Integral equations, Calculus of Variations and their Applications

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Lecture 43

Euler's Equations: Some Particular Cases

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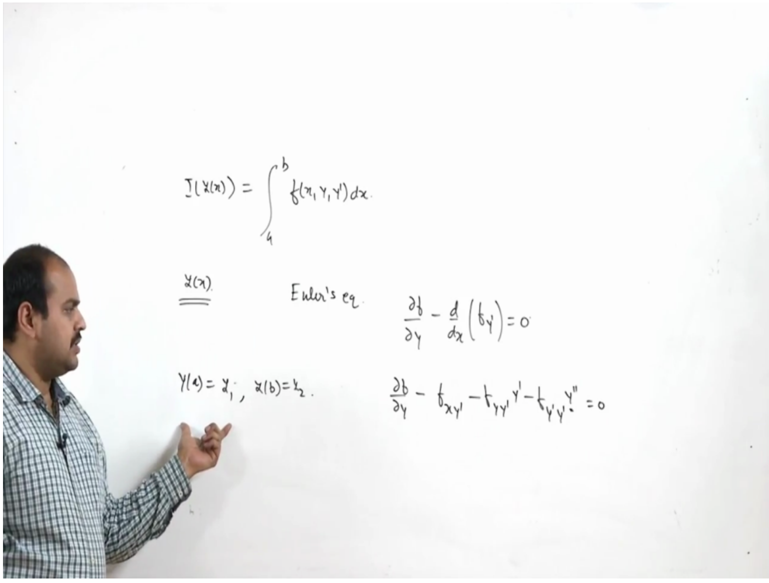

Hello friends welcome to the today's lecture and if you remember in previous lecture we have discussed this that is we have a functional defined as this I of y of x is equal to a to b, f of x y, y dash, d of x here and if y of x is a curve on which this I, this functional achieve either maxima or minima or we can say that on this curve y of x this curve achieve the extremum then the Euler's equation we have derived in last lecture that is dabba of f by dabba of y minus d by d of x, f of y dash is equal to 0.

So it means that if y of x is an extremal of this then it must satisfy this Euler equation and if you simplify this that is nothing but dabba f by dabba y minus here I assume this as f of x, y dash minus f of y, y dash, y dash minus f of y dash, y dash and y double dash. So here I'm assuming that f of y dash is a function of x we differentiate with respect to x.

Here we assume that f of y dash is a function of y, so we differentiated with respect to y and y dash. Here we take the part of f of y dash as depending on y dash. So f of y dash, y dash, y double dash is equal to 0. So it means that if y of x is extremal it must satisfy this (dif) differential equation which is a second-order differential equation in the variable y and we want to find out say solution.

Here please note down that this y of x is a solution extremal of this provided that y of a equal to 0, y of a equal to y1 we have assumed and y of b equal to y2 we have assumed. So this is already given, so it means that find out the extremal, we need to find out the solution of this second-order differential equation satisfying the boundary condition. So in general this is not always having a solution.

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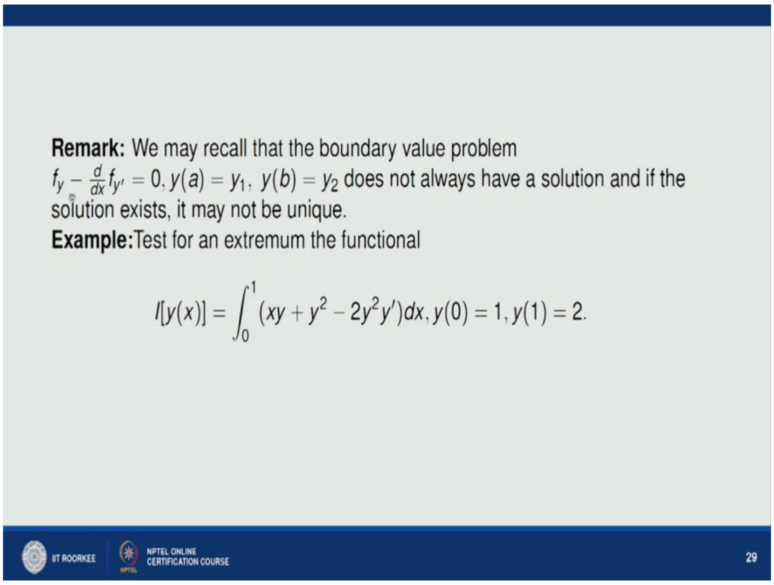

So here we may remark that, we may recall that the boundary value problem fy minus d by dx, f of y dash equal to 0 satisfying the boundary condition that is y of a equal to y1 and y of b equal to y2 does not always have a solution and if the solution exists, it may not be unique at all. So let us take one example and try to see whether this remark is how much it is true?

So for that let us take the extremal of the function I of y x equal to 0 to 1, xy plus y square minus 2 y square y dash, d of x with the boundary condition that y of 0 is equal to 1and y of 1 is equal to 2, so these boundary conditions are already listed in the given problem.

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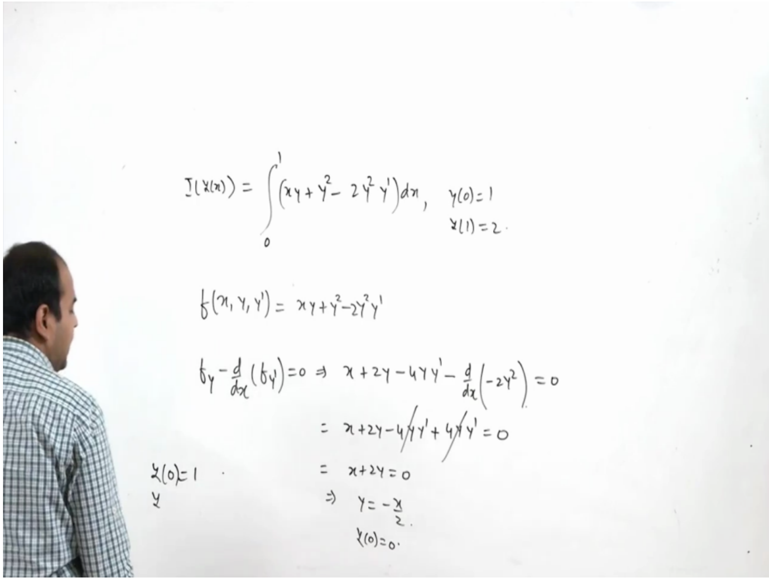

So let us take this particular example 0 to 1, xy plus y square minus 2 y square y dash, d of x, y of 0 equal to1 and y1 equal to 2. So we have written down the functional x this I of y of x is equal to 0 to 1, xy plus y square minus 2 y square y dash, d of x with the boundary condition that y of 0 is equal to 1and y of 1 is equal to 2. So here from this it is clear that f of x, y and y dash is equal to xy plus y square minus 2 y square y of dash.

So to find out a differential equation which a extremal function satisfy we recall the Euler's equation that is f of y minus d by d of x, f of y dash is equal to 0 which gives us differentiation with respect to y, it is x plus 2y minus 4y, y dash minus d by dx of fy dash, so d by dx of fy dash will be that is minus 2 of y square, so which is nothing but is equal to 0.

This simplifies x plus 2y minus 4 y, y dash minus if you write it here then it will be this plus sign 4y and y dash equal to 0, so these 2 will cancel out and you will have that x plus 2y has to be 0. So this implies that y has to be minus x by 2, so it means that if we have an extremal of this functional then that extremal function y of x must satisfy this Euler's equation or we can say that your extremal function must be of this form that y of x is equal to minus x by 2.

Now if you look at the boundary condition then y of 0 equal to 0 and here if you look at the value of y of 0 is coming out to be 0. So it is not, no y of 0 is equal to 0 but if you look at the boundary condition, the boundary condition is y of 0 is equal to 1. So it means that the functional which satisfies the Euler equation is not satisfying the boundary condition given with the functional.

So it means that this y of x equal to minus x by 2 though it satisfied the Euler's equation but it is not a solution of the boundary value problem, it means y of 0 equal to1 is not satisfy in this particular case. So here we say that this functional does not have any extremal, right? Because it is not the solution of boundary value problem, it maybe a solution of only equation but not satisfying the boundary condition. So in this case we say that this functional does not have any extremum, okay.

Now moving on to the next cases, let us consider the case 1 here we are considering the case depending on the form of f, so here since f involves x, y and y dash, so here we are considering the case that f is written only in terms of xy in terms of y dash, in terms of y and all these cases we want to consider and we try to see that simplified version of Euler's equation, this is just us sub cases of this Euler's equation.

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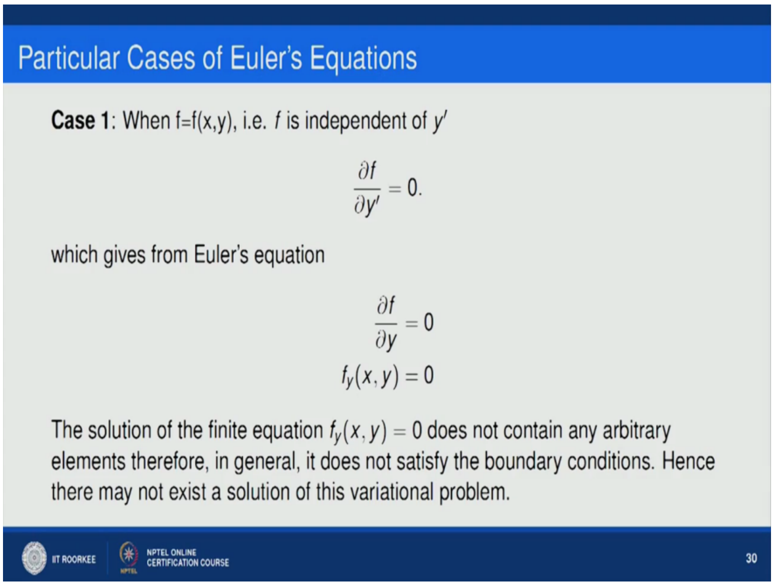

So first let us consider the case 1 when f is equal to f of x, y it means that f is independent of y dash. So if we take f as independent of y dash it means that dabba f by dabba y dash is equal to 0. In this case your equation Euler's equation which is given by dabba f by dabba y minus d by dx of f y dash. Now f y dash is equal to 0, so second of Euler equation will be 0.

So it will reduce only in the first that is dabba f by dabba y equal to 0 and if you solve this, if you integrate with respect to y you will say that you will get certain functions f. Now here this fy, x, y equal to 0 the solution of this finite equation does not contain any arbitrary element therefore, why? Because dabba f by dabba y equal to 0 means that f is constant with respect to your variable y and it is already given that f is independent of y dash.

So it means that this f is basically independent of both say y and y dash and here it may happen in general that it does not satisfy the boundary condition. So hence there may not exist any solution of this Variational problem also. It may happen a solution exist only in the condition that the solution of this finite equation that is fy, xy, equal to 0 pass through the boundary points. So in that case only we may have a solution otherwise it will not have a solution.

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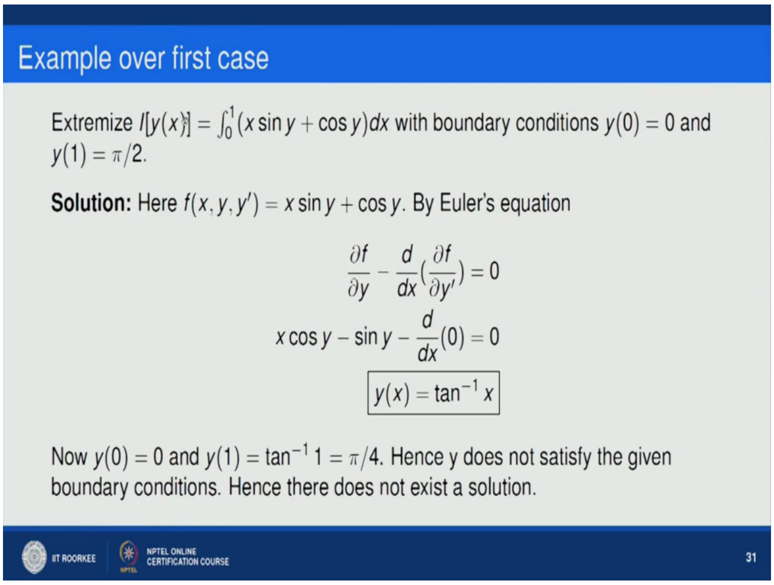

So let us move to one example based on this and try to see what we are trying to explain here. First let us look at here the extremize of I of y of x equal to 0 to 1, x sin y plus cos of y, d of x with the boundary condition, y of 0 equal to 0 and y of 1 equal to pi by 2. So here if you look at the f of x, y, y dash is given as x sin y plus cos of y. So here f of x, y, y dash is equal to x sin y plus cos of y. So it is independent of y, y dash.

So by Euler's equation this term is simply vanish, so we will have dabba f by dabba y is equal to 0. So calculating dabba f by dabba y it is nothing but x cos of y minus sin of y is equal to 0. So it is simplify it is coming out to be that y of x is equal to tan inverse x. Now so we are able to find out the solution of Euler's equation, now coming onto the boundary condition.

So boundary condition is y of 0 equal to 0 and y1 equal to pi by 2. So if you take that x equal to 0 then tan inverse 0 is 0 but if you take x equal to one then tan inverse one is equal to pi by 4. So here the solution does not satisfy the second boundary condition that is y1 is equal to pi by 2. So here for this particular problem we don't have any solution or they can say that y of x equal to tan inverse x is not a solution of this is not going to extremize the functional with the boundary condition.

But if you look at here if we have y1 equal to pi by 2 is replaced by say pi by 4, so in that case the extremal y of x equal to tan inverse x pass through the boundary points means 00 and 1 pi by 4, so in that case when the functional comes with the boundary condition defined at 1 as pi by 4 in that case we can say we have a solution otherwise in this particular case we don't have any solution. So in the case when f of x, y, y dash is independent of y dash in that case in general they may not be a solution.

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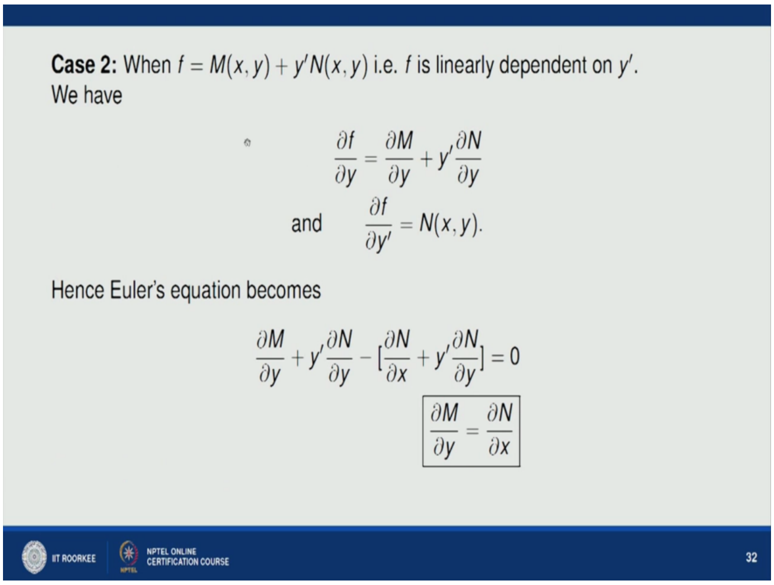

Now moving on to the second case, here we are assuming that f is a linear function in terms of y dash or we can write that f is equal to M of xy plus y dash N of x,y. So in previous case we have seen that N of x,y is coming out to be 0 only. So here we say that it may happen that f is not say independent of y dash but it is a linearly dependent on y of dash.

So in this case if you find out say Euler's equation then calculating dabba of y, dabba y which is nothing but dabba M by dabba y plus and y dash dabba N by dabba y, okay. So and dabba f by dabba y dash is calculated as N of x, y. So if you look at the Euler's equation then Euler equation is what? Dabba f by dabba y minus d dx of dabba f by dabba y dash. So we can say that dabba f by dabba y is this dabba M by dabba y plus y dash dabba N by dabba y minus d dx of N x,y.

So d dx of N x,y will be dabba N by dabba x plus y dash dabba N by dabba y is equal to 0. Now if you simply this term and this term will be cancel out. So this term and this term will be cancelled out and it reduces to dabba M by dabba y minus dabba N by dabba x is equal to 0. So it means that dabba N dabba y is equal to dabba N dabba x is the form of Euler's equation.

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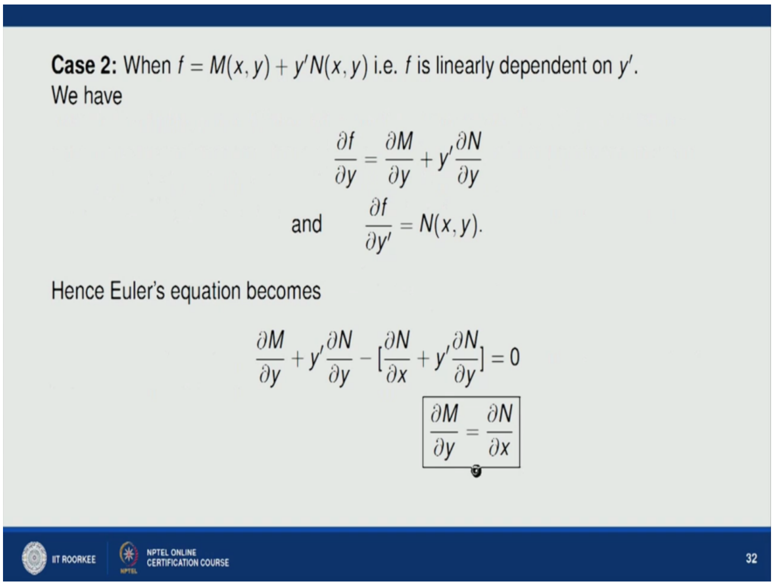

Here we have seen that if Euler equation gives us that the condition dabba M dabba y equal to dabba N by dabba x and we can say that this is not a differential equation rather than it is a finite equation. Now it may happen that this curve which we have achieved is an application of Euler's equation may not satisfy the boundary conditions if it is not satisfy and the boundary condition we simply say that they Variational problem does not have any solution in the class of continues function.

But if this dabba M by dabba y is unequal to dabba N by dabba x or you can say that this My minus Nx is unequal to 0, then the expression then the Euler equation is ideally equal to ideally satisfied and in this case this equation Mdx plus Ndy isn't exact differential and we can write it I as xnot to x, M plus N dy by dx, dx and it is nothing but x not to x1 Mdx plus Ndy and we already know from the theory of coordinate differential equation that Mdx plus N dy is an exact provided My minus Nx is equal to 0.

So in that case it is an exact derivative of some function and we can say the I, the value of I is given as the value at the end points and in that case whatever path you will choose your value of I is coming out to be a constant value and in that case your Variational problem becomes meaningless.

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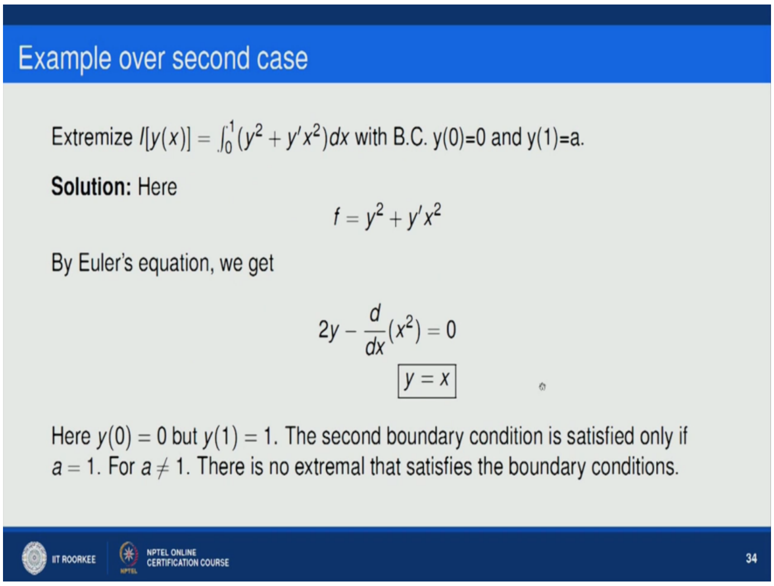

Let us try to understand the theory, let us understand more about this theory by taking a particular example. So let us take this example here we take functional I of y of x is equal to 0 to 1, y square plus y dash, x square d of x with the boundary condition that y of 0 equal to 0 and y of 1 equal to a. So here f can be written as y square plus y dash x square, now if you look at it contains all the terms x, y, y dash but if you look closely then you can say that f is linearly dependent on y of dash.

So if you write down the Euler's equation, Euler's equation is f of y, so f of y is to y minus d by dx of f of y dash which is nothing but x square. So we can simplify and you can write it that the solution of this Euler equation is nothing but y equal to x. So it is not a differential equation but it is just a finite equation. Now let us look at the boundary condition now. So if you look at the boundary condition y of 0 is satisfied but if you look at the second condition that is y of 1 equal to 1 here which is not matching with the given boundary condition that is a.

So if a equal to 1 we can say that y equal to X is a extremal of this functional, in fact this is going to be a minimal curve for this functional but if a is not equal to 1 we can say that we don't have any extremal for in this case. So we can say in case of a equal to 1, y equal to x, basically minimize this functional but if a is not equal to 1 we say that we don't have any extremal at all.

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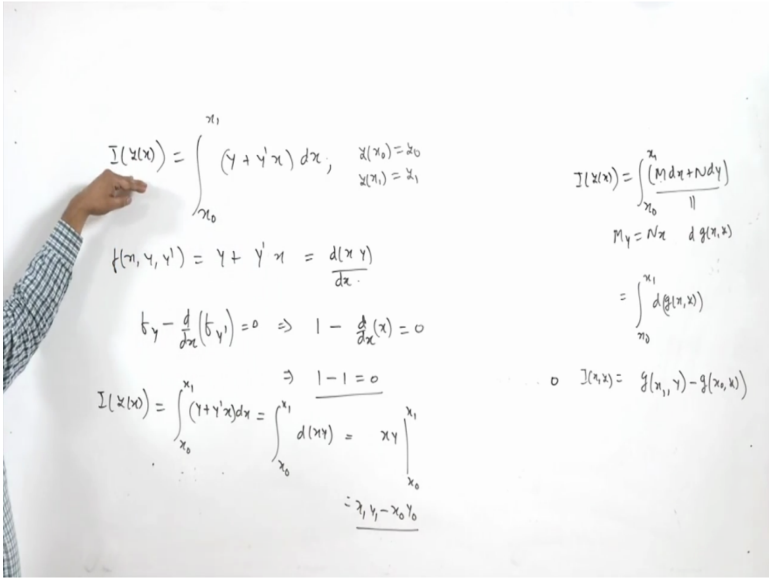

Now let us consider the other example for which this Euler equation gives you an identity. So let me write it here, so here we want to consider the case when My is ideally equal to N of x. So let us consider a function like this I of y of x equal to say x not to x1. Let us take the simple example say y plus y dash, x, d of x, okay. So this is a simple example, now in this case your f of x, y, y dash is given as y plus y dash, x y dash into x.

So here if you find out say Euler equation, Euler equation is fy minus d by dx of fy dash which is equal to 0. So this implies that f of y is simply 1 minus d by dx of y dash that is simply x. So d by dx of x equal to 0 and if you look at this is nothing but 1 minus 1 is equal to 0. So here in this case we can say that your My is unequal to N of x. So in this case if you look at, this is what? This is nothing but d of x of y with respect to d of xy, with respect to d of x.

So in this case your I of y of x is nothing but x not to x1 and y plus y dash x t of x is written as x not to x 1, d of xy and N is nothing but xy evaluated at x not to x1, so this is and here please remember here the condition is also given that let us say y of x not is equal to y nopt and y of x1 is equal to y1. So using these boundary conditions this can be written as say x1, y1 minus x not, y not. So it means that this is just a constant value.

Now here please remember that here we can take any curve, any curve which satisfy this condition y of x not equal to y not and y of x1 equal to y1 and if you take any curve your value of the functional is coming out to be the this constant value that is x1, y1 minus x not, y not. So here the value of I of y I of y of x is fixed and it is equal to this. So here the question of say extremizing the function is now meaningless because this integral is independent of the path between joining these 2 points x not, y not and x1 and y 1.

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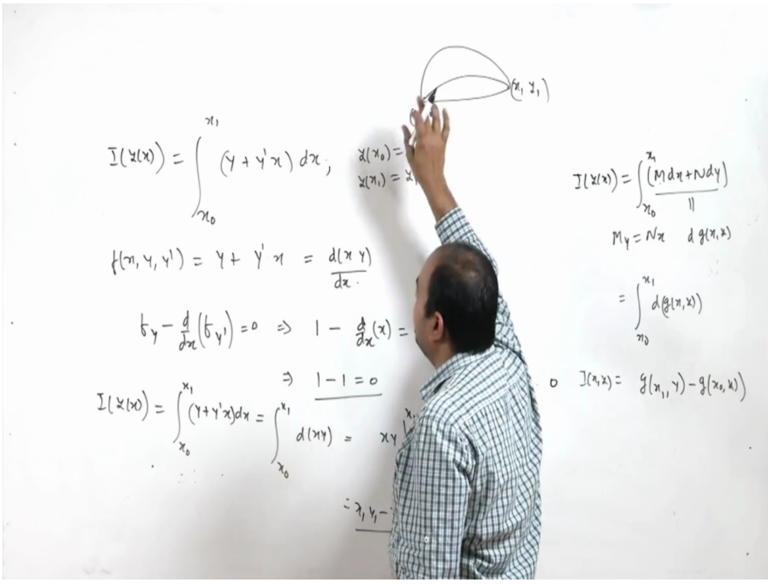

So it means that this is a point x not, y not and this is a point x1, y1, you can take any path, any path whether you take this path or that path or whatever path the value of this I of yx is nothing but the value given here that is x1,y1 minus x not y not. So it means that every curve is extremal. So we can say problem of extremum is meaningless here. Now going into third case, here we are assuming that f is depending only on y of dash.

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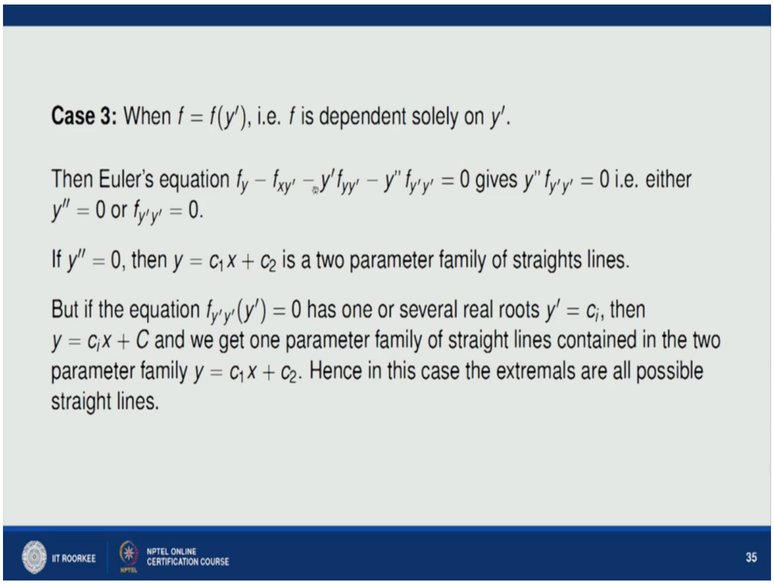

So in this case your Euler's equation f of y, f of x y dash minus y dash f of y, y dash minus y double dash, f of y dash y dash equal to 0 reduces to because you f depending only on y dash. So it is independent of y independent of x, so here we can write this will reduce to only this term that y double dash f y dash y dash is equal to 0. So either this term will be 0 or this term will be 0.

So if y double dash is equal to 0, we can solve this differential equation then we can get y equal to C1x plus C2x and it is a 2 parameter family of state lines but if we look at this f of y dash y dash which is a function of y dash is equal to 0 we can solve this and we can find out the root and that rouot is in terms of y dash. So if we take the routes of this equation y dash equal to Cj we can solve then we can get that y equal to Cix plus C as a solution. So in this case the extremals are all possible state lines.

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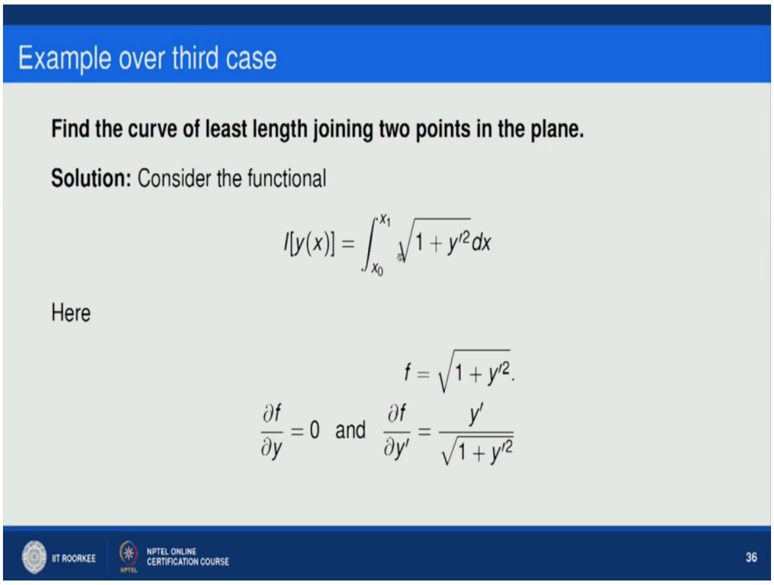

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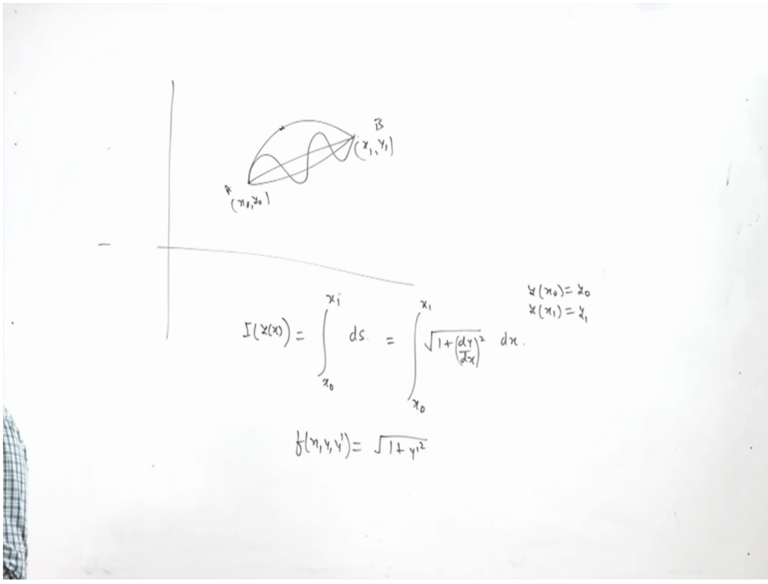

So let us take one example based on this and the example is quite interesting. The example is basically finding the curves of list length joining 2 points in a plane. So we have a plane xy plane and we want to find out say the minimum curve of minimum length. So let me write here the problem is this that we have 2 points x not, y not and we have one more point called x1, y1 call this point as A and this point as B and we want to connect these 2 points by several say curves.

Several curves you can say find out joining these 2 points A and B, what we want to define is a function like this which says that between x not to x1 we have say for every curve we have a length of the curve and we want to show, we want to find out the curve which minimises the length of the curve. So here you can say that d of s is say length element and we can write it as x not to x1 this is nothing but 1 plus dy by d of x whole square and d of x.

So we can say that in this case your functional is the functional is to minimize or minimize or maximize the length of the curve joining the point x not and x1. So here the condition is given that y of x not is equal to y not and y at x1 is equal to y1. So what we want to find here? We want extremize function given here. So in this case your f of xy, y dash is nothing but your under root's 1 plus y dash square. So this is totally depending on only y dash no term x and y is present here. So this is the case lying in your this case, case 3 where f is dependent of x and y it is totally depending on y of dash.

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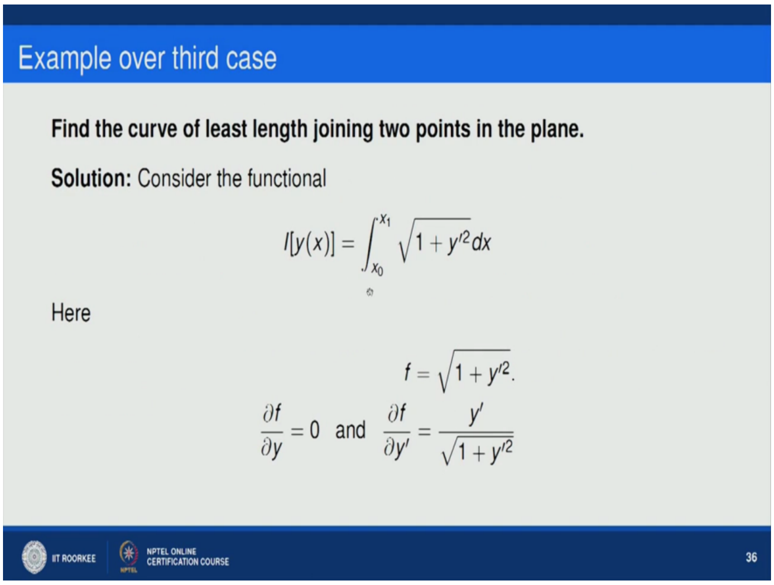

So here this is the functional which we have just defined I of y of x equal to x not to x1 under root 1 plus y dash square d of x and we want to find out the curve which extremizes this functional. So here f is equal to under root 1 plus y dash square. So dabba f by dabba y is equal to 0 and you can say dabba f by dabba y dash calculate it is nothing but y dash upon under root 1 plus y dash square.

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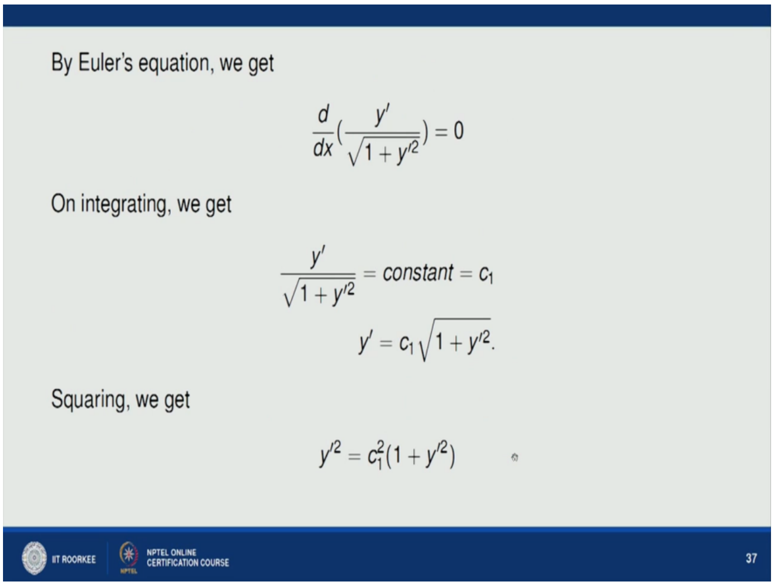

So Euler equation is reduced to this that d by dx of y dash upon under root 1 plus y dash square is equal to 0. So we can integrate with respect to x and we can say that y dash divided by under root 1 plus y dash square is equal to C1.

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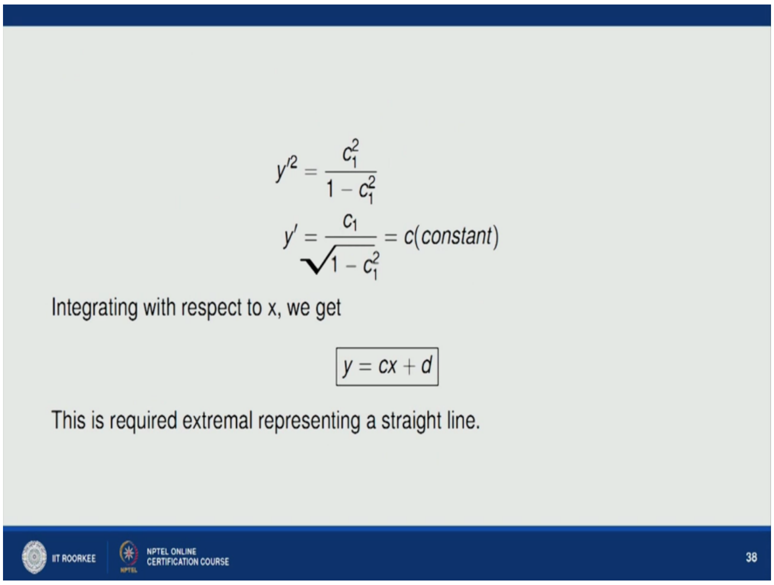

And we can simplify and you can get that your y dash is nothing but C1 upon 1 minus C1 square and which is nothing but another constant which we can call this as C, so y dash is coming out to be C and if we integrate this with respect to x we can get y equal to cx plus d and if you look at this is nothing but a straight line. So it means that the straight line is the curve which extremizes this functional.

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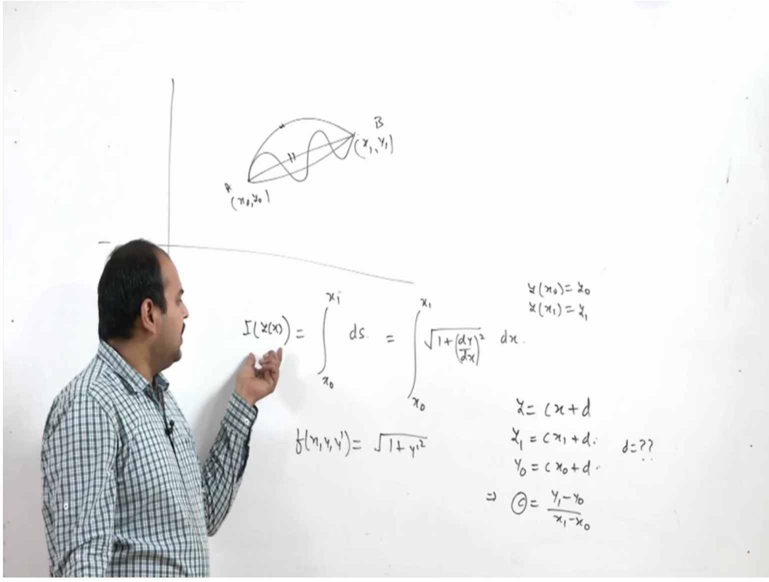

So in this particular case we can say that the shortest curve joining these 2 points A and B is a straight line. So here we wanted to prove and we already know that the a curve having the shortest length joining any 2 points is nothing but a straight line and with the help of this case we are able to prove that it is coming out to be the straight line only. So in this case your solution is coming out to be y of cx plus d and if you want to find out what is the value of c and d you can use the boundary condition and you can say that x equal to x1 it is y1, so y1 equal to cx1 plus d and y not is equal to when x is equal to x not.

So you can say that you subtract and you can get C as say y1 minus y not divided by x1 minus x not which simply says that c is the slope of a straight line joining these 2 points x not, y not and x1, y1. Similarly once we know this C, you can find out d from any of these 2 equations, so you can find out d. So it means that in this case the straight line is the curve at which this functional I of y of x is extremized. So in this particular case the extremal is the curve of minimum.

So we can say that at straight line minimises the length of the curve joining these 2 points x not, y not and x1, y1, okay.

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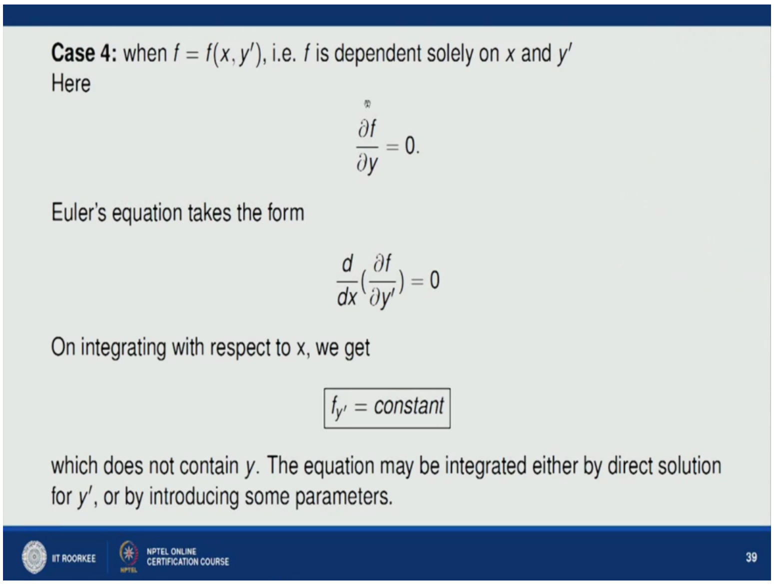

So now moving on to the next case, the next cases that f of x, y dash that f is written as f of x, y dash, so it is independent of y here. So if it is independent of y dash that the first of the Euler's equation dabba f by dabba y is simply 0 and in this case your Euler's equation reduced to d by dx of dabba f by dabba y dash is equal to 0, we can integrate with respect to x and we can have your dabba f by dabba y dash is equal to constant. Now here we can solve this equation which does not contain y either directly for y dash or you can use some kind of parameter to solve this.

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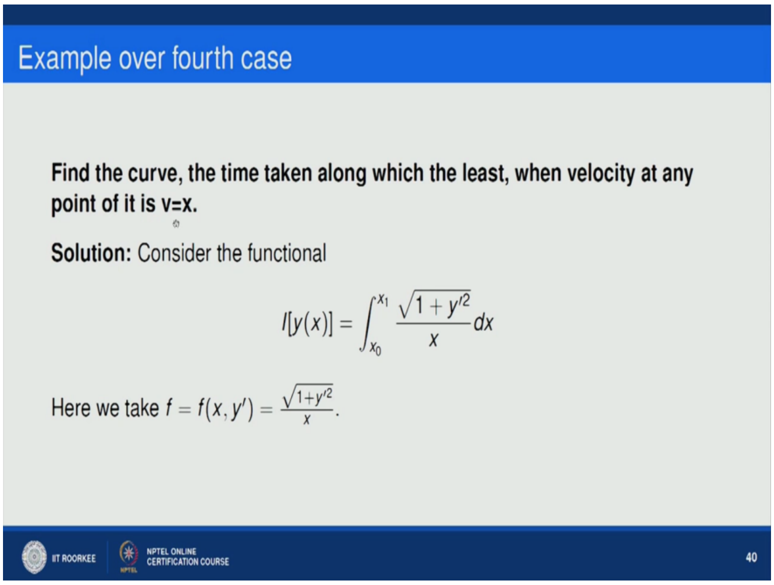

So let us take one example to have more understanding of this. So let us take this example and it is the basically a functional which consider the time taken by a particle moving from one point to another point, so it is what find the curve on which the time taken along which is the least when velocity at any point of it is v equal to x. So basically what they're trying to do here?

This I of y of x is actually calculating the time because if you look at here in our right hand site this under root 1 plus y dash square d of x is nothing but d of s. So we can say that it is I of y of x is equal to x not to x 1, ds by velocity. So ds by velocity is nothing but say time calculated from x not to x1. So it means that this functional is nothing but time taken along with which particle is moving with the velocity v equal to x.

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By Euler's equation, we get

$$\frac{d}{dx}\left(\frac{y'}{x\sqrt{1+y'^2}}\right) = 0$$

On integrating, we get

$$\frac{y'}{x\sqrt{1+y'^2}} = constant = c_1'(say) \qquad (11)$$

Put

$$\frac{dy}{dx} = \tan t \qquad (12)$$

From (11) and (12), we get

$$\frac{\tan t}{x \sec t} = c_1'$$

$$x = c_1 \sin t, \quad \text{where} \quad c_1 = \frac{1}{c_1'} \qquad (13)$$

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So here if you look at f is nothing but f of x, y dash it is independent of y here. So here in this case we can solve this then you can have Euler equation which simplify here as y dash upon x plus under root 1 plus y dash square is equal to constant. So let us take this constant as C1 dash. Now here we have different different ways to solve this, one way is to take this y dash is equal to some kind of parameter.

Another way is to take simplify this for y dash and then try to solve. So let us, let me write, let me solve this by taking the sum parametric representation of y dash. So here we can take dy by dx equal to Tan of t, if we take dy by dx equal to Tan of t where t is some kind of parameter then equation 11 reduces to this, y dash is tan of t and under root 1 plus y dash square is reduced to sec of t, so you can find out the value of x which is nothing but C1 sin t, where C1 is nothing but the reciprocal of this, C1 dash.

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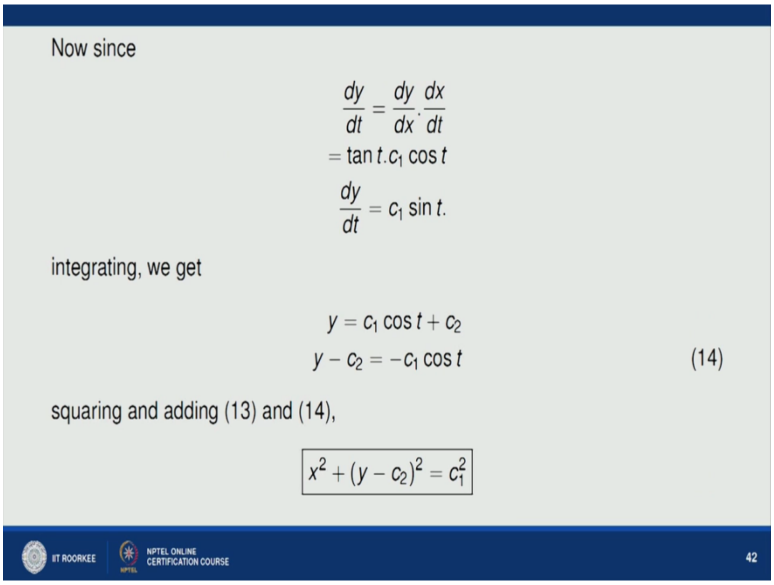

So here x can be written as C1 t, using the value of x you can simplify dy by dt. You can calculate the value of dy by dt that is this is equal to dy by dx and dx by dt. So dx by dt is already known to us that is derivative of this x that is C1 sin t and so you can find out dy by dt equal to dy by dx that is tan t, dx by dt is C1 cos t, if you simplify this coming out to be C1 sin t, so you can find out say y in terms of t by solving this first order differential equation.

So when we solve this is coming out to be y equal to C1 cost t plus C2. Now here again I am using this C1 as minus of C1 , so here I say that this is the solution of the Euler equation, so here given in a parametric form. So here we can simplify and we can write y minus C2 equal to minus C1 cos t and if we write it here then we can eliminate the parameter here.

So y minus C2 is equal to minus C1 cos t and x is equal to C1 sine t, so we can simplify this and we can remove the parameter t and we can say check that this is nothing but the solution of this x square plus y minus C2 whole square equal to C1 square which is nothing but the equation of family of the circles here.

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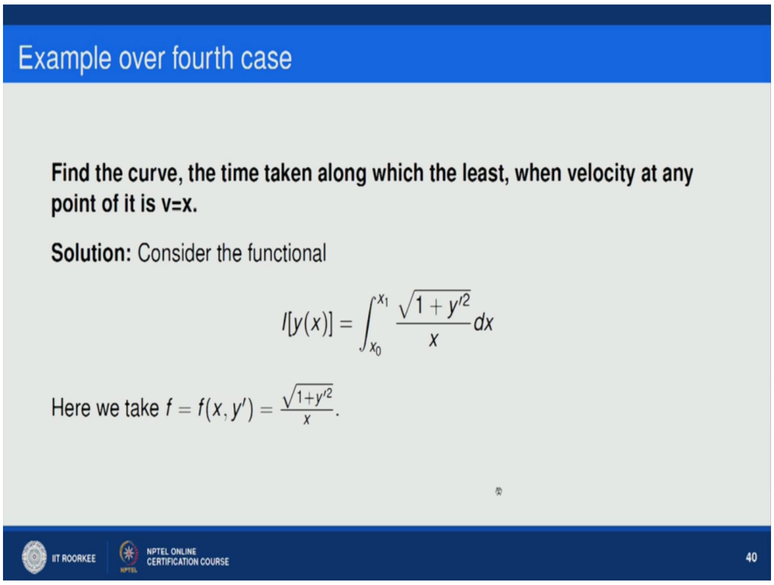

So it means that here in this case the curve on which the time taken is the least and when the velocity is given as v equal to x.

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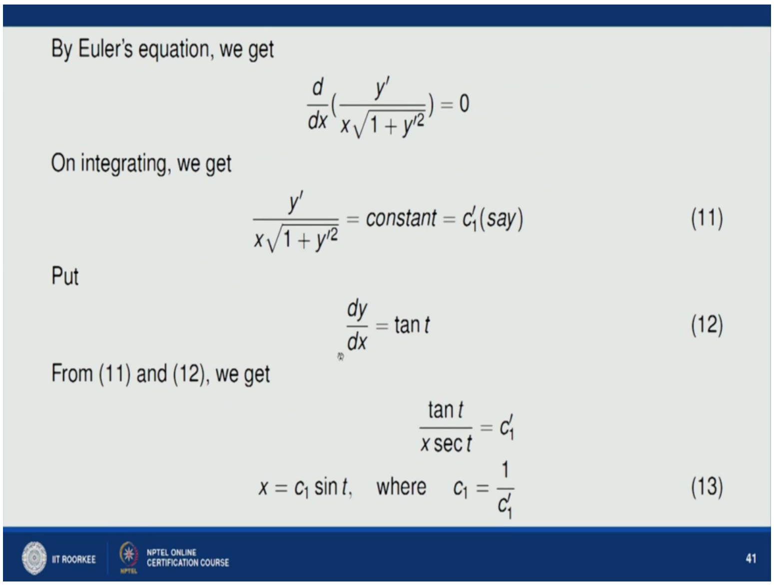

Let me we solve this equation number 11 directly by direct method. So here we have already seen when we take dy by dx is equal to tan t then we already know that we can solve these in terms of parameter, in this case your x equal to C1t and y is also given as C1 cos t plus C2. Now let us take this problem given in equation number 11 and try to solve without taking the help of parameter.

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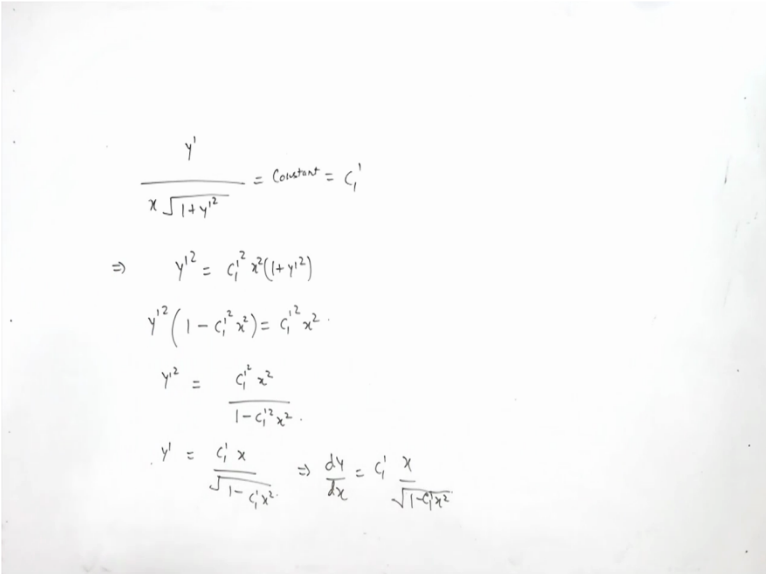

So here we have seen that this equation number 13 we have solve with the help of taking parameter value for y dash but we can also solve in a different way we can say direct way. So here let us take this constant as say that we have taken it as C1 dash, so we can take C1 dash here, so here we can square it and we can say y dash square is equal to C1 dash square, x square 1 plus y dash square and if you simplify you can take y dash square as 1 minus C1 dash square, x square equal to C1 dash square, x square here.

So y dash square is equal to C1 dash square, x square upon 1 minus C1 dash square, x square and you can write it y dash is equal to C1 dash x upon under root 1 minus C1 dash, x square. So it can solve this by, so here y dash is equal to this and we can solve this by writing dy dy by dx is equal to say C1 dash, x upon under root 1 minus C1 dash, x square and this you can solve by simple integration and you can check that here we are getting the similar kind of a solution , okay.

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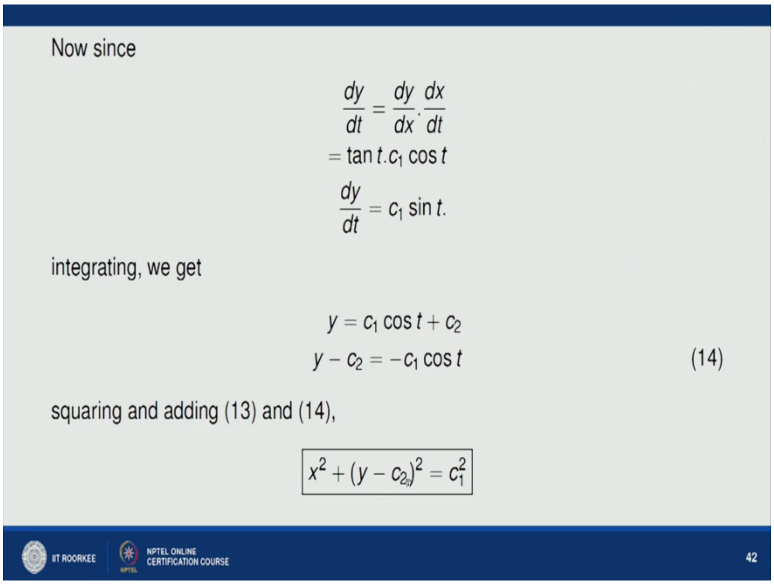

That is the solution is similar to the family x square plus y minus C2 whole square plus C1 square. So here you may get different constants for integration. So here you can write it that y is equal to integration you can write it C1 dash, x upon under root 1 minus C1 dash, I think this is square here and this is square here, this is square here C1 dash square, x square, d of x plus some kind of C2 here.

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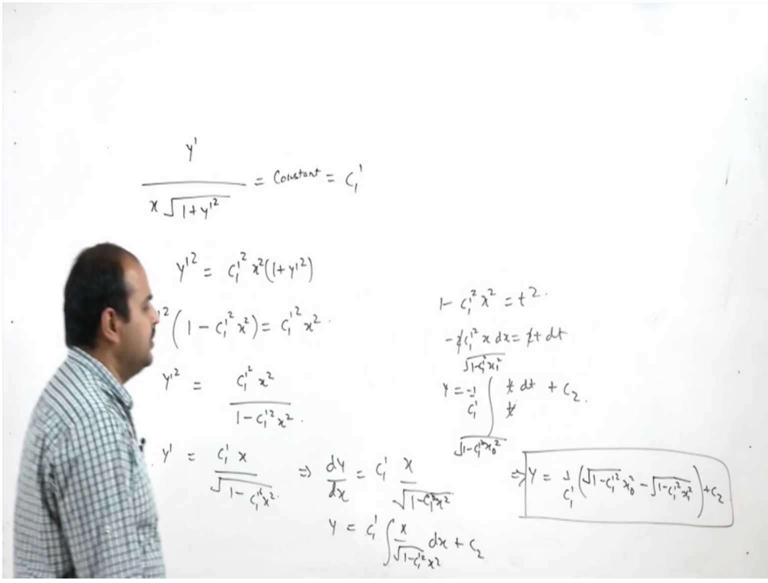

So you can simplify, you can simplify by writing 1 minus C1 dash square, x square, st, so let me write it 1 minus C1 dash square, x square say t square, so you can write it minus 2 C1 dash square x, d of x equal to 2t dt, so2 will be cancel out and you can simplify that is y equal to, so you can say that minus C1 dash is coming inside. So you can write 1 upon C1 dash is equal to this is nothing but t of dt and this denominator is also written as t here plus C2 and here your constant will be this integration with respect to x not to x1.

So the corresponding value of t you can calculate, so t is nothing but under root 1 minus C1 dash square, x not square, similarly here we have 1 minus C1 dash square, x1 dash square, right? So you can say that this is nothing but y equal to 1 upon C1 dash and here you will get there is one minus sign also because of this so here it is. So here I write it here as 1minus C1 dash square, x not square minus under root 1 minus C1 dash square, x1 square and plus C2.

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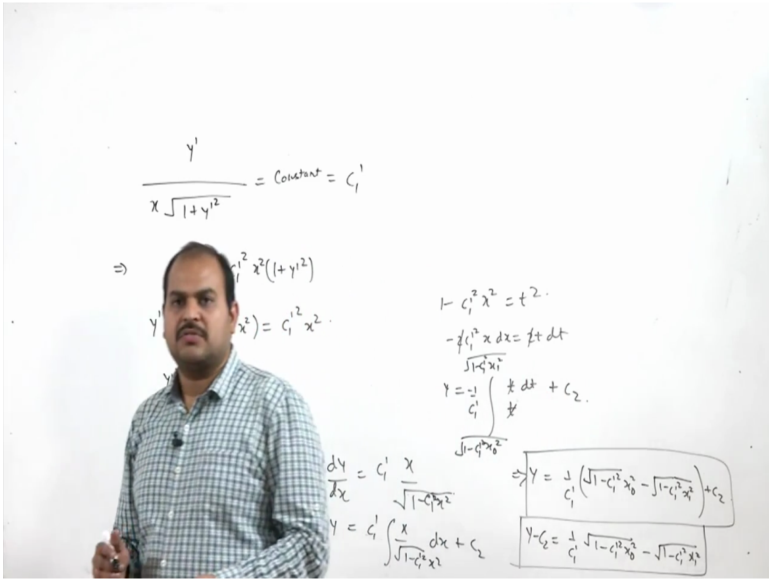

So here you can say that your solution is also of this type or you can say that y minus C2, so if I take say x1 as variable thing that it is nothing but y minus C2 is equal to 1 upon C1 dash under root 1 minus C1 dash square, x not square minus under root 1 minus C1 dash square, x1 square. So it is also the previous type, if we assume that x1 is a variable point x then you can say that this is nothing but a family of circle. You can square it and you can say that it is also a family of circle.

So here we have seen certain cases of Euler's equation when f of x, y and y dash is assuming a very simple say particular cases and there is one more case left that is a function f of x, y, y dash is independent of x that is quite important but we will discuss in next class. So today what we have discussed? We have seen the particular example based on Euler's equation and we say that, how to find out say extremal of a given functional.

So we will continue this in our next class, thank you very much for listening us. We will meet in next class, thank you.