Integral equations, Calculus of Variations and their Applications
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Lecture 44
Euler's equation: A Particular Case and Geodesics

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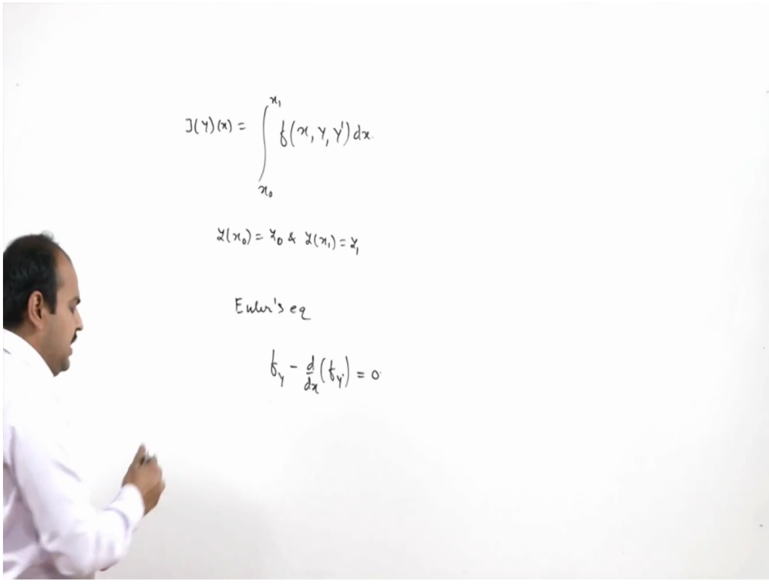

Hello friends welcome to the today's lecture. In previous class if you remember we have discussed the this kind of functional, let us call it J of y of x is equal to say x not to x1, f of x, y, y dash, d of x such that y of x not is equal to y1 and y of x1 equal to y of x not equal to y not and y of x1 is equal to y of 1 and we have discussed the Euler's equation which gives the necessary condition that the curve y is an extremum for this functional Jy and it is coming out to be that Euler's equation for this is reduced to your f of y minus d by dx of f of y dash is equal to 0.

So basically it is a second-order differential equation in terms of y and so this equation represents the solution of this equation represents the curve which extremizes the given function this, okay. So in previous class we have derived this equation and we have discussed certain subcases of this, in fact we have discussed the following cases.

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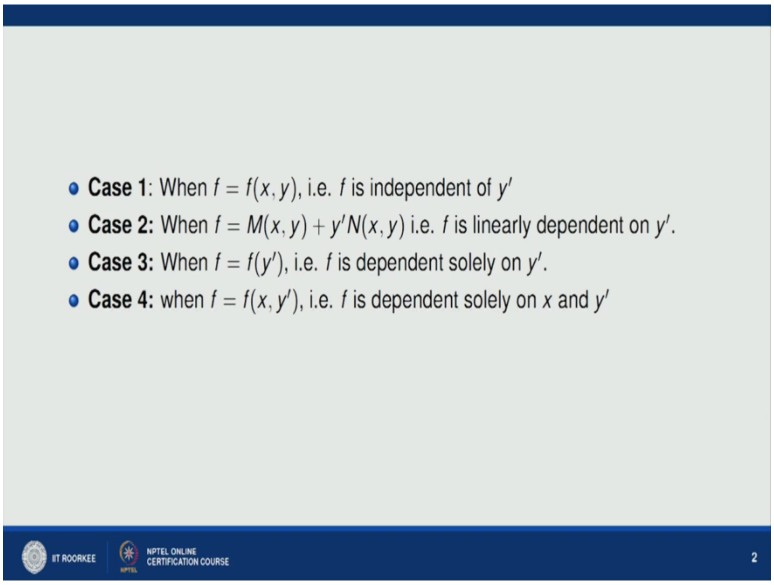

If you look at the case 1 when function f is equal to f of x, y it means that f is independent of y dash and in this case your Euler's equation reduce to your simple finite equation that is derivative of f with respect to y is equal to 0 and we have seen that in general it may not have a solution but if the finite equation which we have obtained passes through the boundary points then it will have a (()) (2:20) solutions and we have seen.

And in the second case when f is a linearly dependent function on y dash, in this case also we have discussed to sub cases, one that Euler equation reduces to My equal to N of x and if this My equal to N of x satisfied boundary condition we have a extremal otherwise if My is equal to N of x is identically equal to 0 then or we can say that My is equal to N of x is there then the extremal is basically coming out to be a constant and it does not depend on any say curve y of x.

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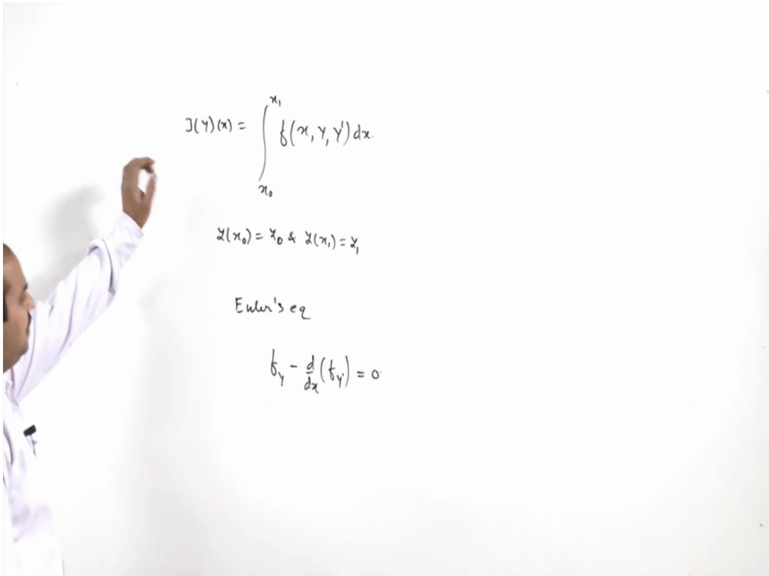

So in this case when My is equal to N of x then this extremal has this extremal of function has no meaning because it depends only on the end point.

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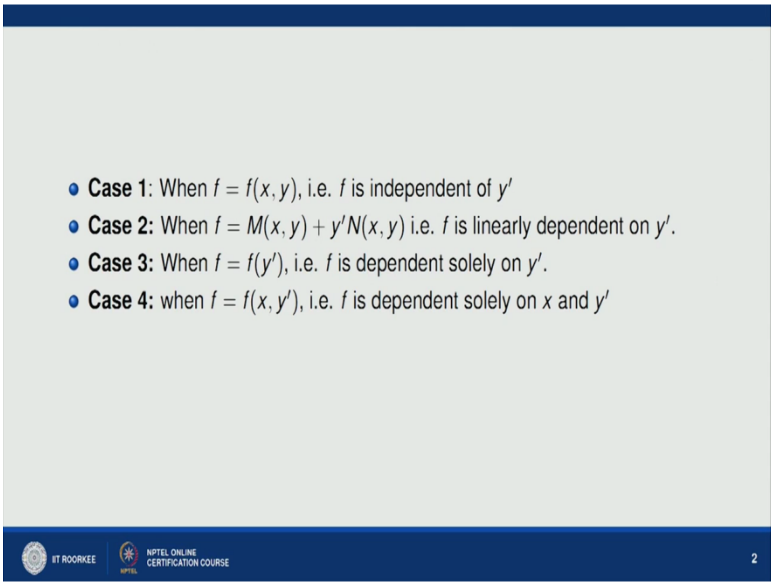

So in this case, case of 2 there are 2 sub cases and we have discussed these 2 cases, sub cases and examples also based on this and in case 3 when f is depending only on y dash, here also we have seen that the already extremal lines are coming out of the straight line is only. So this also we have discussed 1 example also and gays for when f is equal to f of x, y dash that is f is dependent only on x and y dash and here we have seen that this, in this case your Euler's equation reduce to differential equation depending on y dash and we have discussed certain cases also.

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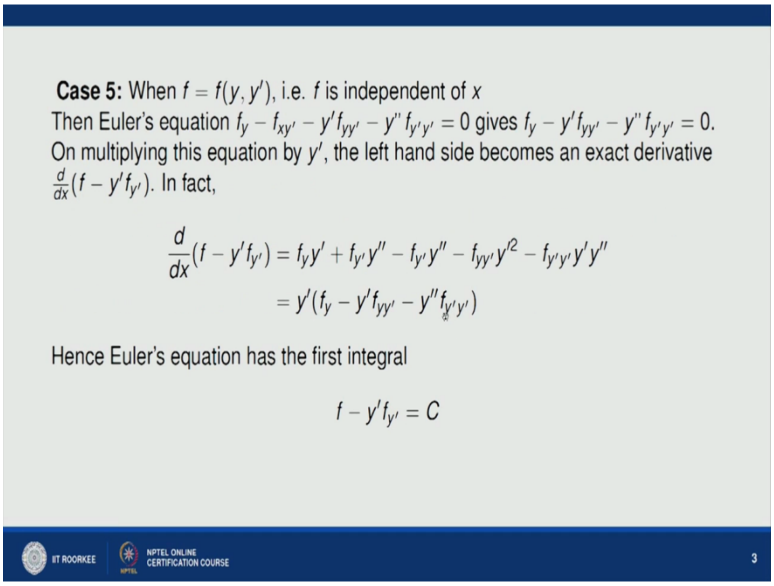

Now in this lecture will start with the case 5. So here since f is independent of x then this term f of xy dash is coming out to be 0 and this reduces to f of y minus y dash, fyy dash minus y double dash f y dash y dash equal to 0. So here if we observe carefully than if you multiply that, after multiplying this equation this differential equation by y dash this becomes an exact derivative of f minus y dash, f of y dash.

Here we can check that if we expand this form d by dx of f minus y dash, fy dash then we can say that since f is a function of y and y dash only then derivative of with respect to x will be f of y, y dash plus f of y dash, y double dash. So that is the total derivative of f with respect to x minus now we are finding out d by dx of y dash , f of y dash . Now f of y dash and we are taking the derivative of y dash with respect to d with respect to x then it is coming out to by y double dash minus f y, y dash, y dash square minus f of y dash y dash, y dash and y double dash.

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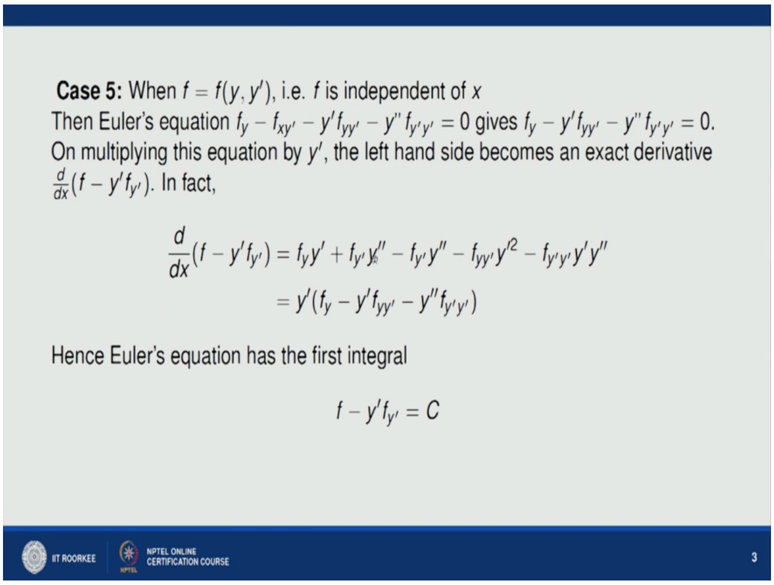

Now here these 2 terms, this and this will, these 2 terms this and this will cancel each other and this reduces to this form where we can take out this y dash common in and it is nothing but this. So Euler's equation when f is independent of x is precisely the value of this quantity is equal to 0. So it means that if this quantity is equal to 0 this implies that d by dx of f minus y dash, fy dash is also equal to 0.

So if you solve this, if we integrate this d by dx of f minus y dash of y dash equal to 0 with respect to x, we can say that we have this equation f minus y dash, f of y dash equal to C. So this is our first order differential equation in terms of y dash and this does not contain your the variable x in explicit manner. We can solve this equation with respect to y dash and with the usual method with the introduction of any kind of parameter.

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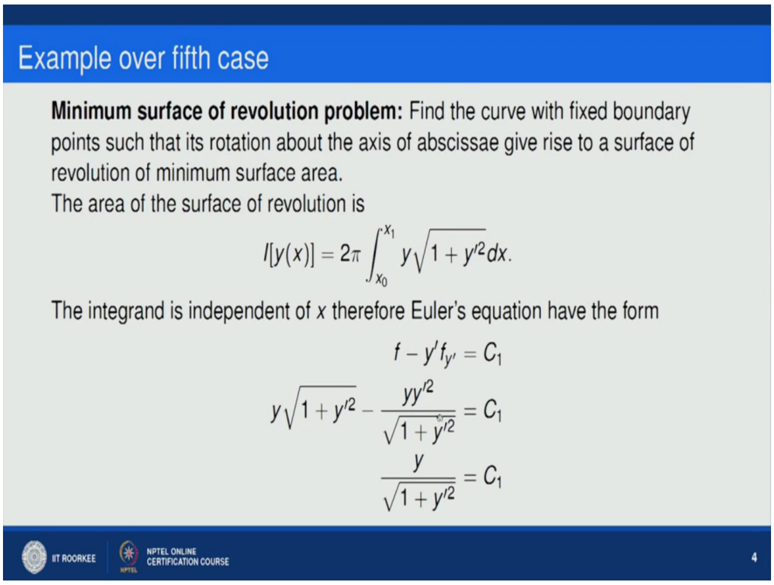

So let us take an example based on this and see how important is this case is? So the example is minimum surface of revolution problem. So here the problem is to find out the curve with fixed boundary points such that its rotation about the abscissae gives rise to a surface of revolution of minimum surface area.

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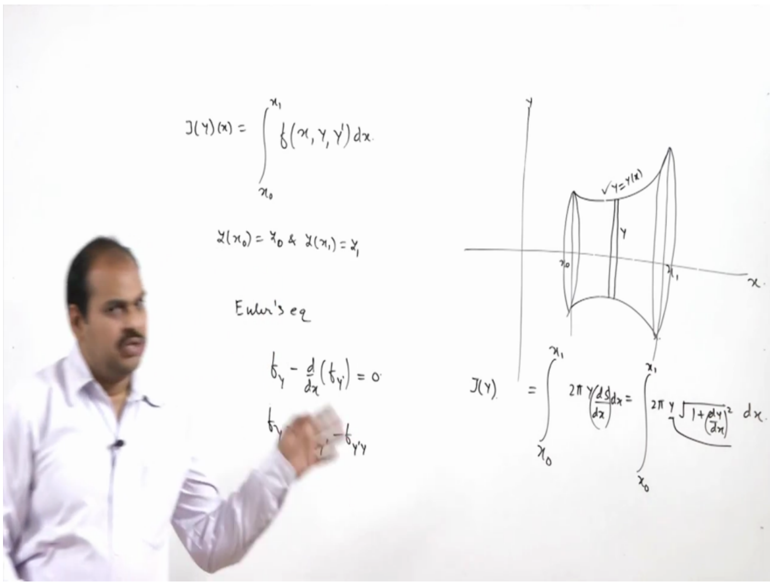

For example if I look at here, so we have this I think have already discussed this problem. Let us say we have a curve like this, this is your point x not and this is a point your x1 and if we this is y of y of x and here if we revolve this curve along this axis x then you will get a surface like this and we want to find out that under what curve, this under what curve this ye equal to y of x, the surface area of this shape is minimum.

So here to find out the surface area, it is what we try to do, we simply take a small element and this is your y of this is y. So surface area is going to be your 2pi y d of s, so you can say that your surface area between x not to x1 you can calculate it like this, x not to x1, 2pi y of ds and let us call this as a functional based on y. So J of y represents the surface area on this revolve shape and it is given by x not to x 1, 2pi y of ds.

So it means that if you change your y, your surface area is going to change and hence the value of this integral will also change. Now we want to find out say curve y such that this integral or you can say this functional is minimal. So here let us simplify this, so here we have x not x1, this 2pi y and ds you can write it ds by dx and d of dx here and this ds by dx can find out as 1 plus dy by dx whole square and d of x values of x is x not to x1.

Here, in fact you can write it here the points say s1 and s2 which is corresponding to the point when x is equal to x not and x is equal to x1, so we want to minimize this functional. So here if you look at this is nothing but a functional this is the functional independent of say x. So here your x is is not present, so here this case is coming under the sub case 5 and here in this case we have defined here I, we have discussed his (()) (9:34).

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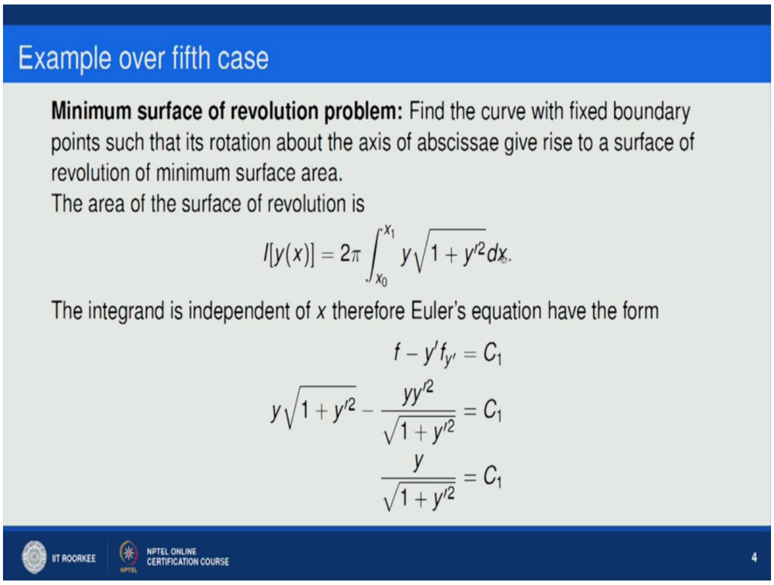

So here we have seen that the surface of the revolution, the area of the surface of the revolution is given by J of y of x equal to 2 pi, x not to x1, y under root 1 plus y dash square d of x. So as we have discussed here that this Euler equation is reduced to this that f minus y dash, f of y dash is equal to C1. So here f is y under root 1 plus y dash square.

So if you simplify this f is y under root 1 plus y dash square y dash and then if you differentiate this with respect to y dash you will get this and after simplifying you have y upon under root 1 plus y dash square is equal to C1. So basically this is a kind of a first order differential equation, we may further simplify it by squaring it on both the sides.

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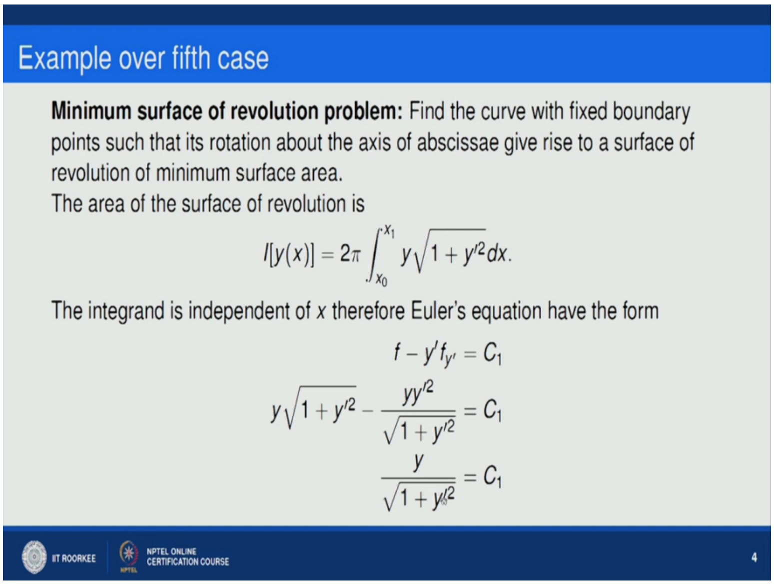

So here we have this we can solve by squaring it and then find out a differential equation in terms of y dash or a simpler method we can evolve by saying that if we take y dash in a way that let us say y dash is equal to sin of hyperbolic t, so if we take y dash as sin hyperbolic t then 1 plus sin, sin square hyperbolic t is basically cos square hyperbolic t, so here under root is there, so you can take that if y dash is sin hyperbolic t then y is coming out to be C1 cos hyperbolic t.

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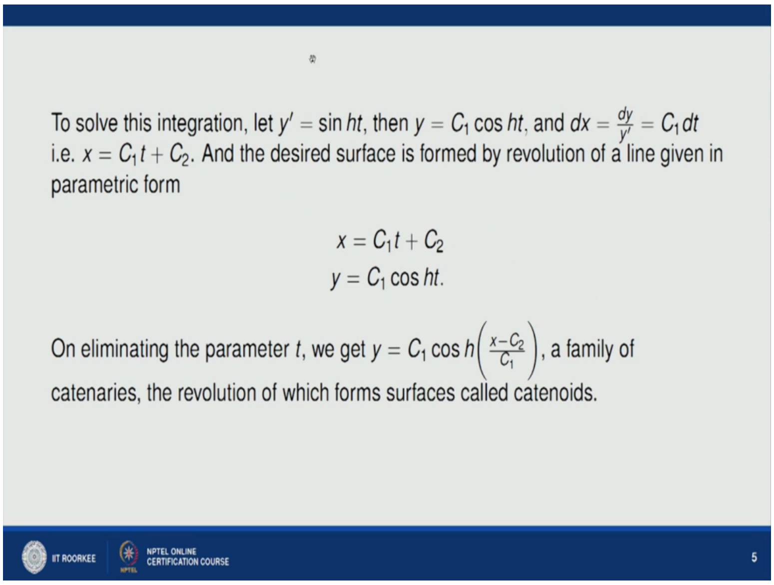

So you can write it here let y dash equal to sin hyperbolic t then y is coming out to be C1cos hyperbolic t and we can find out say variable d of x as dy by upon y dash and dy you can calculate, dy sin hyperbolic t and y dash you can simply say that it is coming out to be C1dt and when you integrate it, it is coming out to be x of x is equal to C1t plus C2. So here your curve is given in terms of parametric form that is y equal to C1 cos hyperbolic t and x equal to C1t plus C2.

So this after removing, so this is a parameter representation of a curve which minimises the surface area of revolution and we can simply find out the relation between y and x by removing the parametric t here. So you can find out the value of t here as x minus C2 divided by C1. So you're equation of the extremal curve is given as y equal to C1, cos hyperbolic, x minus C2 upon C1.

And this curve basically represent the family of catenaries and shape which we obtain by the revolution this catenaries is called get Catenoids. And here if you remember here we have 2 constants C1 and C2 that we can obtain satisfying the condition that y of x not is equal to y not and y of x1 is equal to y1. So this C1 and C2 you can obtain in terms of x not and y not and please remember here we can solve these equations, this equation y upon under root 1 plus y dash square equal to C1 by squaring and simplifying for y and y dash.

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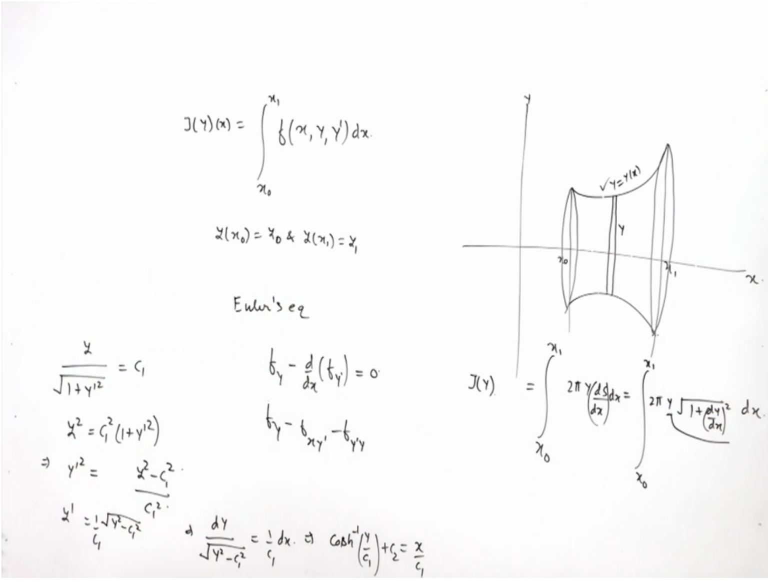

So this also you can do, in fact let us see here this is your y upon under root 1 plus y dash square. I’m just giving you a small hint how you can solve it, right? If you square it you have y square equal to C1 square 1 plus y dash square, right? And you can simply say that it is y dash square is equal to you can say that it is y square minus C1 square and divided by basically C1 square.

So here you can say that y dash is equal to under root y square minus C1 square 1 upon C1. So here this you can say that it is dy by under root y square minus C1 square is equal to 1 upon C1, d of x and if you integrate on both the sides then you will get this as what is that? This is cause hyperbolic inverse y by C1 plus C2 is equal to say x by C1. So if you simplify you will get the, that the similar kind of relation you’re getting and you can simplify and you can get the value of y, is it okay? So here this is example based on the case 5.

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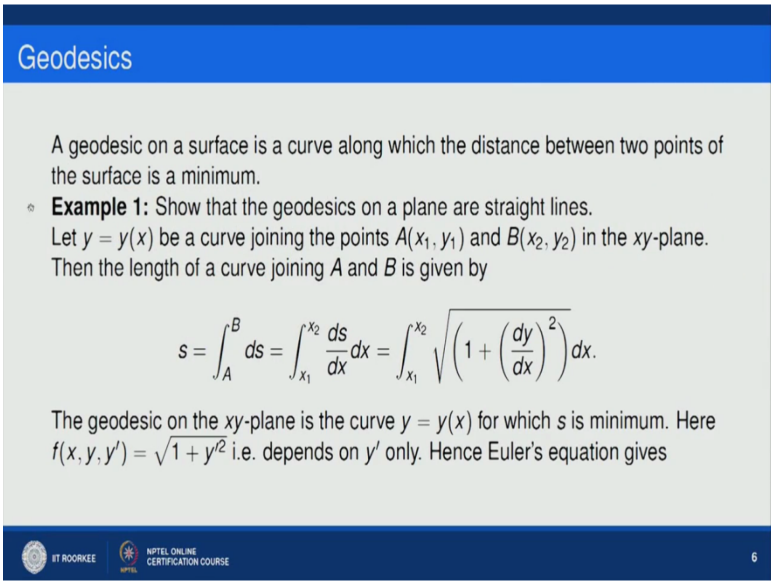

Now let us take very important example of this Euler's equation and that is the finding the curves which minimises the distance between any 2 points are even surface and there is known as Geodesics. So Geodesics you can define it like this, that Geodesics on a surface is a curve along which the distance between 2 points of the surface is minimum. So what it means?

That you have a surface take 2 points and then you can join these 2 points by some curves which are lying on your surfaces only. So out of those () curves on your surface you need to choose the curves which minimises the distance between A and B along the surface and whatever curve you will get as a minimizer of the distance that curve is known as Geodesics.

So we have seen the first example earlier also but let me famed this example here also. So first example show that the Geodesics on a plane or straight line. So here your surface is your plane. So if we have a planar surface and if you want to find out the geodesics that it is coming out to be a straight line. So how we can form? How we can solve this using the Euler's equation?

So here we have a curve which joints the point A and B where a is denoted as x1y1 and B as x2y2 in the xy plane that they can find out see length between A and B as A to B, d of s and d of s you can write it as under root 1 plus dy by dx whole square and so you can say that the functional s represent the length between the point A and B whose x coordinate is given as x1 and x2.

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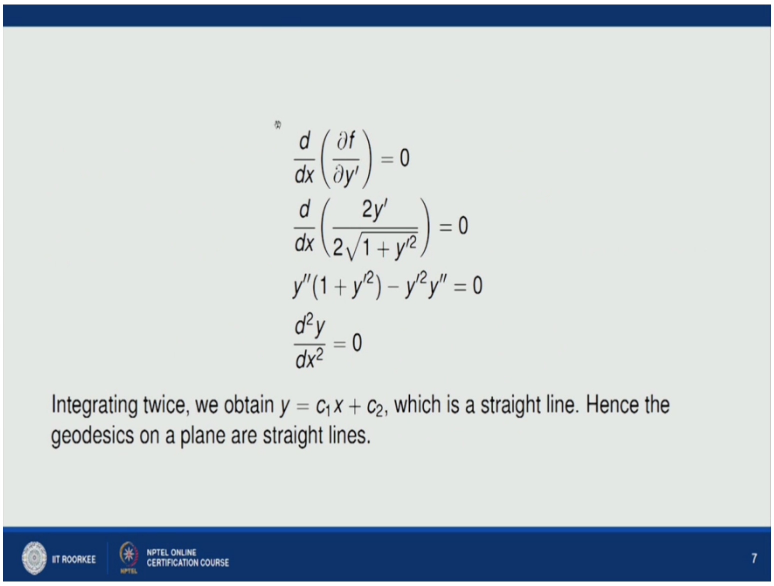

So we have to minimise this functional, right? So the Geodesics on the xy plane is curve y equal to y of x for which this functional s is minimum. So here if you look at the integrand is what? Integrand is under root 1 plus y dash square and which is independent of x and y all we can say depends only on y dash and hence we can use our subcase and we can get the Euler equation is nothing but d by dx of dabba f by dabba y dash equal to 0.

So here we can again pose this equation has dabba f by dabba y dash equal to constant which we have discussed but if you look at the value of dabba f by dabba y dash it is coming out to be 2 y dash upon 2 under root 1 plus y dash square. So when you simplify this than it is coming out to be a second-order differential equation like this but after simplifying it is nothing but only this y double dash is equal to 0.

So this is a second-order differential equation, a very simple second-order differential equation whose solution is nothing but this C1x plus C2. So here this is nothing but only a straight line and C1 and C2 you can obtain by the condition that y of x1 and y of x2 is equal to y 2. So this C1 and C2 you can obtain and you can say that this is nothing but a straight line which passes through your point A and B. So here Geodesics on a plane are nothing but straight lines. So this is something kind of a, we know from long back but here is the mathematical proof for the fact which we already know.

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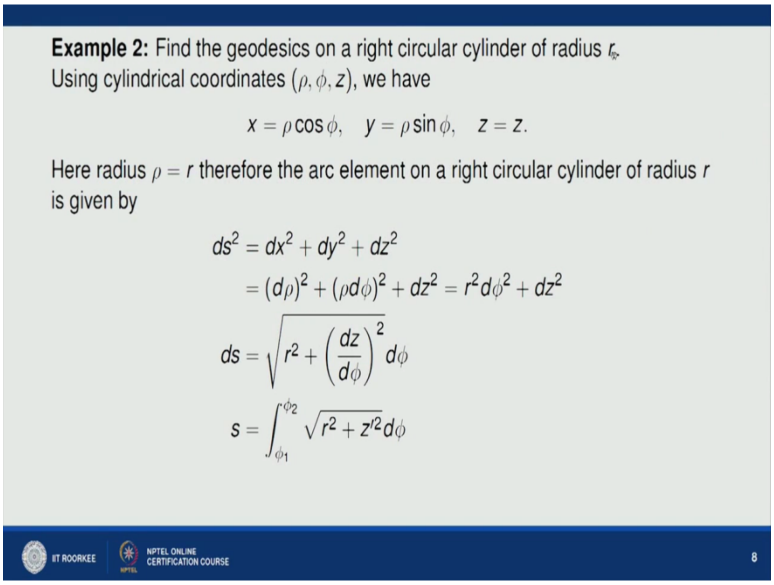

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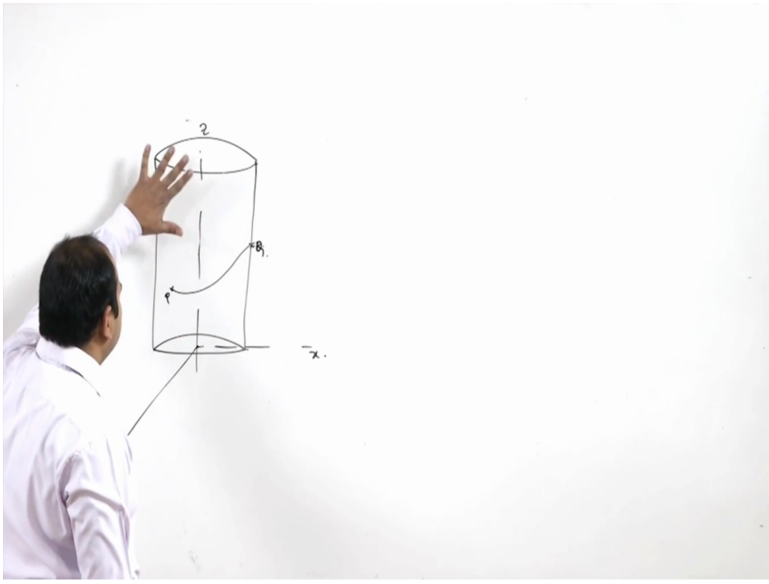

So let us move to next example which is a little bit nontrivial kind of thing and so example is that find the Geodesics on a right circular cylinder of radius r. So let me use her, so here we have a right circular cylinder and we have take just 2 points P and Q here on the cylinder and we want to we can connect this P and Q by some curves which are lying on the surface only.

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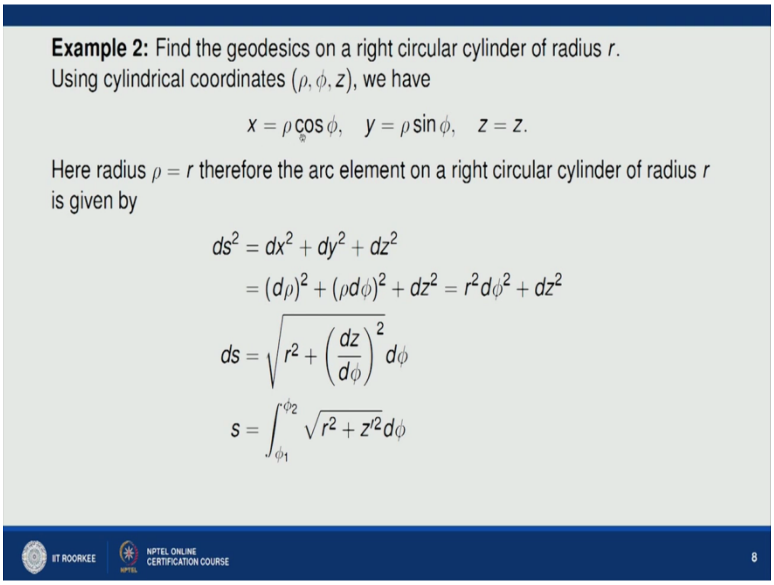

So there are several possible curves here and you want to find out say the curve which minimises the distance between P and Q which we are travelling only on this the cylinder, okay. So here we can simply say that how we can find out? So let us use cylindrical coordinates and its Rho, phi and z and here the x coordinate you can write it Rho cos phi and y coordinate is Rho sin phi as z is equal to z.

So any point on your cylinder, cylinder with radius r is given by r cos Phi, r sin phi and z. So here your P is given by the can write it r cos phi, r sin phi and the z here. Now to find out say the surface here let us use this thing that ds square is equal to dx square plus dy square plus dz square basically it is this and we want to find out say length between length of the curve of joining P and Q.

So here if you simplify this dx square dy square plus dz square you can write it in terms of say d square, let me write it here, g11 plus g22 d of phi square plus g of 33 dz square and you can find out gii is equal to say dabba x by dabba rho whole square plus dabba y by dabba rho whole square plus dabba z dabba rho whole square similarly you can write it g22 as dabba x by dabba phi whole square plus dabba y by dabba phi whole square and dabba z upon dabba phi whole square and with the help of this you can easily find out say ds square.

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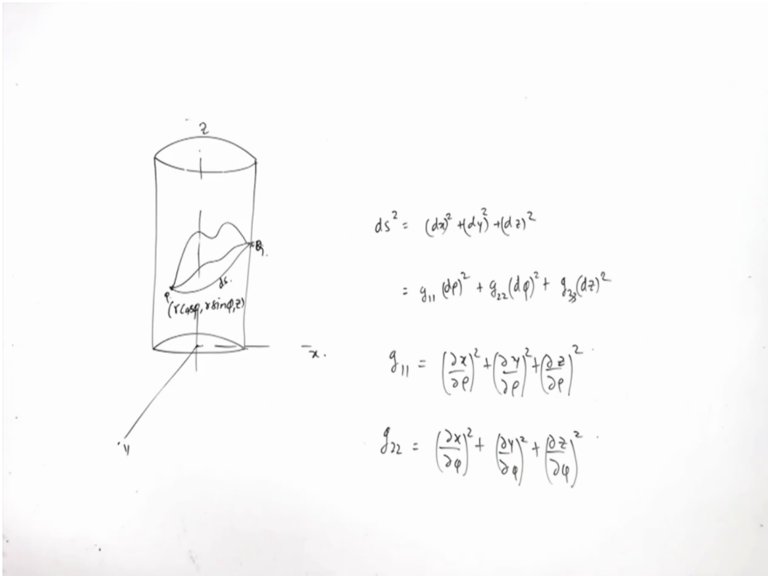

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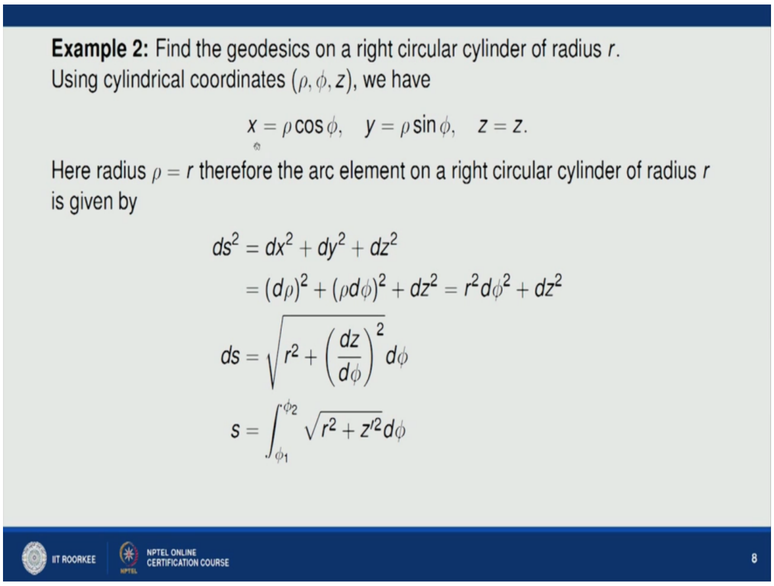

So your this thing is coming out, so for right circular cylinder of radius r your ds square is coming out ds square is equal to dx square plus dy square plus dz square and here x is given as rho cos phi, y is equal to rho sin phi, z equal to z. So using the formula which we have listed here you can find out say g11, g22 and g33 and in this case your g11 is coming out to be dabba x by dabba rho, dabba y by dabba rho and dabba z by dabba rho.

So dabba x by dabba rho is coming out to be cos phi and dabba y by dabba rho is equal to sin phi and dabba z by dabba rho is simply 0. So in this case your g11 is coming out to be this cos square Phi plus sin square Phi that is equal to 1, so g11 is coming out to be 1 here.

Similarly you can find out g22 as dabba x by dabba phi which is minus rho sin phi and dabba y upon dabba phi is your rho cos phi and dabba z by dabba phi is nothing but 0.

So when you find out g22 it is the square of dabba x by dabba phi which is nothing but rho square sine square phi plus dabba y by dabba phi square that is rho square, cos square phi when you add these 2 it is coming out to be rho square. So you're g11 is coming out to be rho square. So similarly you can find out g33, so you can say that your ds square is given as d rho square plus rho d phi square plus dz square.

Now for this your rho is fixed, so rho is given as r. So you can say that the arc element on a right circular cylinder of radius r is reduced to only r square, dphi square plus dz square because the component along this, this is simply 0 because rho is a fixed quantity. So ds square is given as r square d phi square plus dz whole square. So ds you can write it as under root r square plus dz by dphi whole square, d of phi.

So here we are assuming that z is a function of phi and it is simplified as your functional arc length is given by S equal to phi1 to phi 2 under root r square plus z as square d of phi. Now if you look at this is nothing but a function on phi and zs and here this is independent of say phi and z. So you can take the help of Euler's equation

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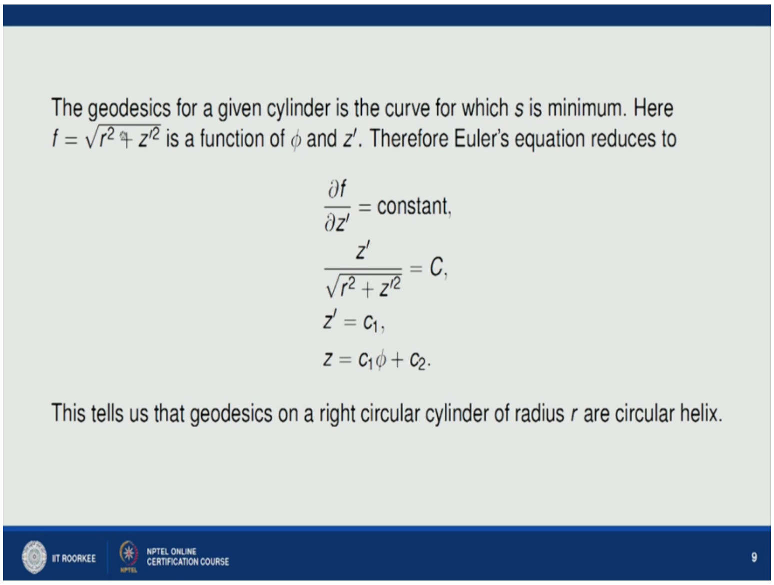

And we can say that the Geodesics for a given cylinder is the curve for which S is minimum and you can say that Euler's equation reduces to dabba f by dabba z dash is equal to constant and f is given by under root r square plus z dash square. So when you plug in this value you will have zs upon under root r square plus z dash square equal to C. Now you can simplify this.

So here we have z dash upon r square plus z dash square equal to some constant C, so you can square it and you can solve it. So after squaring it you have z dash square equal to C square r square plus z dash square. So you can simplify this, so z dash square is coming out to be C Square, r square upon 1 minus C square. So here C is constant, r is constant, so here this whole quantity is constant.

So you can write it z dash is equal to some other constant call it say z1, so when you solve this then it is coming out to be z is equal to C1 phi plus C2, please remember this z dash here is nothing but dabba z bu dabba phi, okay. So when you integrate this z dash as C1 you will get Z equal to C1 phi plus C2. Now again this C1 and C2 you can obtain with the help of the initial values that P and Q are the initial values, is it okay?

So now if you look at carefully then what is this? So here if you vary your Phi then of course your height will also change and it is coming out to be that the path which minimises the distance between any 2 point P and Q is nothing but Helix on the right circular cone. So here in this case the Geodesics on a right circular cylinder of radius r is nothing but circular Helix which is represented by z equal to C1 phi plus C2.

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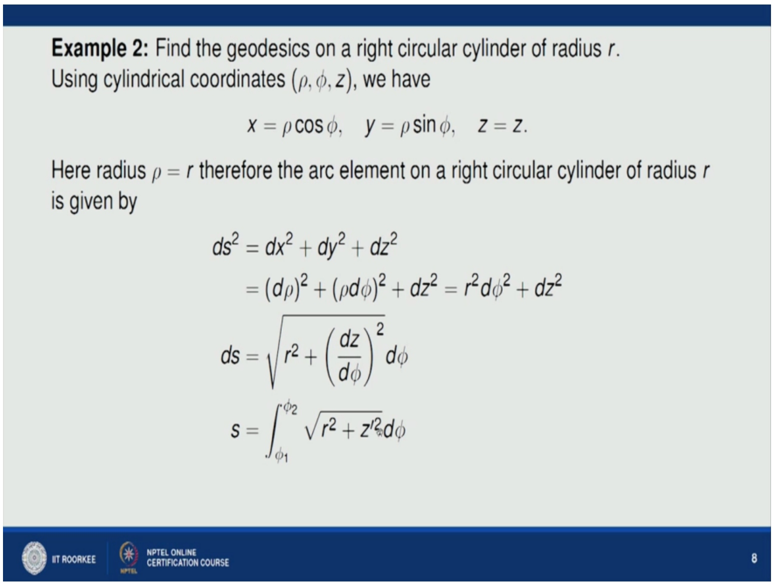

Where C1 and C2 is maybe obtained by the initial condition here, here is the initial condition that z of phi1 is equal to z1 and z of phi is equal to z2, okay. So moving on say next example.

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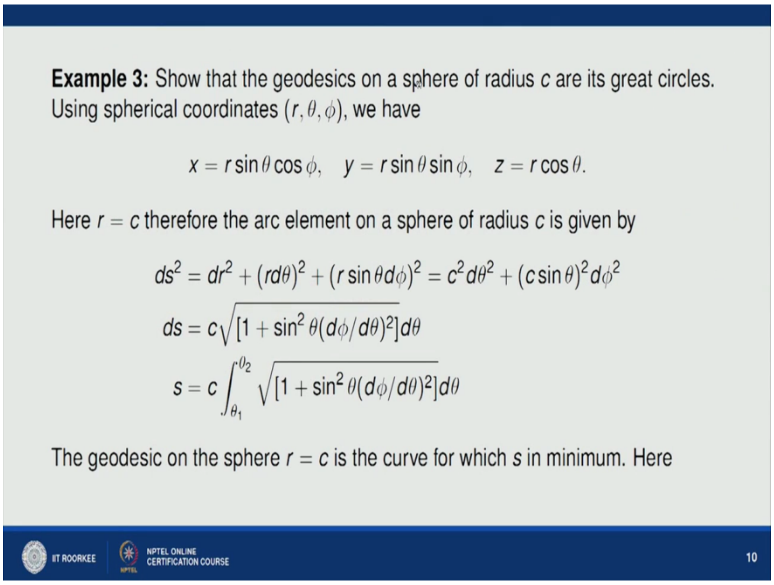

So here we want to see that that geodesics on a sphere of radius C and we want to show that it is nothing but great circle of this sphere. So here to solve this problem we use the spherical coordinates system and that is r, Theta, phi, so here in terms of r, Theta, phi the can write down your x coordinate as r sin Theta cos phi, y as r sin theta sin phi and z equal to r cos theta.

So here theta is the angle made by the radius factor z axis and phi is angle of the projection of radius factor and the angle from the x axis, okay. So here r is equal to C, so similarly as we have done in the previous case your arc element on this sphere is given by ds square as dx square plus dy square plus dz square and that you can find out as g11 dr square plus g22 d Theta square plus g33 d phi square and you can easily calculate g11, g22 and g33, formula is already given in the previous example.

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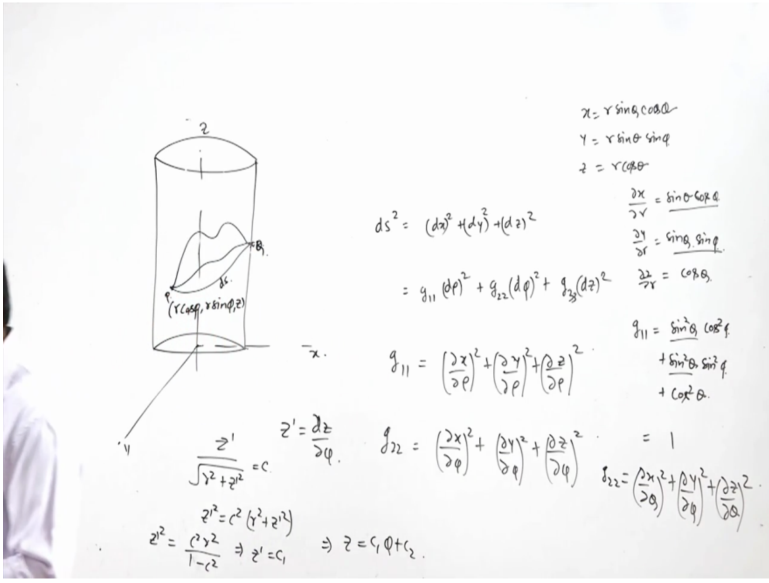

So but let us take very simple 1, g11. So g11 is what dabba x by dabba r whole square plus dabba y by dabba r whole square plus dabba z by dabba r whole square. Let me do it here, so here you can write it here your x is equal to r sin theta, cos of phi, cos of phi and y equal to r sin theta sin phi and z equal to r cos of theta, right? So here you can find out g11 here.

So g11 is dabba x by dabba r, so that is sin theta and cos phi and dabba y by dabba r is equal to sin theta and and sin phi and dabba z by dabba r is basically your cos of theta. So when you find out g11, g11 is basically square of these 3 conditions, so when you square these 2 you will get sin square theta. So sin square theta cos square Phi plus sin square theta and sin square Phi and plus cos square theta.

So here when you take sine Theta square common then this is nothing but cos square Phi plus sin square Phi that is 1 and ultimately you will get sin square theta plus cos square theta that is 1. So here in this case you can get your g11 and similarly you can find out g22 and g33. So here g22 is nothing but with respect to theta you're finding, so your g22 is coming out to be dabba x by dabba theta whole square plus dabba y by dabba theta whole square plsu dabba z by dabba theta whole square. So you can find out the value of g22 and similarly g33.

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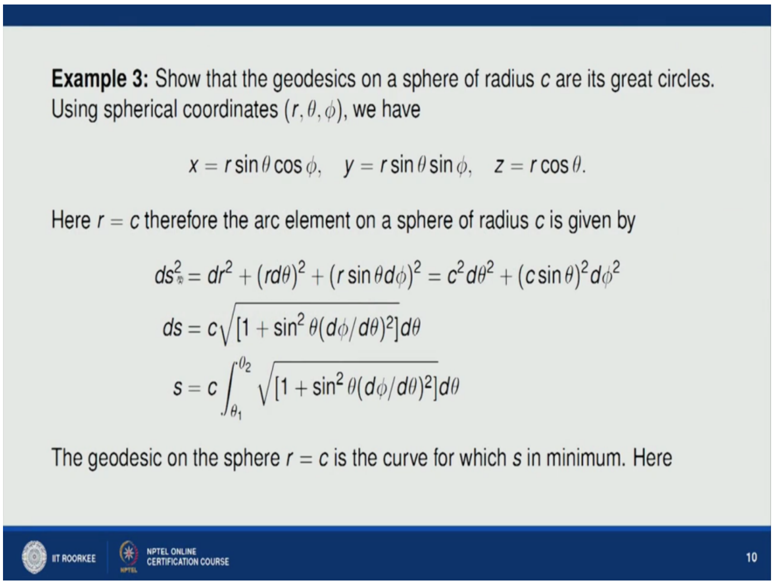

So you can say that here we have calculated and it is coming out to be that ds square is equal to dr square plus rd theta whole square plus r sin theta d phi whole square. Now here as we have taken that radius is constant and it is given as C. So this dr is coming out to be 0 and so ds square is coming you can write it C square d theta square plus C square sin square theta d phi square.

So this is the arc element on the sphere of radius C and you can simplify and you can form the functional which gives the arc length between 2 given points A and B and you can say that S is given as C theta 1 to theta 2 under root 1 plus sin square theta d phi by d theta whole square d theta. So the Geodesics on the sphere r equal C is the curve for which this functional S is minimum.

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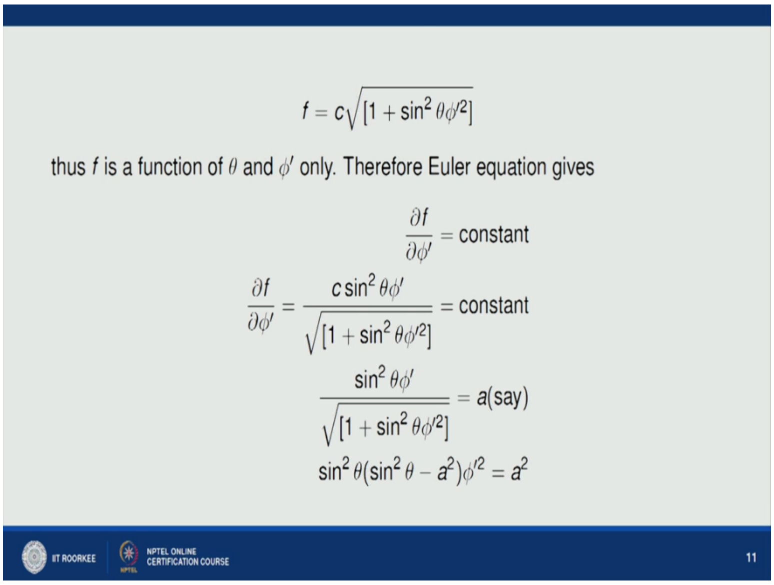

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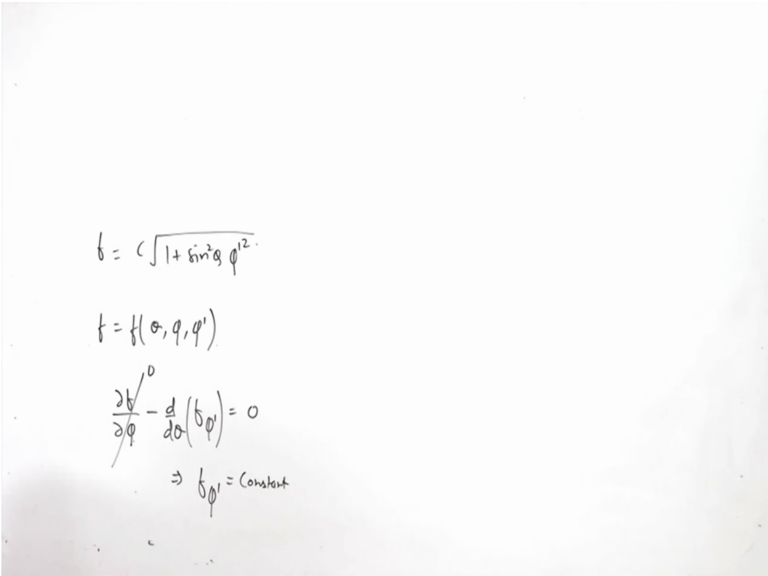

So here your f is nothing but C under root 1 plus sin square theta, phi dash square. So let me write it here your f is your C under root 1 plus sin square Theta and phi dash square, so Phi dash square. So here you can say that f is a function of say your site and phi and phi dash. So here now this is independent of your phi, so here your Euler equation for this integrand is basically what dabba f by dabba phi minus d by d theta and f of phi dash equal to 0.

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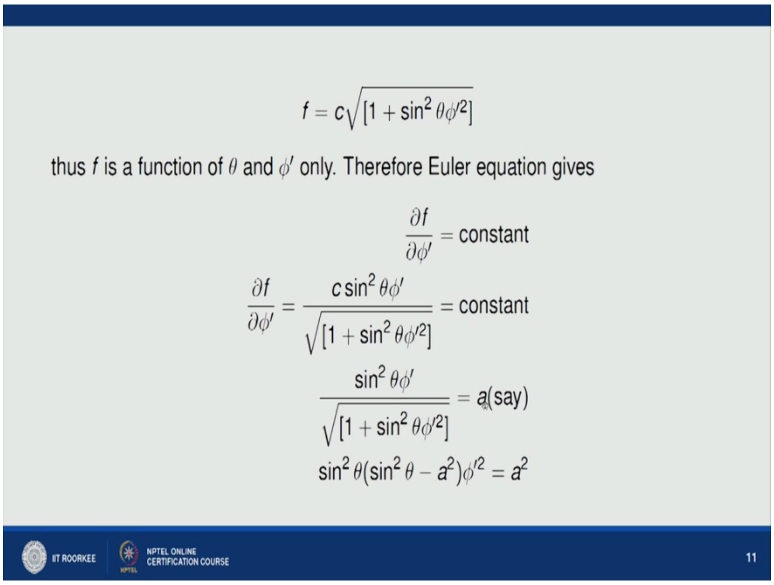

Now since this integrand is independent of Phi, so this term is simply 0 and you can simply write this as f of phi dash is equal to constant. So here we are using this and you can say that Euler equation reduces to dabba f by dabba phi dash is equal to constant. So f is given you can find out say value of dabba f by dabba phi dash is coming out to be C sin square theta, phi dash divided by 1 plus sin square theta, phi dash square equal to constant.

Now let us take this constant as sum a and you can simplify this so when you, this is constant divided by C is your a. So when you simplify this you can square it and simplify for phi dash square, so Phi dash square is coming out to be a square upon sin square theta and sin square Theta minus a square.

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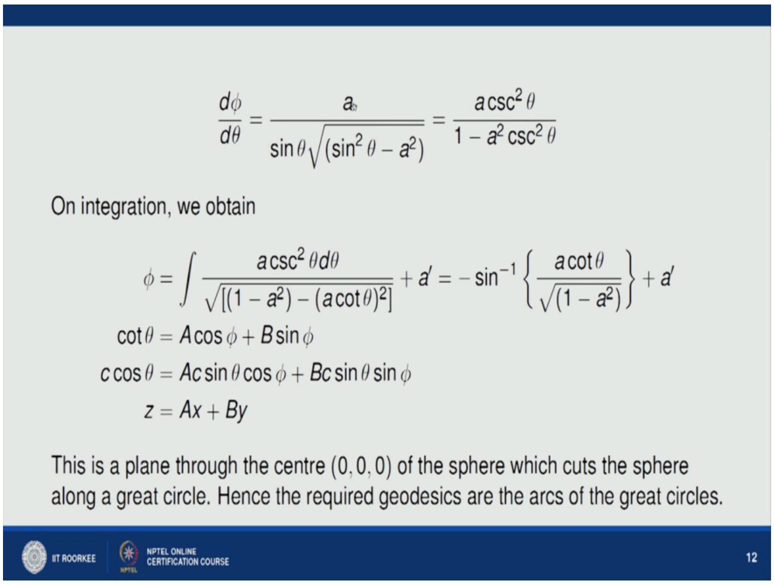

So that is this, so d phi by theta is given by a divided by sin theta under root sin square theta minus a square and which is further simplified as a cosec square theta divided by 1 minus a square, cos square Theta. So now you integrate this, so d phi by 1 and this a cosec square Theta upon 1 minus a square, cosec square theta, d theta. So when you integrate will get Phi equal to integrand of this.

Now to find out simplifying this, what you can do, you can write cos square Theta as 1 plus cot square Theta and you can write it and this is simple integral and here you can take a cot Theta as some variable and you can simplify this integral as minus sin inverse a cot theta upon under root 1 minus a square and plus a dash where a dash is integration constant. So here when you simplify this, so phi is equal to minus sin inverse, a cot theta upon under root 1 minus a square plus a dash. So you take a dash here and take the sin on both the sides, you will get a cot theta equal to under root 1 minus a square and let me write it here this is quite clumsy here.

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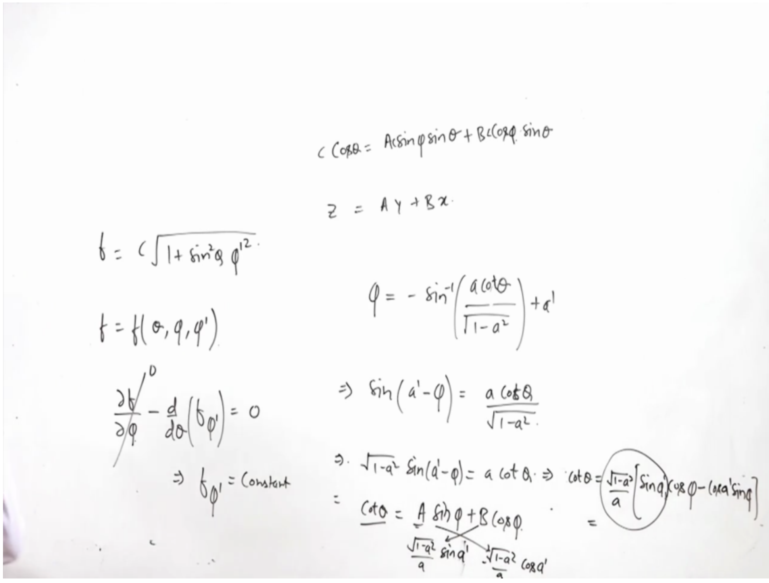

So let me write it here, this is your phi here equal to minus sin inverse a cot theta upon under root 1 minus a square and here we have a dash. You can write it here that it is a sin of you can take this as you take minus here and sin of a dash minus phi is equal to a, this is cot theta , this cot, this cot theta, you can take this as a cot, cot theta is under root 1 minus a square.

So here you can simplify this and you can multiply by 1 minus a square here also, 1 minus a square sin of a dash minus phi equal to a cot theta. So here many simplify using the formula sin of a, a minus b you can write it here you can take a also. So here you can write it cot of theta equal to you can write it some constant of sin of phi plus B of cos of phi.

In fact this is what, this you can simplify as a cot Theta is equal to under root 1 minus a square and you can write it here sin of a dash cos of phi minus cos of a dash, sorry, cos of a dash sin of phi, right? So here you can simply say that you can divide it by a also and you can write it here a and this, so here you cot theta and you can call this as your a, so a is nothing but under root 1 minus a square divided by a and your sin of a dash.

Similarly your B is nothing but under root 1 minus a square upon a into cos of a dash, is it okay. So your b is given by this, so by taking A and B these constants you can write it cot of theta is equal to A sin phi plus B cos phi and you if you simplify you can take cos of theta is equal to this a sin phi sin theta plus B cos theta, ssin of theta and this is what, this is cos of phi .

So this is cos of phi and if you multiply both side C then this is what? This is your z here and this is your y here and this is your x here, there is something wrong here, yes this is x, right? So here we can say that it is z equal to Ay plus B of x. Here I have to look at here the coefficient of sin phi is a and if you look at this is interchanged, so your A is this and B is this so that you can simplify.

So it is just a constant A is given by minus under root 1 minus A square upon A, cos of A dash but anyway you can always write it cot of theta equal to A sin phi plus B cos phi where A and B are suitable constants and when you simplify your equation is coming out to be some z equal to Ay plus B of x.

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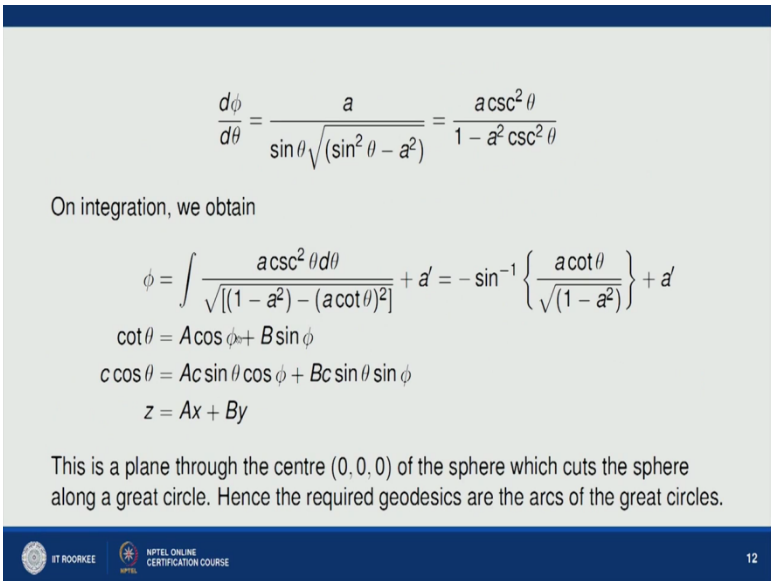

So here in calculation if I look at here, sorry, cot of theta is equal to A cos plus B sn phi and when you multiply by C it is coming out to be Ac sin theta cos phi plus Bc sin theta sin phi and which is nothing but A plane passing through the origin 000, right? So we can say that the curve which minimises the extreme minimises the functional is basically a plane through the centre of this sphere and which cut the spear along a great circle.

So w here we can say that the required Geodesics are the arcs of the great circles, so here we stop here but before stopping here you can give 1 point on this example, you can simplify this and whether it is z equal to Ax plus By or it is given as Ay plus Bx but anyway it is a plane passing through the sphere. So here we stop and we will meet in next lecture where we can discuss the extremal or say Euler equation for in place of curve, it is the Euler equation for the sphere, okay.

So here we stop and we will meet in a next lecture, thank you for listening us, thank you.