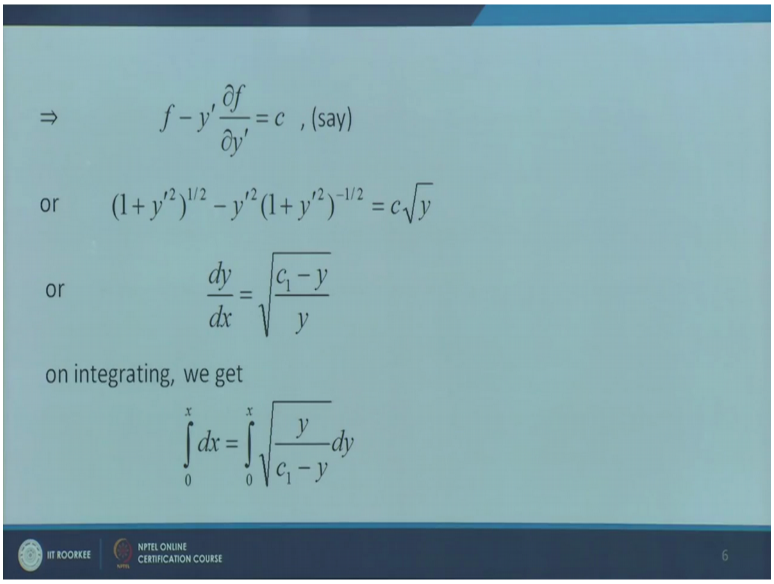

So this is what we have f minus y dash delta f over delta y dash is equal to C.

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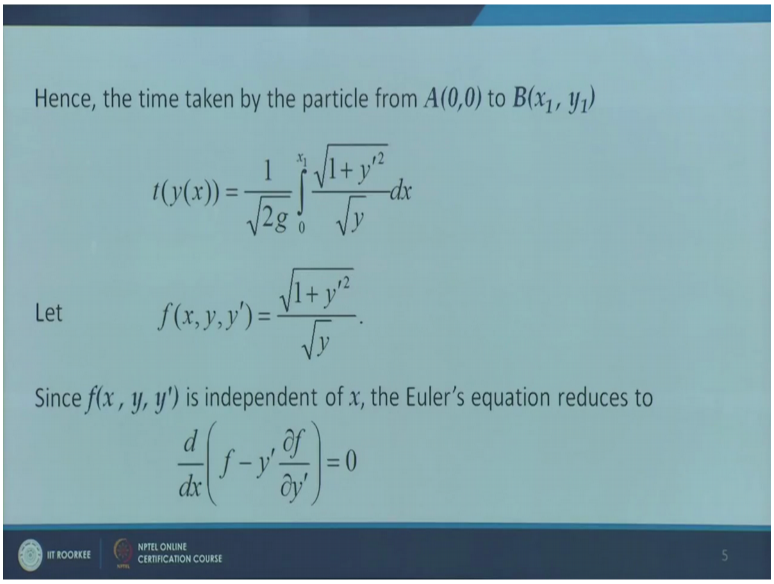

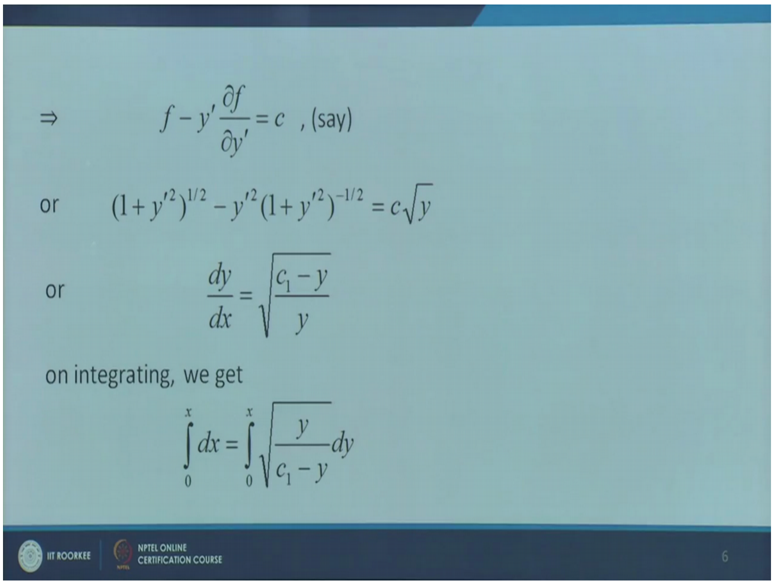

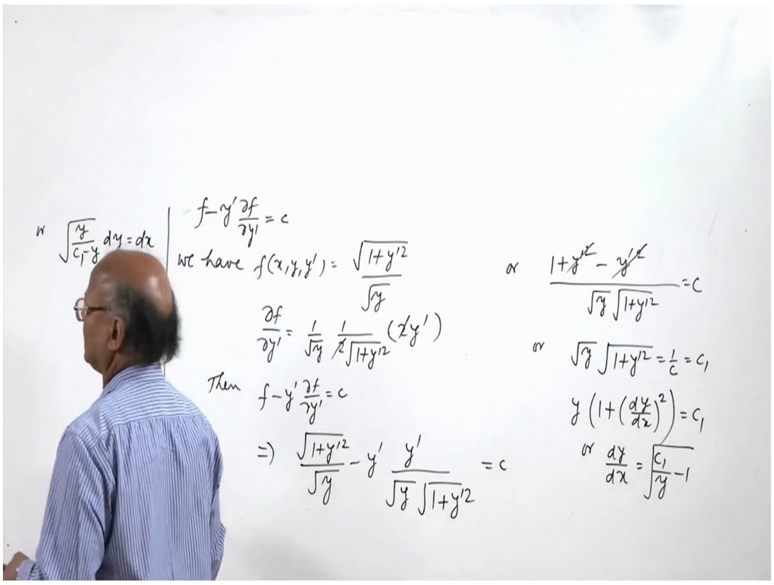

Now f is equal to Euler's simplify this equation f minus y dash we have f equal to under root 1 plus y dash square upon root y. So we find the partial derivative with respect to y dash from here. So we shall have 1 over root y 1 over 2 square root under root 1 plus y dash square into 2y dash. So this 2 will cancel with this 2 here and then f minus y dash delta f over delta y dash is equal to C gives us under root 1 plus y dash square divided by root y minus y dash times , y dash divided by root y into under root 1 plus y dash square equal to C.

So what we will get? Root y into square root 1 plus y dash square and then we shall have 1 plus y dash square minus y dash square equal to C. So this cancel’s and we get or under root y into , under root 1 plus y dash square equal to 1 by C. So what we get is we can solve it. So let us see, this is y times 1 plus dy by dx whole square 1 over C if we put as C1 so then this is C1. So or we can say dy by dx is equal to C1 over y minus 1 square root. Now as x increases y also increases. So we get this so C1 this is so the subtracting the variables we will get or root y over C1 minus y or into dy is equal to dx.

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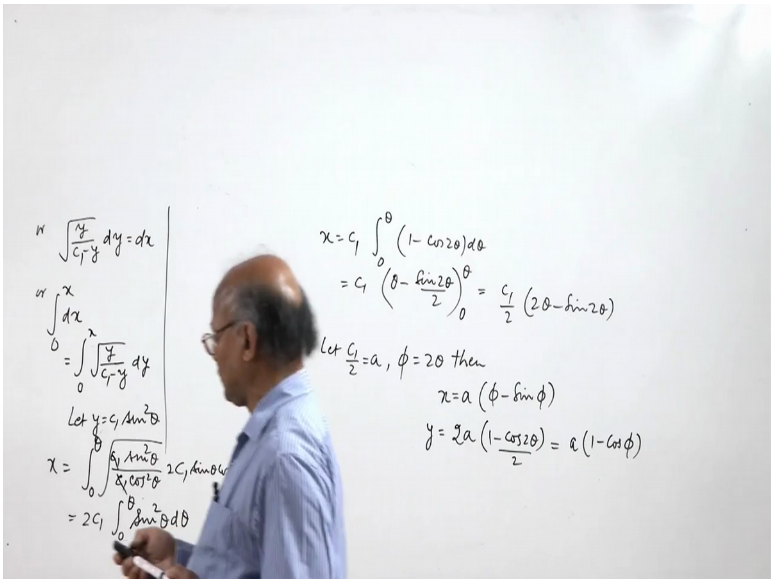

So let us integrate with respect to let us integrate both sides. So integral this will give you integral 0 to x dx equal to integral 0 to x. So let us now put let y be equal to C1sin square theta. So left hand side becomes x and we get here 0 to dy will become 2C1 sin theta cos theta dtheta and this is one x is x y is y, so I think we should put here y here. So this will give you 0 to pi by 2 and or we can say 0 to theta not pi by 2 at this point 0, it should be theta not 0 not pi by 2, it should be theta. So this x here, let me write x here. So this is C1sin square theta and what we will get C1 minus C1sin square theta C1 cos square theta, dy is 2C1 sin theta cos theta d theta, we have to put the limit for theta. So we have written it theta.

So this is C1 cancel out here and what we will get here, 2C1 integral 0 to theta and then we shall have cos theta will cancel with cos theta we shall have sin square theta dtheta. So what we have is this. so this is now we can integrate this. So this is x equal to C1 times 0 to theta and then 1 minus cos2theta dtheta. So this is equal to C1 then we have theta minus sin 2theta by 2, the integral of 1 minus cos 2theta. So 0 to theta will give you C1 by 2 2theta minus sin 2theta. Now let us say, let C1 by 2 equal to a and phi be equal to 2theta then, x is equal to a times phi minus sin phi and y is equal to C1 sin square theta. So y is equal to C1, C1 is equal to 2a and sin square theta is 1 minus cos 2theta divided by 2. So this will be a times 1 minus cos phi.

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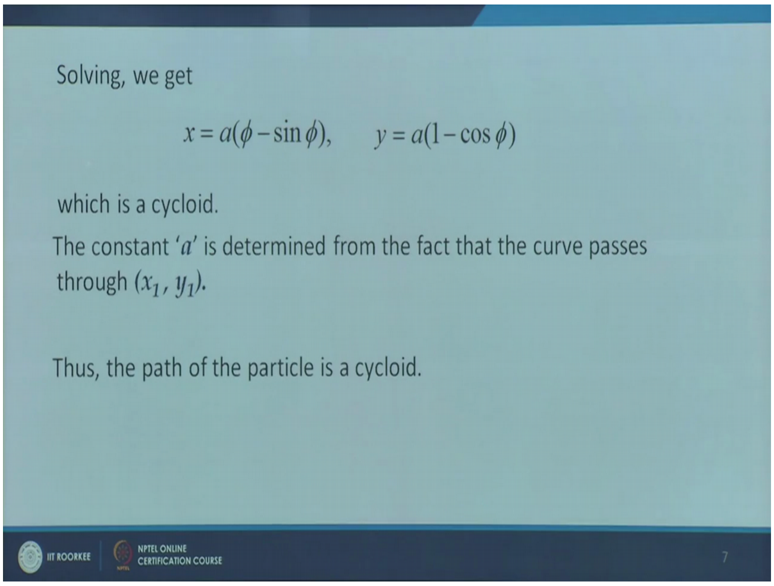

So I x equal to a times phi minus sin phi, y equal to a times 1 minus cos phi and it is very well known that the parametric equations of the cycloid are given by x equal to a times phi minus sin phi, y equal to a times 1 minus cos phi. So when the path key (())(19:20) slides under the action of gravity where the resistance and the friction of the medium are neglected, it travels along the satellite. Its path is that of a cycloid. The constant here is determined from the fact that the particle passes through the point (x1, y1). So that will give you the values of a and phi.

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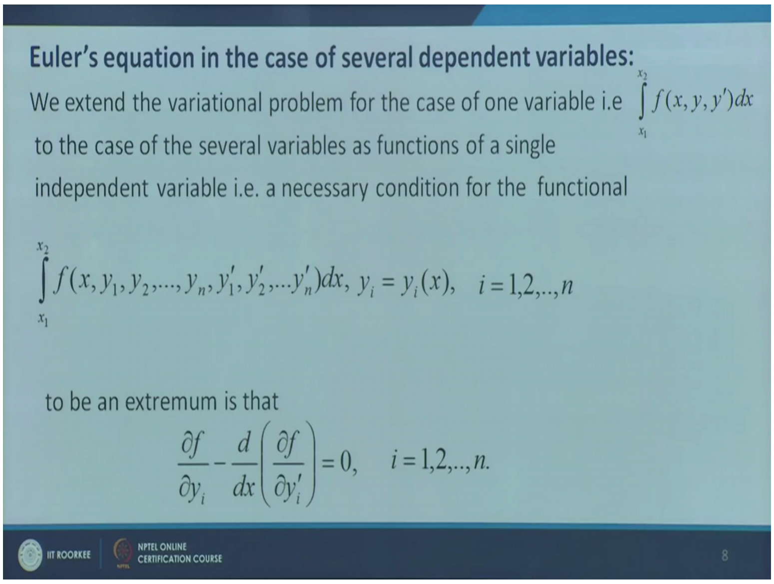

Now let us show the path of the particle is a cycloid. Now let us consider Euler's equation in the case of several dependent variables, you have earlier done the Euler’s equation in the case of one dependent variable. Now we shall be doing the Euler’s equation in the case of several dependent variables. So in previous lectures you have studied how to find the extremum value of this functional, where there is one independent variable x and there is one dependent variable y. So we discuss the case of the Euler’s equation in the case of 1 dependent variable. Now we shall extend the study to the case of several dependent variables. So the variational (())(20:35) problem will be now (())(20:37) to the case of several variables as function of a single independent variable.

let us consider for example, this functional integral x1 to x2 f(x, y1, y,…yn, y1dash, y2dash, yn dash) dx this functional we want to extremize this functional where yi is our dependent variables, which are depend on the one single independent variable x and I varies from 1 to n. So there are n independent variables all depending on 1 single independent variable x and the condition for this functional to be an extremum is that the partial derivative of f with respect to yi, yi is ith dependent variable, minus d over dx partial derivative of f with respect to yi dash is equal to 0 and this must hold for all values of i, i equal to 1 to n. So there will be system of n equations here corresponding to the n dependent variables yi.

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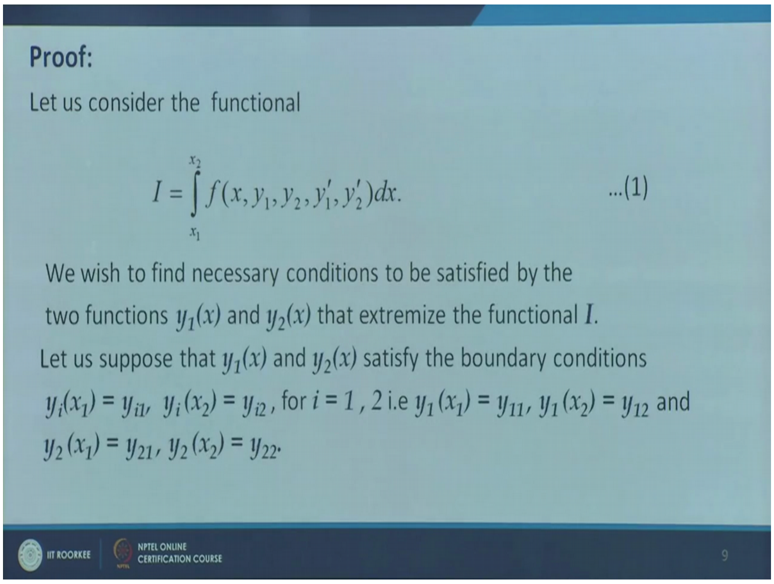

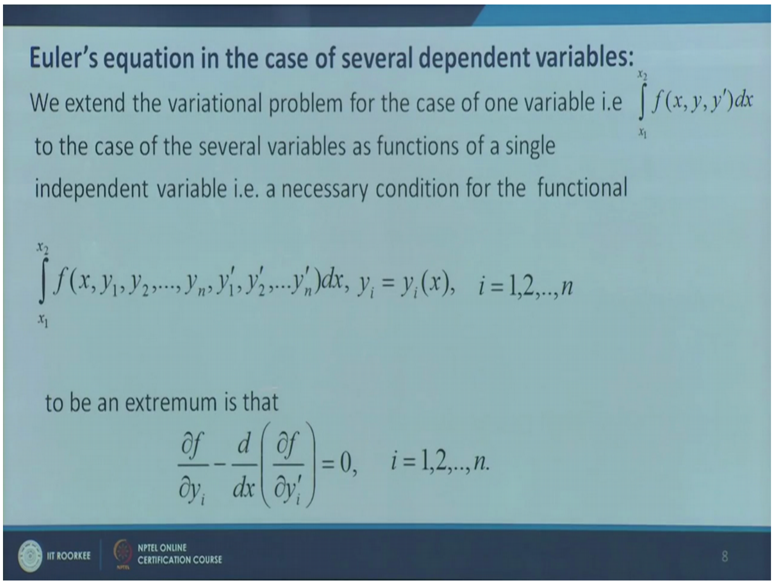

Now le us we shall prove this result we shall find this necessary condition for this functional for the case n equal to 2. The proof can be really extended to the case of any arbitrary n.

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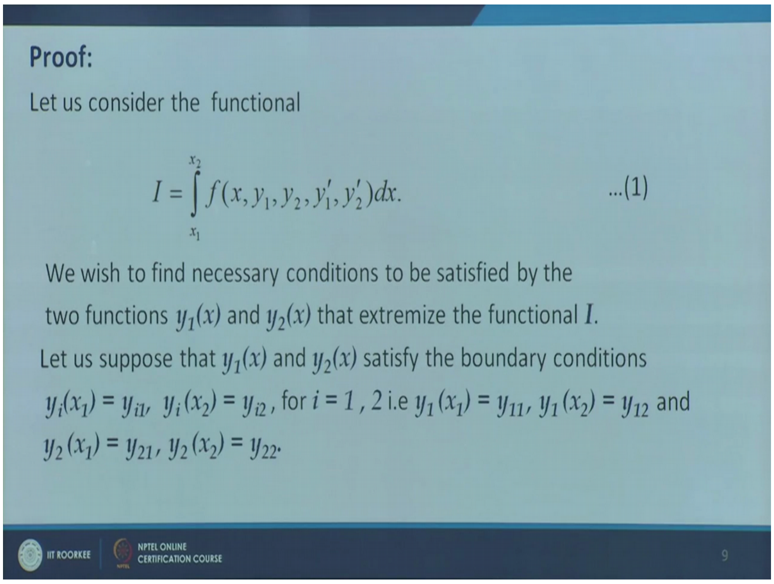

Now let us consider this functional I equal to x1 to x2 f(x, y, y2, y1 dash, y2 dash). Now we (())(21:56) to find the necessary condition to be satisfied by the two functions y1 and y2, which depend on the single independent variable x and those values of y1, y2 which extremize this functional i. So now suppose that y1 and y2 x satisfy the boundary conditions at the point x1 by y1 x1 is equal to y is equal 11 and y1 at x2 is takes the value y12 y2 function at x1 takes the value y21 and y2 at x2 takes the value y22.

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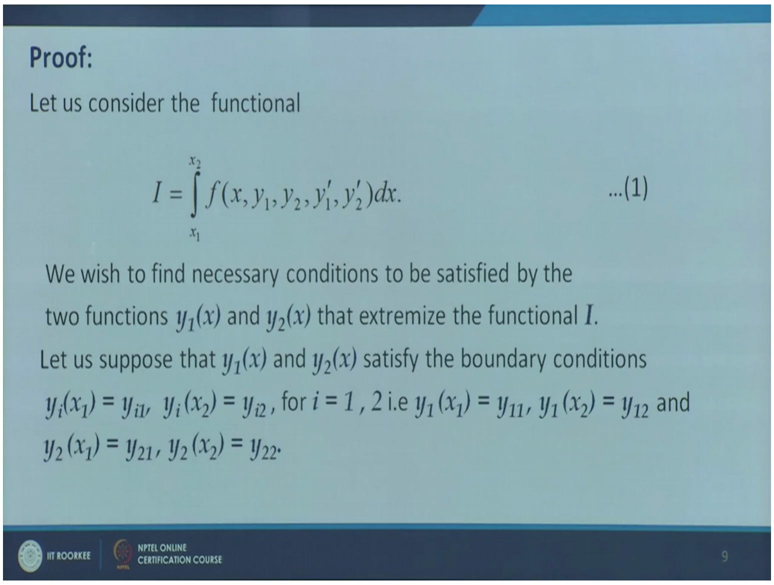

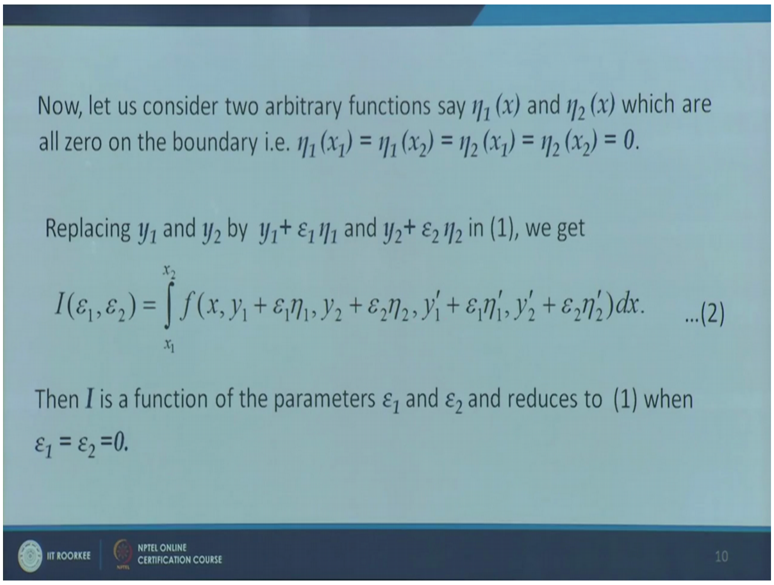

Let us consider 2a arbitrary functions eta1(x) and eta2(x), which assume value 0 at the boundary points eta1 at x1 and eta1 at x2 is equal to 0. Similarly eta2 at x1 and eta2 at x2 is equal to 0. Now what we will do is? In the given functional we shall replace y1 and y2 by y1 plus epsilon1 eta1 and y2 by y2 plus epsilon2 eta2, then the functional that we get will be a function of epsilon1 and epsilon2, we shall call it as I epsilon1, epsilon22. So the function will take the form x1 to x2 f(x, y1 plus epsilon1 eta1, y2 plus epsilon2 eta2 and then y1 dash plus epsilon1 eta eta1 dash y2 dash epsilon2 eta2 dash. Now this I then is a function of 2 parameters epsilon1 and epsilon2 and it we you can see that it reduces to the previous given functional this the this one, when epsilon1 epsilon2 take value 0s.

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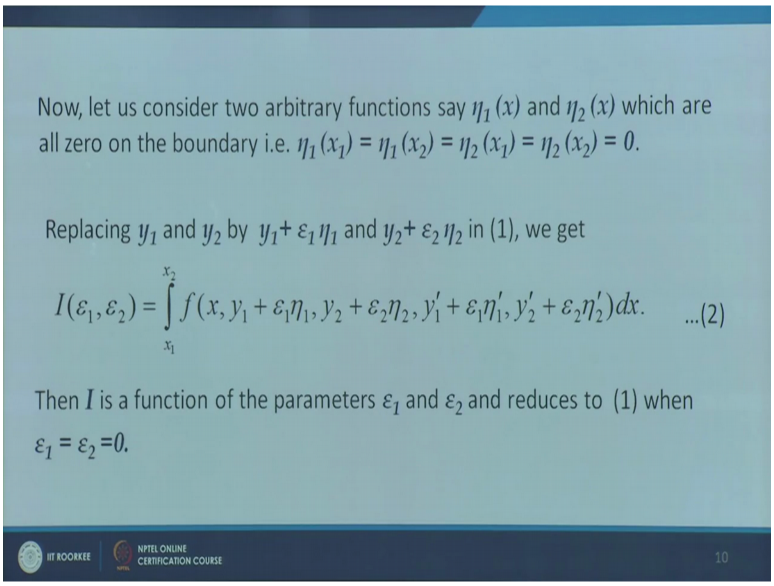

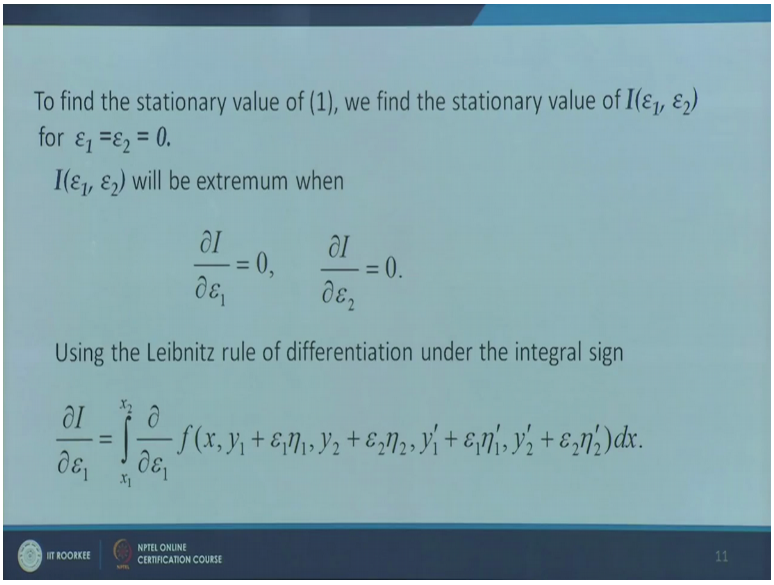

Now to find the stationary value of this function**,** since I is a function of 2 variables epsilon1 and epsilon2, it **it** will assume extremum value provided its partial derivative with respect to epsilon1 and epsilon2 are 0 that is those are the necessary conditions for I epsilon1, epsilon2 to have an extremum value. Now when we differentiate I with respect to epsilon1 let us see how what we get.

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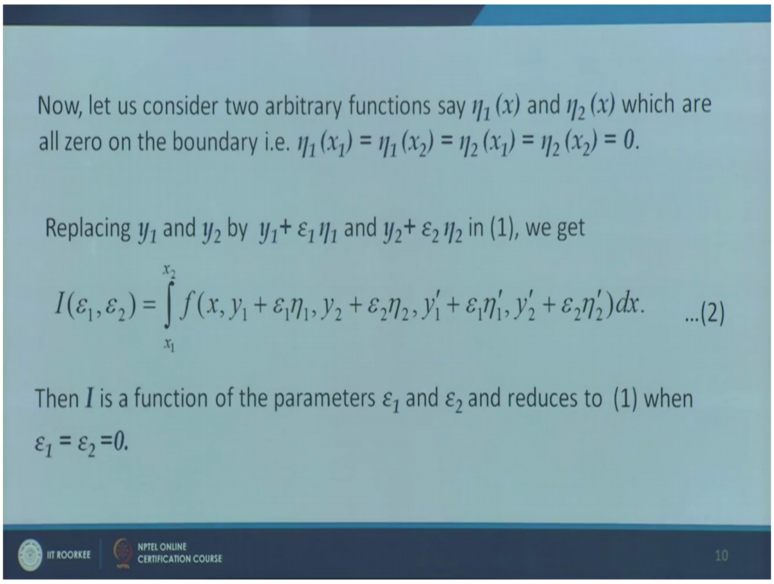

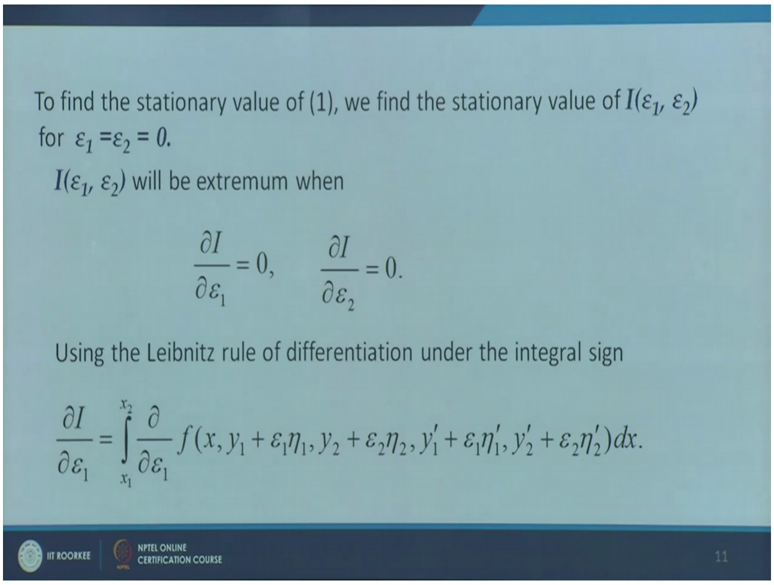

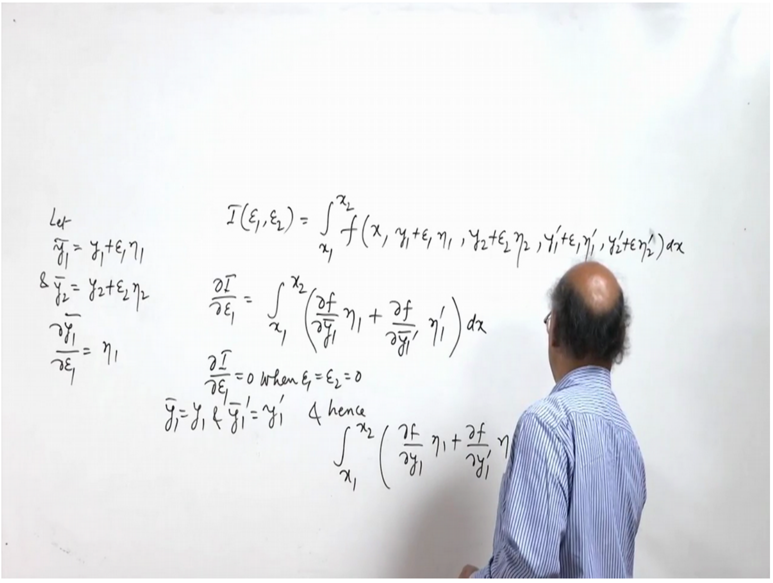

So this is I, when we differentiated with respect to x with respect to epsilon1. Since the integrant is a function of I1 epsilon1, so we will have to use the (())(25:05) of differentiation under the integral sin and the here the limits of integration are constant. So we shall have x1 to x2. Now let us say that y1 bar be equal to y1 plus epsilon1 eta1 and y2 bar be equal to y2 plus epsilon2 eta2, okay. Then since the limits of integration do not depend on x1 and x2 upon epsilon1 and epsilon2, the partial derivative of y with respect epsilon1 will be x1 to x2, the partial derivative of this integrant with respect to epsilon1 and when we differentiated with respect to epsilon1, we shall have, so delta f over delta y bar delta y bar y1 bar and delta y1 bar over delta y1 bar over delta epsilon1 will be equal to eta1. So we shall have eta1 and then delta y1 f over delta y1 bar dash into epsilon1 dash.

When we differentiate it partially with respect to epsilon1, the this will be differentiated with respect to epsilon1.This is y1 bar, this is y1 bar dash. So partial derivative of f with respect to y1 bar into partial derivative of y1 bar with respect to epsilon1, which is eta1. Partial derivative of f with respect to y1 bar dash. So this is this and then partial derivative y1 bar dash with respect to epsilon1 will be eta1 dash dx. So this is what we have. Now we have delta I over delta epsilon1 is equal to 0 when epsilon1 is equal to epsilon2 equal to 0. So when epsilon1, epsilon2 are 0s, y1 bar becomes y1 bar becomes y1 and y1 bar dash becomes y1 dash and hence, we have integral x1 to x2 delta f over delta y1 eta1 plus delta f over delta y1 dash eta1 dash dx equal to 0.

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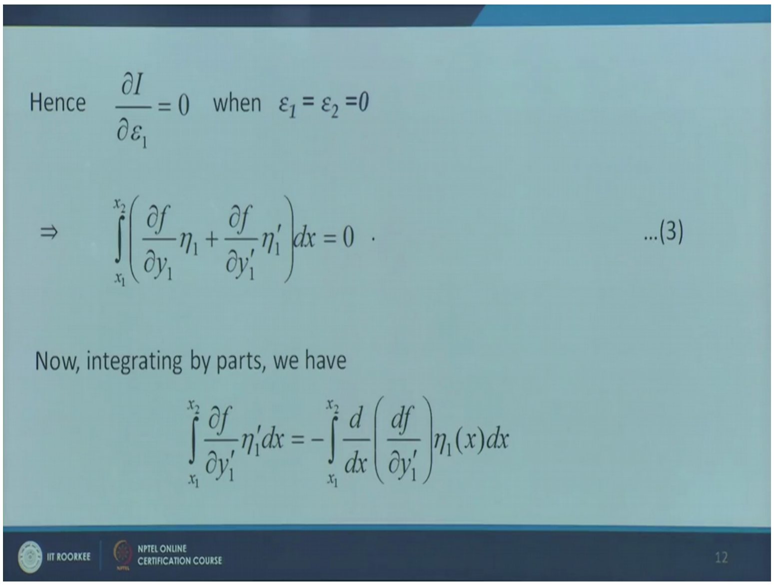

So we have this equation integral x1 to x2 delta f over delta y1 eta1 plus delta f over delta y1 dash eta1 dash dx equal to 0. Now let us integrate by parts. The second term of this on the left hand side.

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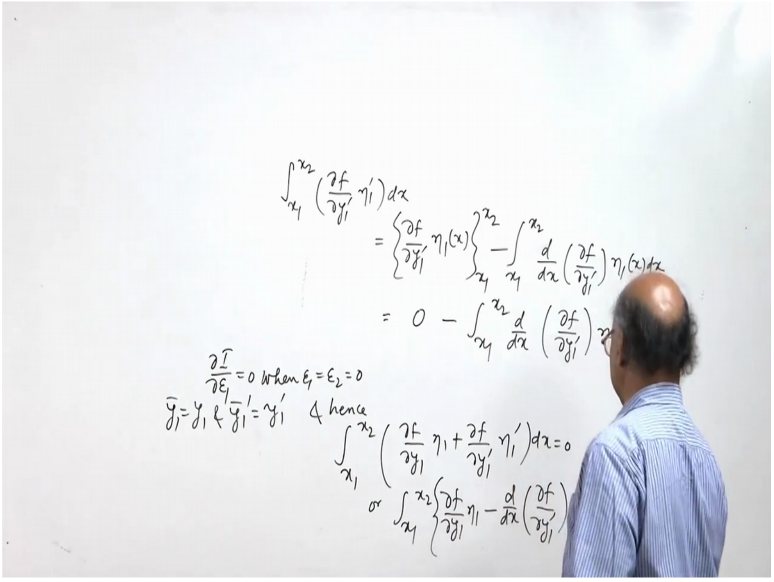

(So when we) so let us see, integral x1 to x2 delta f over delta y1 dash eta1 dash dx. When we integrated by parts, we have the partial derivative of f with respect to y1 dash into eta1 eta12(x) x1 to x2 minus x1 to x2 d over dx of this. Now we have assume that eta1 (())(29:15) that x1 and x2. So since eta1(x) x2 is equal to 0 eta1(x1) is equal to 0, so we shall have 0 minus integral x1 to x2 and d over dx of. So the second integral here second term here second term here becomes this. So this be this gives us or x1 to x2 delta f over delta y1 eta1 minus d over dx of we get this, okay. So thus we have integral x1 to x2 delta f over delta y1 minus d over dx delta f over delta y1 dash multiplied by the function eta1(x) dx equal to 0. So this equation holds true for all choices of eta1(x). Therefore, we must have delta f over delta y1 minus d over dx of delta f over delta y1 dash equal to 0.

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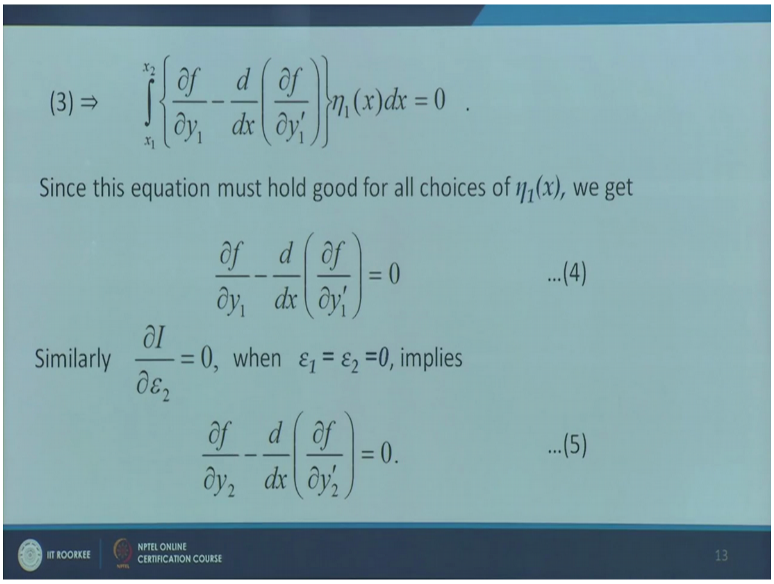

Similarly when we differentiate I1 epsilon1, epsilon2 with respect to epsiln2 by using the (()) (30:37) rule of differentiation under integral sign and put epsilon1 epsilon2 equal to 0 what we will get is delta f over delta y2 minus d over dx of delta f over delta y2 dash equal to 0.

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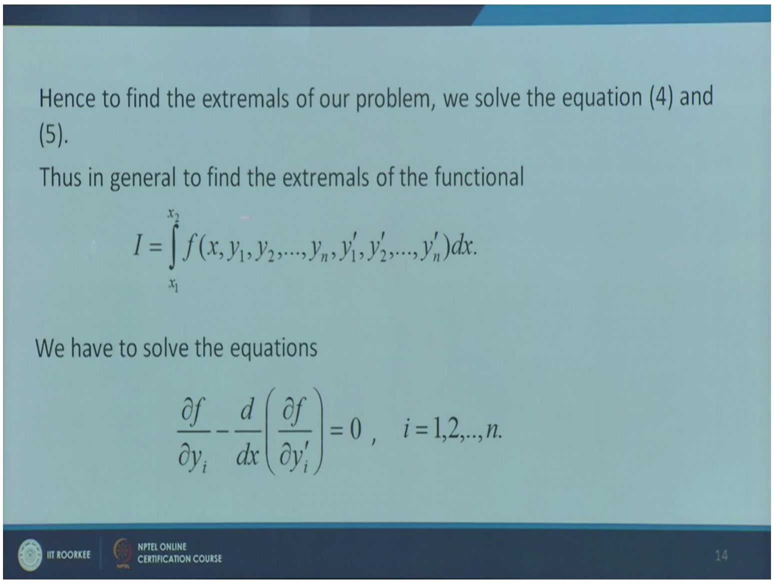

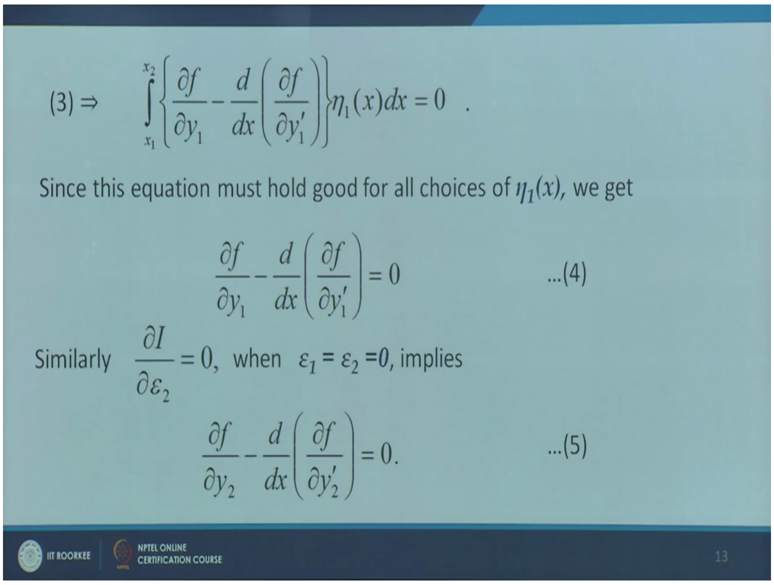

And thus to find the extremum of our problem we solve the 2 equations, this is this equation and this equation. So corresponding to two dependent variables y1 and y2 which depends on x we have 2 equations here, which are to be solved.

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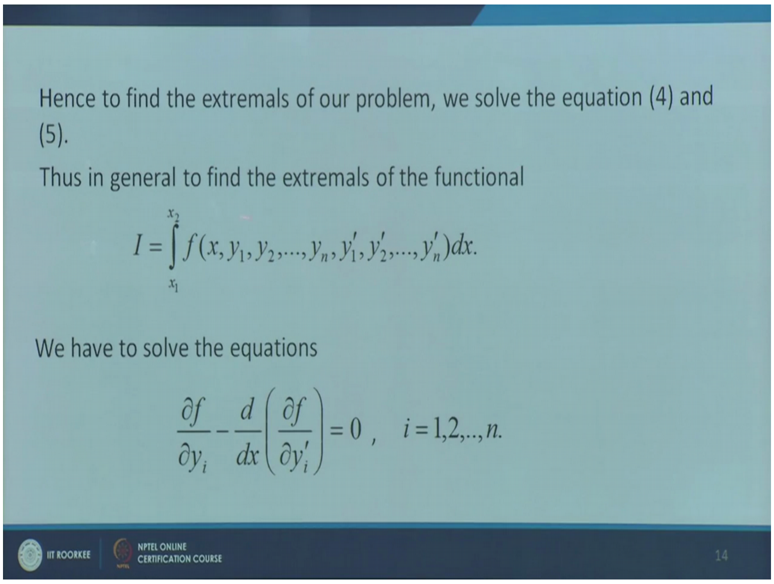

So these two equations we is have to solve to find the extremum of the problem in the case when n is equal to 2. In the general case we can it generalize it to n variables. So in the case on n variables when I is a integral x1 to x2 f(x, y1, y2, ,,,yn, y1 dash, y2 dash,…yn dash) dx, then similarly we can say we shall have to solve n equations, delta f over delta yi minus d over dx delta f over delta yi dash equal to 0, where i is runs from 1 to n, 1,2,3 and so on up to n. in our next lecture we shall take examples on this Euler's equation (in the) when the number of dependent variables are several and they depend on a single independent variable. So with this I would like to conclude my lecture. Thank you very much for your attention.