Course on Decision Modeling
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Lecture No. 41
Calculus o Variation: Basic Concepts-II

Hello friend, welcome to today's lecture. Here if you recall we have discussed in last class the concept of functional and we have seen that in calculus of functional the branch calculus of variations discuss the methods which discuss the maxim and minima of a given function. And we have seen that the methods of finding the maxim and minima of functional is closely related to the concepts of finding maxima-minima of functions.

So in last class we have discussed say relation between function and functional and also we have seen the definition of increment in terms of functions. So it means that the delta x and then the corresponding analog version in terms of functional that is delta y, where y is the argument of a given function. So here we continue that lecture in today's lecture, so todays next thing we want to discuss is the continuity of a function.

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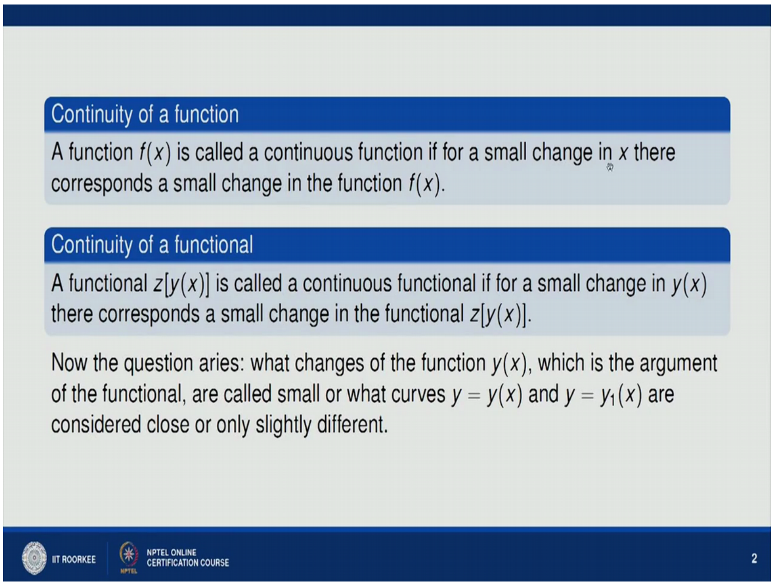

In previous class we has Just given the Just idea of what is continuity, so you look at the continuity of a function then we can see that a function fx is called a continuous function is for a small change in x there corresponds a small change in the function fx, so that is the definition of continuity of a function in then you want to look at the analogs version of the continuity of a functional.

So we can define the continuity of a function in a similar manner and say that a function z yx is called a continuous functional if for a small change in y of x there corresponds a small change in the functional z yx. Now in terms of function it is quite obvious that if there is a change in x, say x 2 x plus h then what is the change in functions, so f of x plus h and f of x.

Now here you want to see that what small change in y of x we are referring. So here we may ask this question what changes of the function y of x is the argument of the functional are called small or what curves y and y 1 are considered close are only slightly different. So it means that if you talk about functions and then in function x a small change is x plus h, but in terms of functional your argument is y of x, so what is a small change in y of x? So it means y of x and the small difference in y of x that is say y 1 x. So how in what way we can see that Y and Y 1 are small enough?

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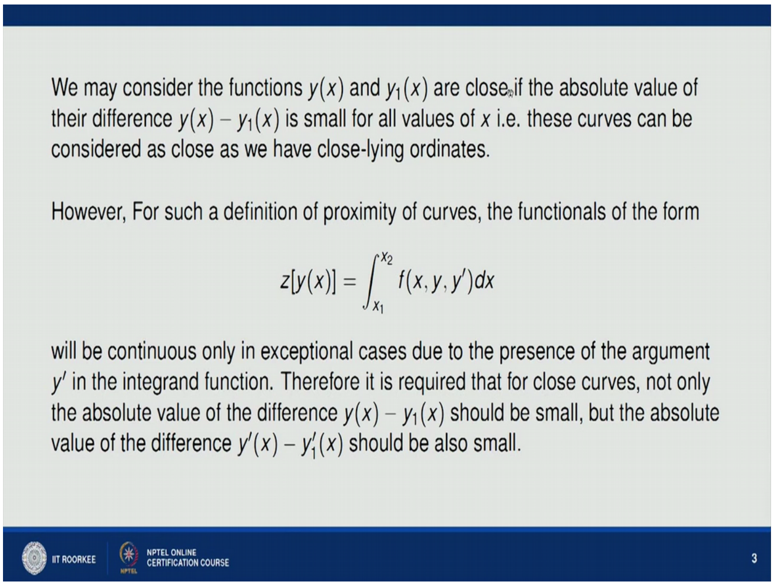

Here regarding the constants we may consider the function yx and y 1 x are close if the absolute value of the their difference is small. So it means that the modulus value of yx minus y 1, x is small for all values of x then we can say that okay these 2 functions are close enough. Now but if we take this definition as closeness then look at this functional, so z yx is equal to x 1 to x 2 f of x y y dash dx.

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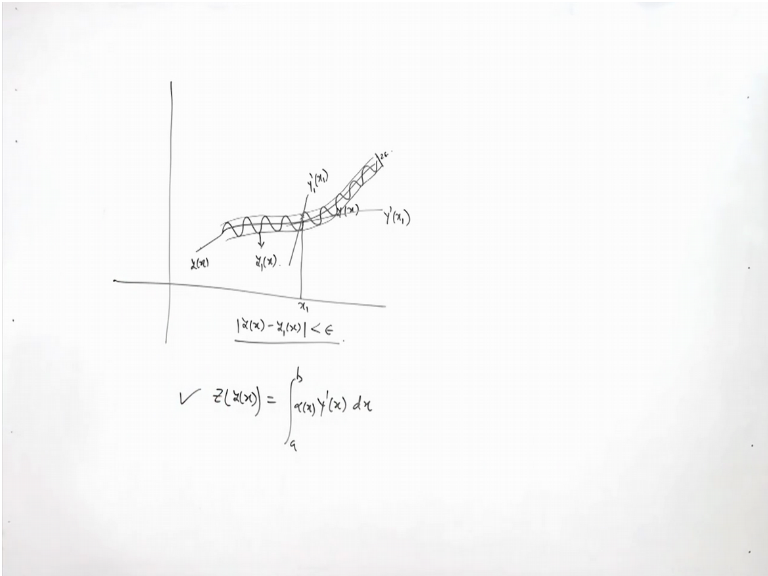

Now it may happen that the function like this which are small which are close enough in this sense may have possibility that their derivatives are quite far or we can say that if you look at this particular graph here that here you Just write down this thing and here we can define this is your suppose this is your y of x and then we can defined your another y 1 x like this. So here we can see that let us say that this is your okay so here this y x is this graph y of x and y 1 is this spiral one so it is y 1 x.

Now I can say that if this is say distance is like 2 Epsilon so we can say that all the time y of x minus y 1 x if I look at their derivatives this their difference is less than say Epsilon. But if I look at the any point say let us say this particular point then though their functional value is quite say quite close to each other but if I look at the value of that derivative so it means that at this particular point your let us say call this point as say x 1, so at this particular point this is your own y dash x 1 and this is the value of y 1 dash x 1.

So here the slope of curve y 1 and the slope of curve y is quite far to each other. So in this sense we can say that this functional in the sense we can say that your y and y 1 are not close enough in this sense of first-order derivative and it may happen that if we consider a functional which involves in terms of say y dash, so if we consider z y of x like this equal to say a to b you take simple y dash x d of x you can take any functional for x y dash.

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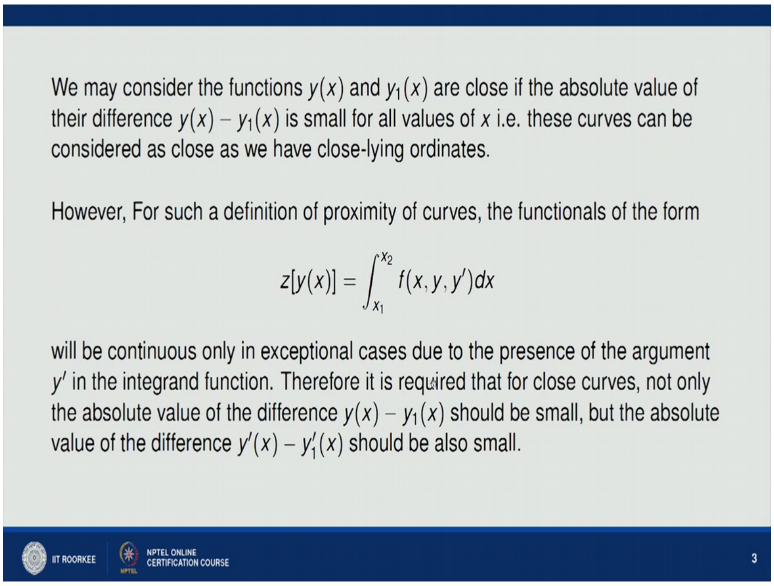

So in this sense if we take closeness only in this sense then this functional may not be continuous at all, so this compel us to define our definition of closeness in a more suitable manner. So we can say that therefore it is required that for close curve not only the absolute value of the difference should be small but the absolute value of the difference should also be small. So it means that not only the their functional value will be same but their tangent is also close enough to each other and it may happen in some kind of functional which may involve say a method for derivative of argument y.

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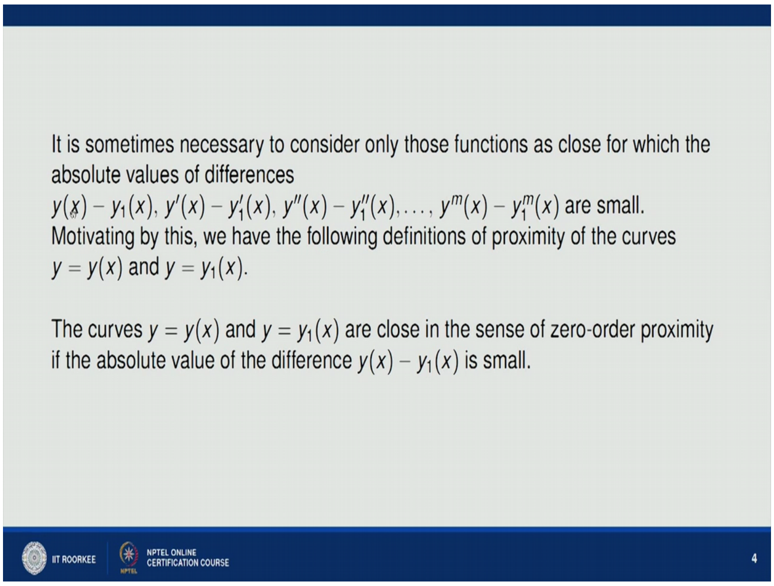

We may define the definition of closeness as follows that the difference the absolute value of the difference is yx minus y 1 x, y dash x minus y 1 dash x till y mx minus y 1 mx are small. So based on these definitions we can define the definition of closeness as follows, the curve y equal to yx and y equal to y 1 x are close in the sense of 0 order proximity if the absolute value of the difference yx minus y 1 x is small.

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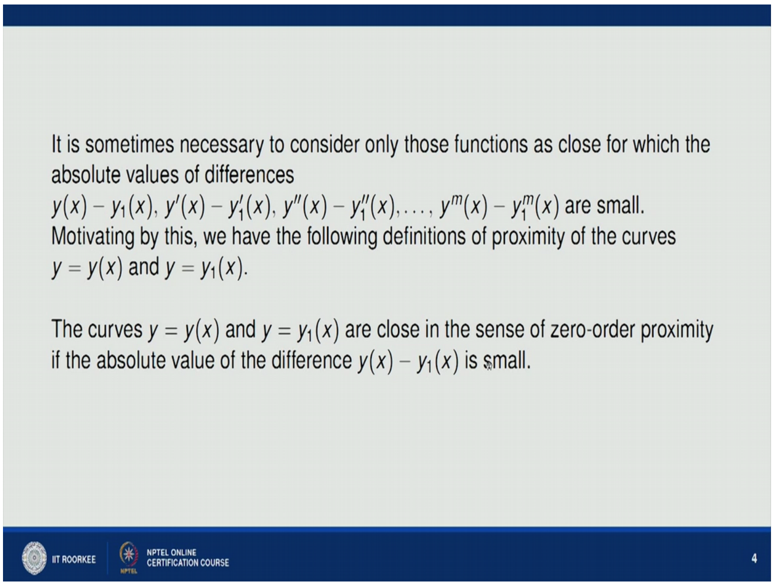

Similarly, we can define the first order proximity that the curve y equal to yx and y equal to y 1 x are close in the sense of first-order proximity if the absolute value of the differences yx minus y 1 x and derivative that is y dash x minus y 1 dash x are small. So keeping this track we can say that the curve y equal to y x and y equal to y 1 x are close in the sense of m-th order proximity if the absolute value of the differences this is the 0^{th} order derivative, so yx minus y 1 x is small, y dash x minus y 1 dash x is small till the m-th order y mx minus y 1 mx are small.

So we may define our closeness in the sense of 0^{th} order proximity, 1^{st} order proximity or upto m-th order proximity depending on the definition of the functional depending on the on the definition of the functional. And we may observe here that if we consider we may observe from these definitions that closeness of 2 to curves in the sense of m-th proximity implies the closeness in any lesser order of proximity.

It means that if we consider that 2 functions say yx and y 1 x are close in the first proximity then it will be close in the 0 order proximity. Similarly if y and y 1 are close in the sense of m-th proximity then they will be close in close in the m minus 1 order proximity and any order which is less then m.

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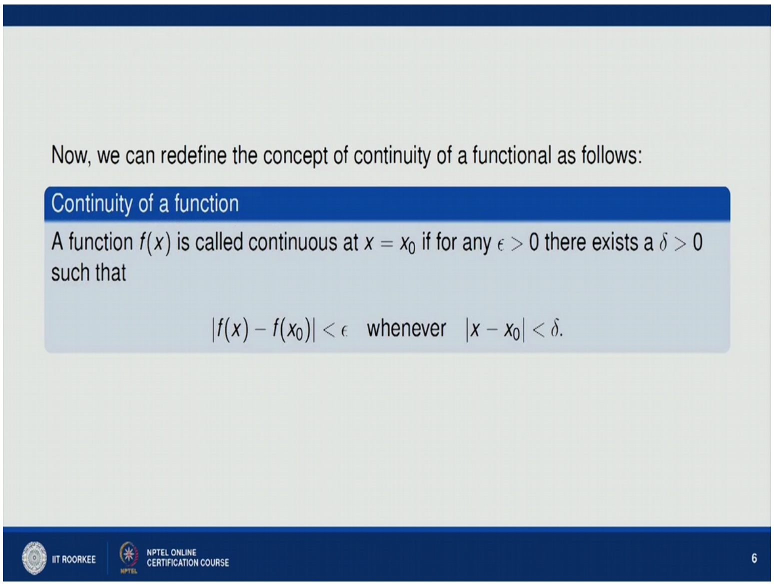

So now with the help of with closeness in the sense of m-th order proximity we may define our continuity of functional as follows. So first we define what is continuity of a function, so a function fx is called continuous at x equal to x 0 if for any epsilon greater than 0 there exists a delta greater than 0, such that modulus of f of x minus f of x 0 is less than epsilon whenever x minus x 0 is less than delta or i can say that the functional value deferred by at max by epsilon whenever the argument is difference of the argument is less than delta.

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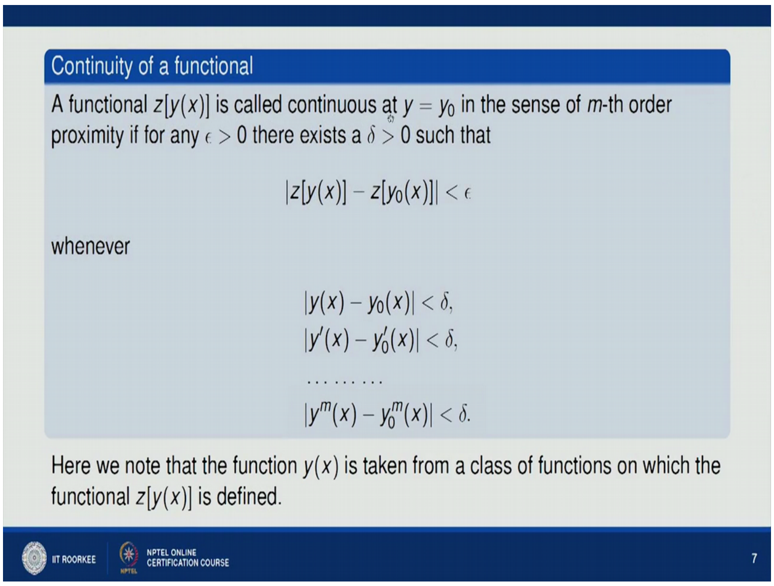

Similarly if we go to the definition of continuity of a functional we can say that a functional z yx is called continuous at y equal to y 0 in the sense of m-th order proximity is for any Epsilon greater than 0 there exists a delta greater than 0 such that the difference in functional value corresponding to y of x and y 0 x is less than epsilon, whenever modulus of y of x minus y 0 x is less than delta, y dash difference between y dash x minus y 0 x is less than delta till absolute value y mx minus y 1 mx is less than delta, so that is the continuity of functional in the sense of m-th order proximity, and we here we have assumed that this y of x is taken from a class on which this functional is defined, so that is the only condition on this y of x.

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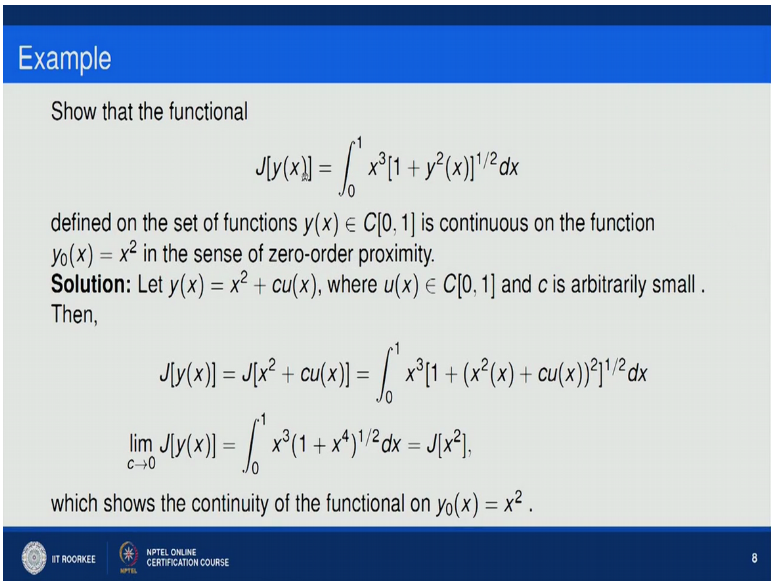

So here we are discussing an example of continuity, so here the examples which says that show that the functional J yx equal to 0 to 1, x cube square root of 1 plus y square x d of x, defined on the set of function y of x equal to c 0, 1 and this c 0, 1 is space of function which is continuous on close interval 0 and. And we want to show that this functional J yx is continuous on functional y 0 x equal to x square in the sense of 0 order proximity.

So it means that let us take arbitral function y of x in the proximity of in the neighborhood of x square, so let us define yx is equal to x square plus c u of x, where c is arbitrary small parameters and here ux is also member of c 0,1. it means that ux is also a continuous function defined on close interval 0 to 1.

And then we want to check the functional value on this y of x, so J y of x which is nothing but J of x square plus c ux and by the definition of function you can write it 0 to 1 x cube 1 plus x square x plus this x square plus c ux square under root of this thing d of x, so here if you take the limit c tending to 0 in that case your y of x will tend to x square, right.

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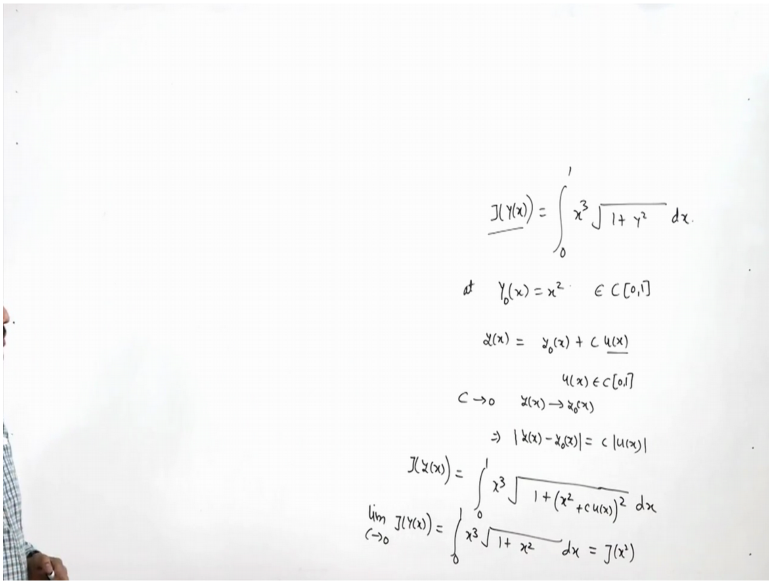

So let me write it here, we are looking at this functional J of y of x that is 0 to 1 and here it is written as x cube under root of 1 plus y square d of x and we want to check the continuity of this functional at y 0 x equal to x square, so we want to check whether this functional is continuous at this or not.

So for this you take a arbitrary function say y of x which in the neighborhood of x square so let us say that simply y 0 x plus say some c times some another function let us say u of x and here since y of x is also a member of c 0, 1, so here y of x is a member of c 0, 1, so we can say that this u of x is also a member of c 0, 1, means ux is a also a continuous function.

And then we try to show that if c tending to 0 then this y of x is tending to y 0 of x, so it means that if i take this c very small then they we can consider that y of x and y 0 x are close enough in the sense of zero order proximity. So we can say that here difference of is what y of x minus y 0 x and that is nothing but c of u of x this thing. So if c is going to 0 than that difference is also tending to 0, so we can say that here y of x is close to x square in the sense of zero order proximity.

Now we want to check that the function is also continuous in the sense of zero order proximity, so for that you calculate the value of J of y of x so this is 0 to 1 x cube and under root 1 plus now y of x is given by this. So it is x square plus c u of x whole square d of x.

Now we try to say that if this y is tending to y 0 then what is the functional value it will change to it will go to the value of J y 0 or not.

So here we want to see the change that limit c tending to 0 now J of y of x will be what? So here you can take the limit inside 0 to 1 x cube and limit if you take the limit inside it is 1 plus this is x square because limit if you take inside the limit c tending to 0 than this is simply gone and it is this and if you look at by definition it is nothing but J of x square.

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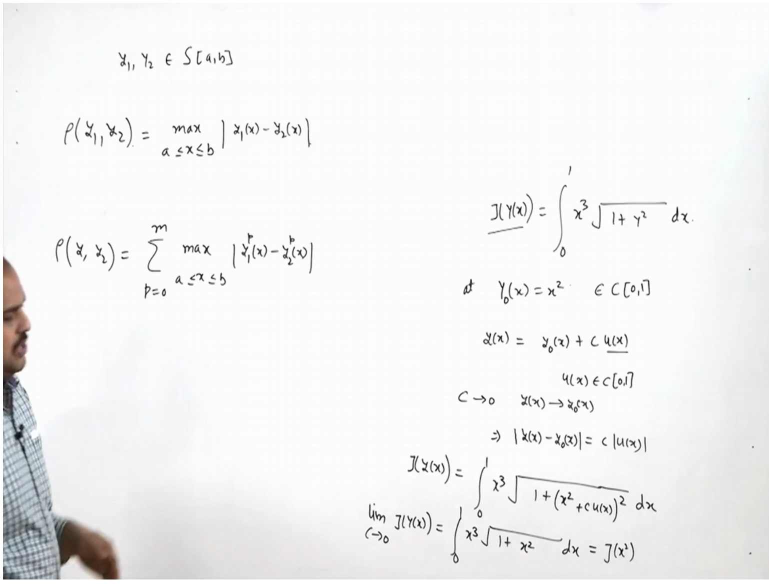

So you can say that functional is continuous at y 0x equal to x square in the sense of zero order proximity. Now if you look at this closeness in terms of m-th order proximity we can also define the distance between 2 function say y 1 and y 2, so let us say define distance between y 1 and y 2 and this y 1 and y 2 are function of some function space let us call it S which is defined on say a to b and we can say that it is nothing but maximum of x between a to be and here we are considering the difference between y 1 x minus y 2 x.

So if we define if you calculate this quantity we can say that this is the distance between y 1 y 2 in the sense of 0 order proximity. Similarly, we can define the distance between in the sense of m-th order proximity in a similar way there we can consider this as submission say P equal to 0 to say m and here it is maximum of x less than or equal to a to b and y 1 x minus y 2 x and here we are considering upto say m-th order derivative here.

So if we calculate this value then we can say that you are y 1 and y 2 distance between y 1 and y 2 in the sense of m-th order proximity. So this is so in the sense of closeness defined in the m-th proximity, we can define distance between 2 function in the sense of 0 order proximity and hence m-th proximity.

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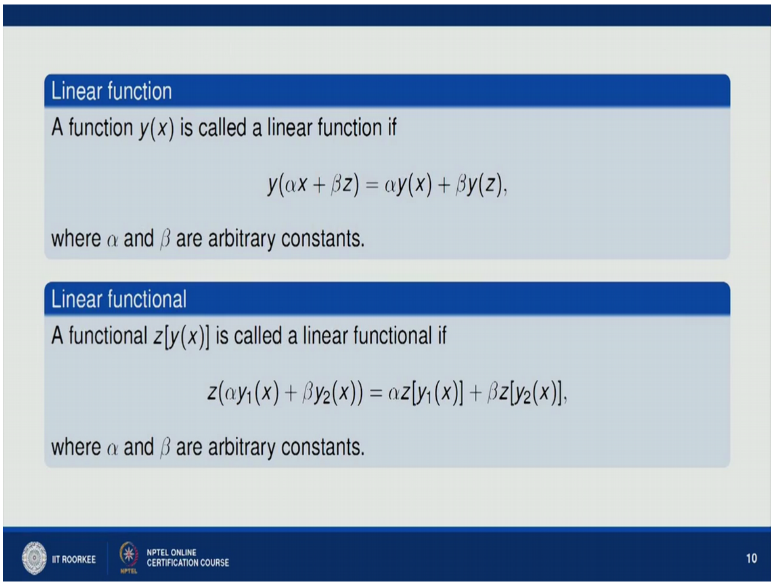

Now Just moving onto the next concept let us define what is linear functional, so please recall that a function is called linear if we have this property that y of alpha x plus beta z is equal to alpha y of x plus beta y of z, where x and z are the point in the domain of y and alpha and beta are Just arbitrary constants.

So in the similar manner we can define linear functional as the functional z yx is called a linear functional if we take 2 function y 1 x and y 2 x from the domain of z and the value of z of alpha y 1 x plus beta y 2 is equal to alpha times z of y 1 plus beta times z of y 2 and here alpha and beta are arbitrary constants. So if you look at these 2 definitions are almost similar.

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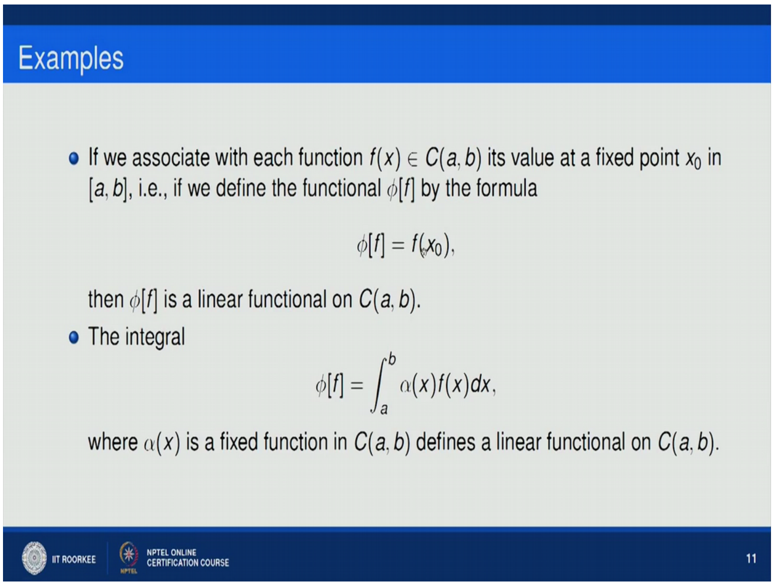

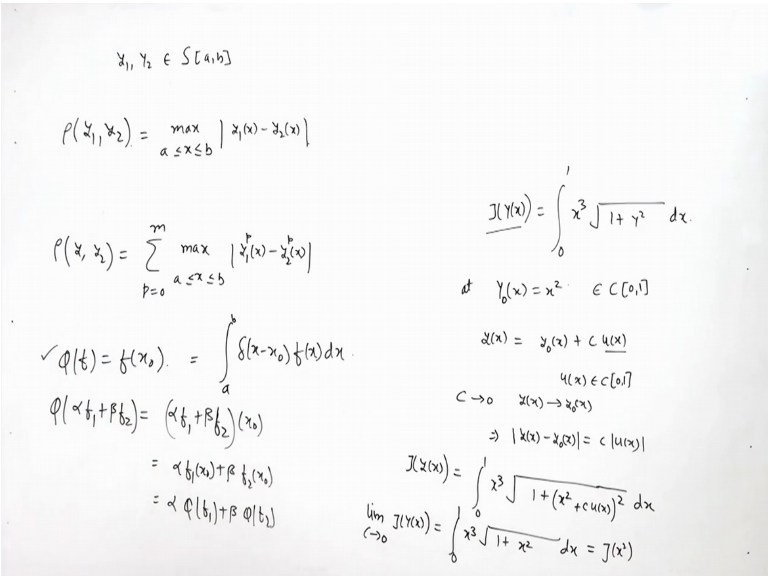

Now going on to example, the first example we can consider as the functional defined on f is the value of f given at a particular point x 0, where x 0 is a any point between this interval a, b. Now this I can define in this manner, so here if you look at phi of f it is defined of f of x 0. So here if I take say alpha f 1 plus beta f 2 then i can write this as alpha f 1 plus beta f 2 evaluated at x 0 and this can be written as alpha f 1 x 0 plus beta f 2 of x 0. And this i can write as alpha phi of f 1 plus beta phi of f 2, so we can say that this functional phi of f is a linear functional. Okay.

Now here we can also see that so far we have represented our function in terms of integrals, so here this also can be represented in terms of integral and we can write this as f of x 0 as a to b, now here we can take the help of derived delta function, delta x minus x 0 f of x dx. So here this function can also be written as definite integral the only thing that here we are using the derived delta function, so this we can see that this is also a functional.

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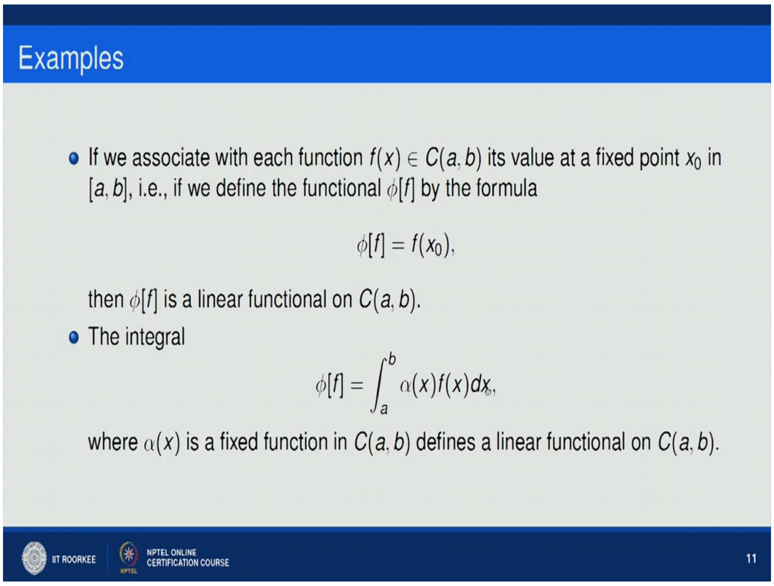

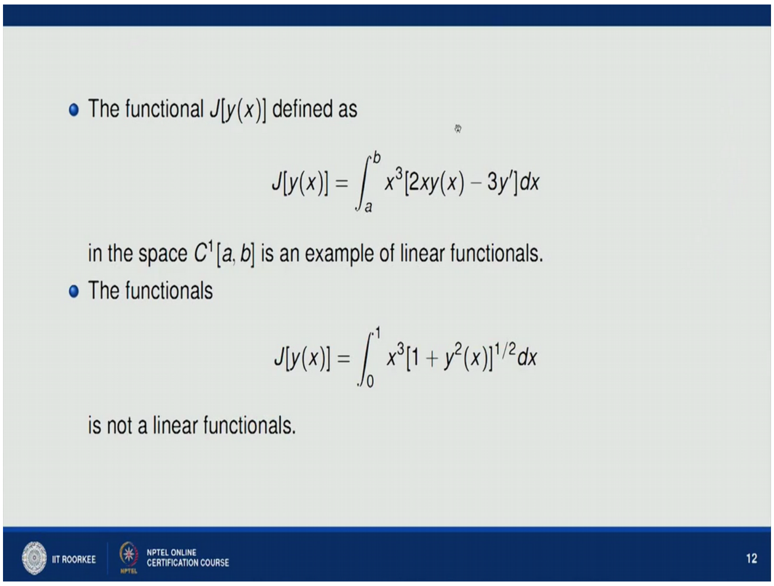

Similarly the integral phi of f which is defined a to b alpha x f of x dx is also define a linear functional, you can check in a same way you Just go and check the definition okay. And similarly this 3rd example and 4th is also, 3rd which is defined as J yx equal to a to b, x cube 2 x y x minus 3y dash dx in the space of C 1 a, b, your C 1 a, b is the space of continuously differentiable function it means that they all those functions whose first order derivative is also continuous.

So in that sense this functional is also a linear function, but if you look at the last example that is J y of x is equal to 0 to 1 x cube 1 plus y square x to the power 1 by 2 dx, it is not a linear functional, because if you replace this yx by say y 1 plus y 2 taking say alpha equal to beta equal to 1 then you can check that it is not equal to J of y 1 x plus J of y 2x you can check easily, okay.

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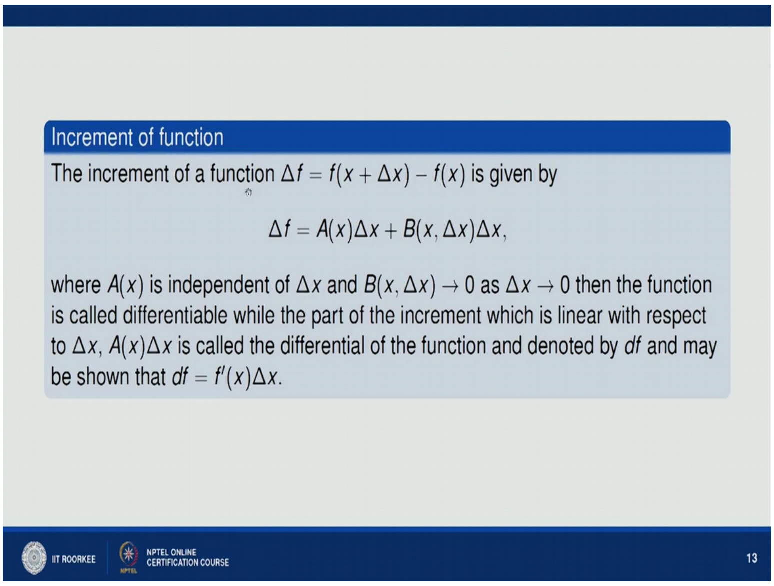

So now let us define let us recall what is the concept increment of function and we can define with the help of this the increment in terms of functional also and that is known as variation. So increment of function is defined as the change in the functional value when the small change in the argument value of the function. So here Delta F is defined as f of x plus delta x minus f of x.

So here argument is change by small difference that is delta of x this is increment in terms of x and if this increment is written as this a of x to delta of x plus bx delta x into delta x, here Ax and bx is defined as follows, that Ax is independent of delta x and B is tending to 0 as delta x is tending to 0 and in this case we can say that this function f is called differentiable and what is the differential of this and they can say that the part of the increment which is linear with respect to delta of x that is this part Ax delta x is called the differential of the function and denoted by d of f and we can say that that d of f is equal to f dash x delta X

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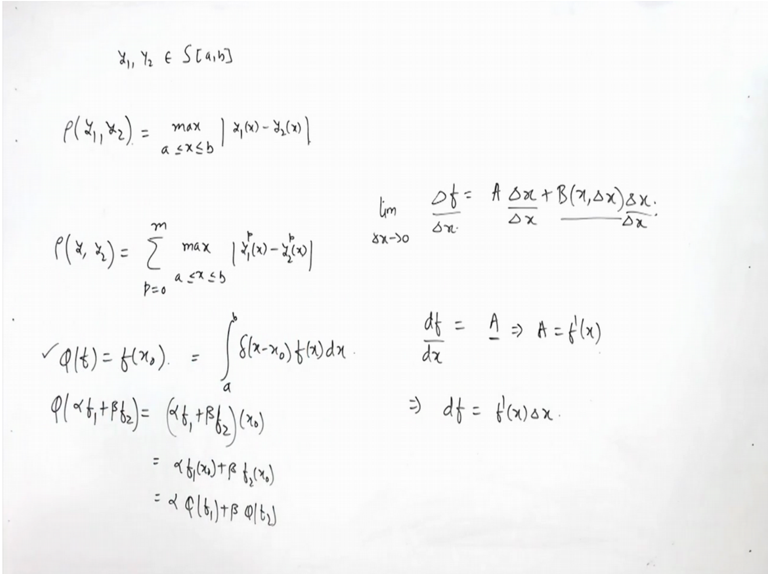

So how we can look at here let me start down this, so here your delta f is given by a delta x plus b x delta x delta of x. So what we try to do here? let us divide by delta x here so if you divide by delta of x here, so that will be simply this and then you take limit delta x tending to 0 then this will this is denoted as d of f and this is nothing but d of f by d of x and it is A, because this part is simply managed to tend to 0 as limit delta x tending to 0.

So we can say that this A is nothing but f dash x, so A is equal to f dash x and we can say that this d of f is given by f dash x delta of x. So that is what how we define differential, so differential of the function which is denoted by d of f is given by f dash x delta of x. So that is how we define increment in terms of function. So in a similar way we can define increment of functional as follows.

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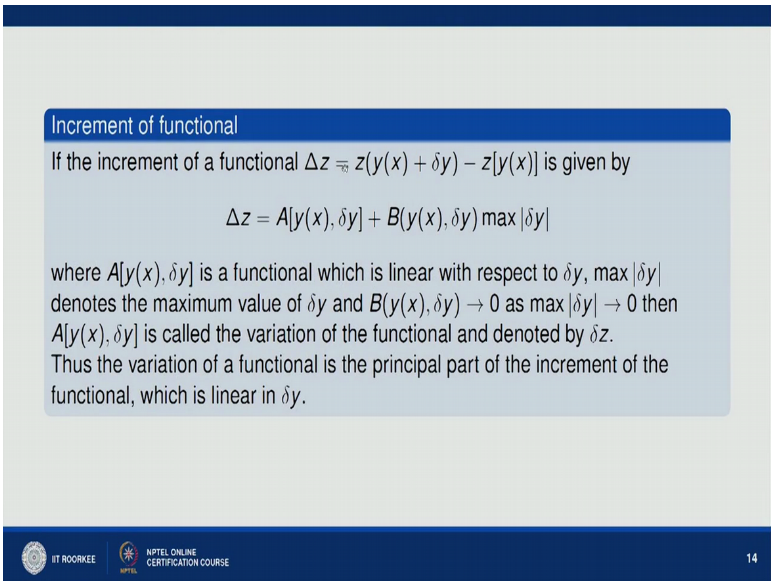

So here increment of function is defined in a similar way that argument is say incremented by say delta y. So and look at the function difference between yx plus delta y and y of x which is given as, delta of z equal to z of y of x plus delta y minus z y of x and if this increment is written like this that is delta z equal to A yx, delta y plus B yx, delta y and maximum of modulus value of y delta y.

And if we consider this function as linear with respect to delta y and maximum of delta y denotes the maximum value of delta y and this B is standing to 0 as maximum delta y tending to 0 then this part which is linear in terms of say delta y and it is a principal part of the increment is called as variation or say increment of the functional z and it is denoted by delta of z.

So we can define variation of functional is the principal part of the increment of the functional which is linear in delta y. So we can say that this is the part it is known as principal part of the increment delta z and it is linear in terms of delta y, so this is A yx, delta y is denoted as variation of the functional z.

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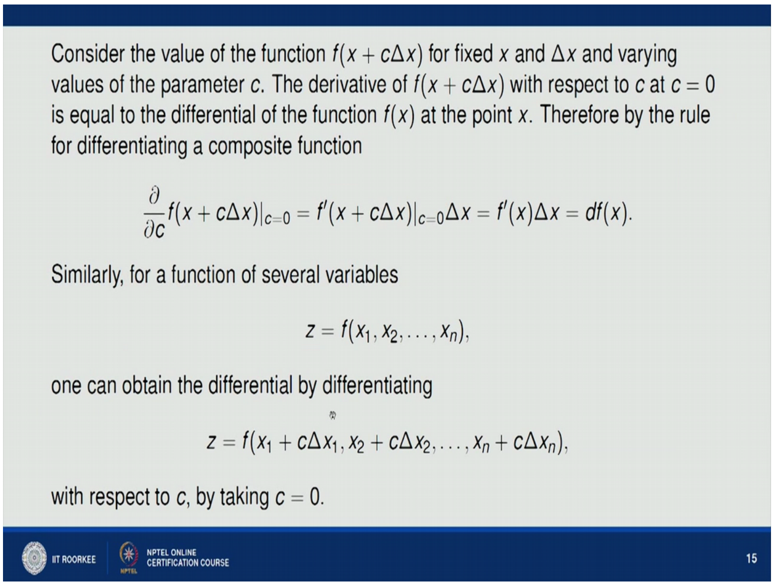

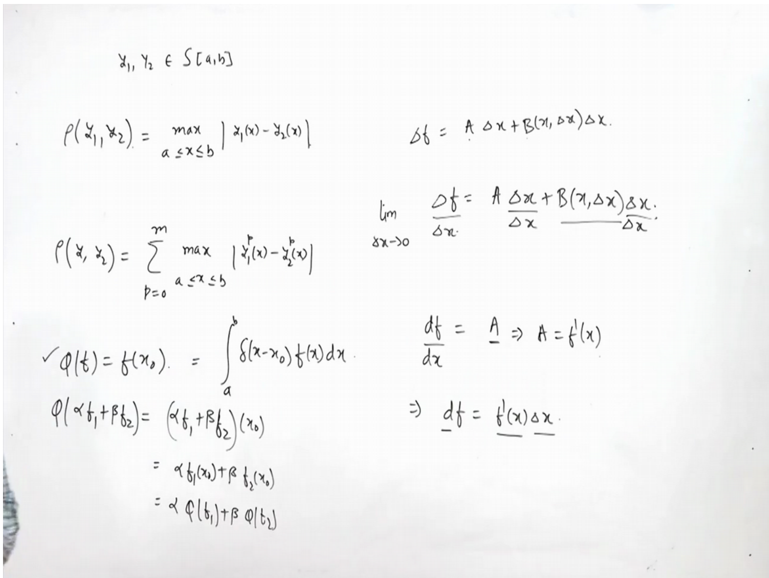

Now we can also define our variation of a functional in other sense, so for that we please recall that if we want to define we have seen how to define the differential of function and differential in terms of functional. So now we considered the alternative way to define increment in function and increment in functional. So if you look at your d of f is defined as f dash x delta of x and which is defined in the sense of this that delta of f equal to A delta x plus B x delta x and delta x, where is A is a linear is independent of delta x and linear part and B is tending to 0.

So here we have an alternative way to define this differential for that we functional evaluated at x plus c delta x, here x is fixed, delta x is fixed, this parameters c is a varying parameters. And if you calculate the derivative of function f with respect to c, at c to 0 then it is coming out to be the differential of the function f of x at the point of x. If you look at just look at here find deba by deba c of f x plus c delta x evaluated at c equal to 0 then we can differentiate this using the differentiation of composite function and it is coming out to be f dash x plus c delta x evaluated at c equal to 0 into delta of x and it is coming out to be that this quantity evaluated at c equal to 0 is nothing but f dash x and we can write it f dash x delta x as d of fx.

So it means that we can also define differential in the sense of this quantity. Similarly for a function of several variables that is z equal to f of x 1 to x n, we can also define differential in same way for that you consider this function f of x 1 plus c delta x 1, x 2 plus c delta x 2 and so on. So here delta x 1, delta x 2, delta x n are all fixed, similarly x 1 to x n are all fixed only thing is that c is the parameters which is variant.

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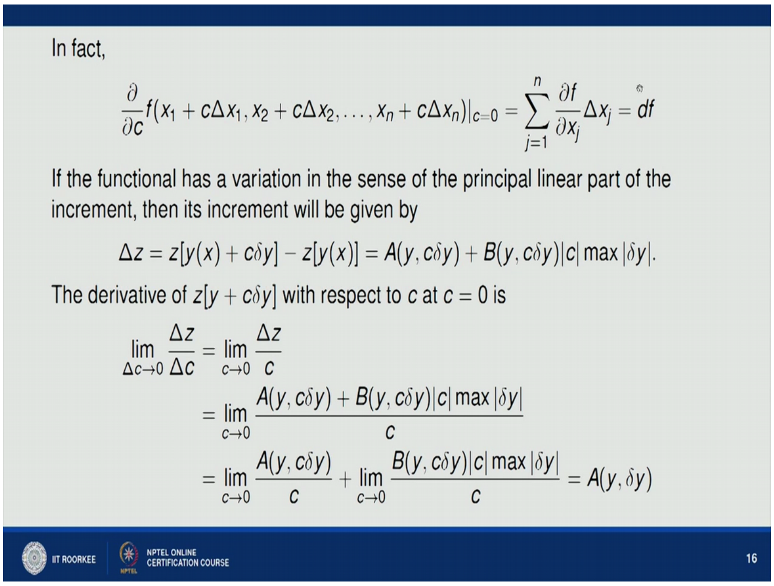

So if you find out say derivative of this with respect to c and take c equal to 0 whatever you will get that is nothing but differentiable differential of this function f. So same way we want to define the increment or variation in terms of functional, so here if the function has a variation in the sense of principal linear part of the increment then its increment will be given by delta of z equal to z yx plus c delta y minus z of yx.

Now here yx is fixed, delta y is fixed only c is variant, so by definition of increment this can be written as A of y, c delta y, only thing is that here this increment is replaced by c delta y so in the definition also delta y is replaced by c delta y, so you can say that it is A y, c delta y plus B y, c delta y, modulus of c, maximum of modulus of delta y.

And if you find say derivative of this with respect to c then it is what limit delta c tending to 0, delta z by delta c which is nothing but limit c tending to 0 delta z by c. So delta Z is already defined by this increment and then you divide by c and take the limit c tending to 0 then this can be written into 2 parts one is corresponding to this linear part and another part is this.

Now here as limited c tending to 0 this B y c delta y is something which tends to 0 as c is tending 0, c is tending to 0 means? c delta y is tending to 0 and hence by definition your B is tending to 0. Whatever we are keeping here that is a bounded functions that modulus of c upon c is bounded by 1 and maximum of delta y is also some fixed point so it is also bounded.

So as limits c tending to 0 by the definition of B it will tend to 0, so what is left here is A y, delta y, because this A is linear with respect to 2^{nd} argument, so c will be taken out and you can see that there is nothing but c times a y delta y divided by c, so c will cancelled out and we have a y delta y. And it is nothing but a variant or increment in the function, so we can say that increment in your functional can be given by derivative of delta derivative of z with respect to c.

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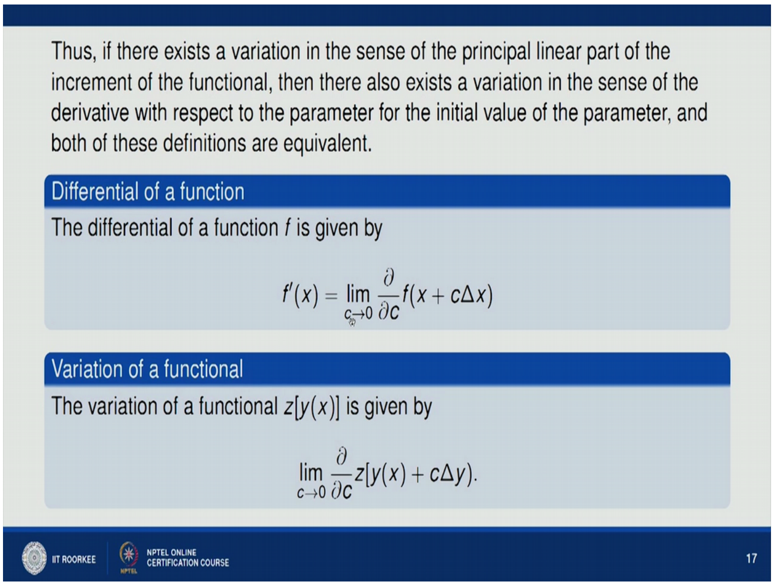

So here we can say that if you summaries this, thus if there exist a variation in the sense of principal linear part of the increment of the functional then there also exist a variation in the sense of the derivative with respect to parameters for the initial value of the parameters and both of these definition are equivalent.

So we can summaries and we can write that the differential of a function can be written as f dash x equal to limit c tending to 0 deba by deba c f of x plus c delta x that is for function, so in a similarly we can define variation of function as this that variation of functional z yx is given by derivative of z of y of x plus c delta y with respect to c at c tending to 0 or you can say limit c tending to 0 deba by deba c z of yx plus c delta y.

So and with the help of increment of variation we can define a point which is maximum point or minimum point, so this portion this will discuss so in next class so here we have a 2 definition of variation of a functional this definition we find it useful because it is not all that time possible to represent your increment delta z as this particular form. So in some cases where your functional is quite complex then this form is quite difficult to achieve so there we find out the increment of function using this later definition which we have just shown, okay.

So we will use this as a definition of finding variation of functional. So in next class we will continue after this and we will define, what is maximum and minimum value of the function and discuss the necessary condition of for maximizing the functional that is Euler question, so here we today bind up our lecture and we will discuss the necessary condition in next class. So thank you for listening us and thank you.