In part and Integral equations, calculus of variations and their applications

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Lecture 40

Calculus of Variations Introduction: Baisc Concepts-1

Hello friends, welcome to the today's lecture on calculus of variations, if you see we have discussed in previous lecture the concept of functional. So here we started with the definite integral I of y defined as x1 to x2 f of x y and y dash d of x and we have seen that the value of this definite integral is depending on the function y of x and which is a function of (()) (0:55).

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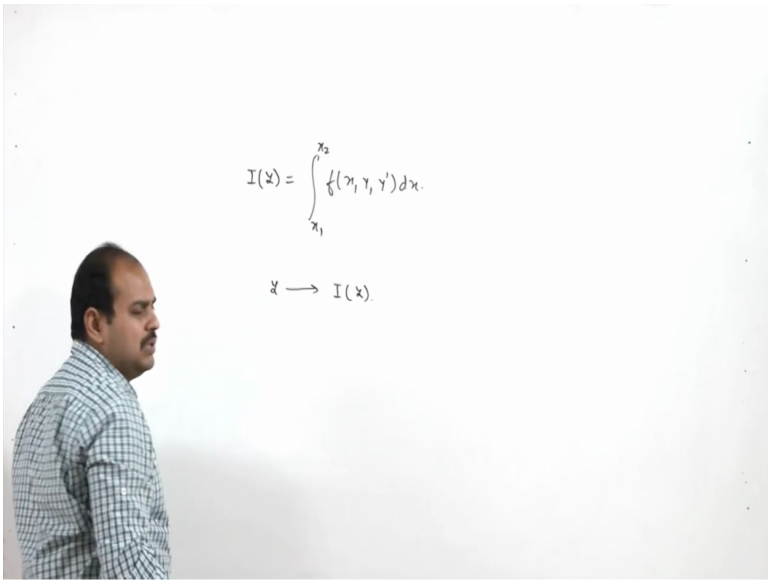

So we can say that this definite integral can be considered as a map from this function y to the value I of y which can be considered as a function of this function y and which assigned the function y to the value denoted by I of y, so we can consider this as a function whose argument is function itself. So this is the concept termed as functional and we have seen that the calculus of variations is a particular branch of calculus of functionals in which we try to discuss the function try to discuss the method to find out the extremum and extremum values of these kinds of functions.

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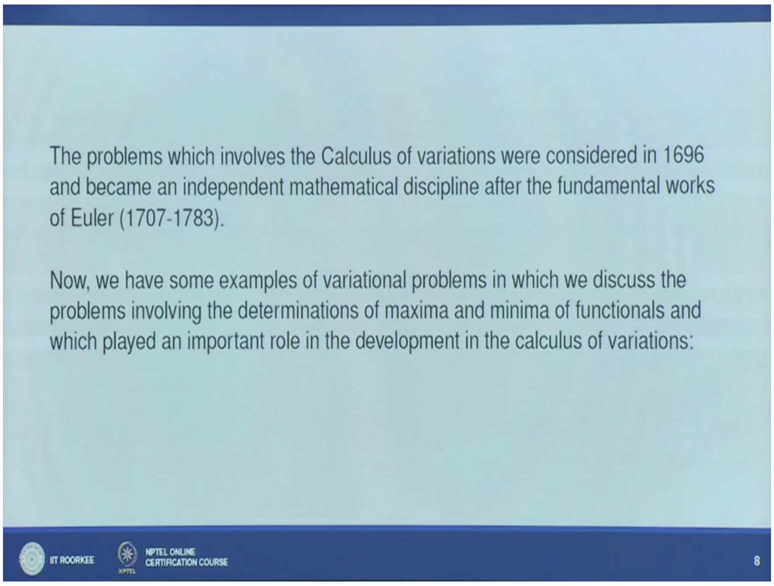

So here we can say that the problems which involve the calculus of variations were first considered in around 1696 and became an independent mathematical branch after the fundamental work of Euler in around 1737. Now we try to discuss several problems of calculus of variation which is supposed to be say beginning point of the classical calculus of variations, in fact it is the work of Johann Bernoulli he posed a problem in June 1696 and that problem is known as the problem of Brachistochrone problem, so that is going to discuss as follows.

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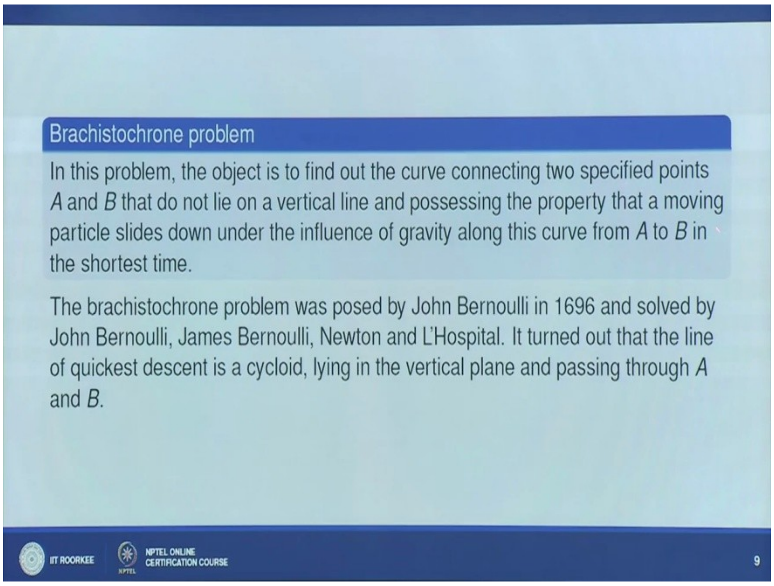

So the first problem which he discussed is this, that the object is to find out the curve connecting specified points A and B that do not lie on vertical line and processing the property that a moving vertical slides down under the influence of gravity and having no fiction along these curves from A to B in the shortest time. So it means that let me just give you a small idea here.

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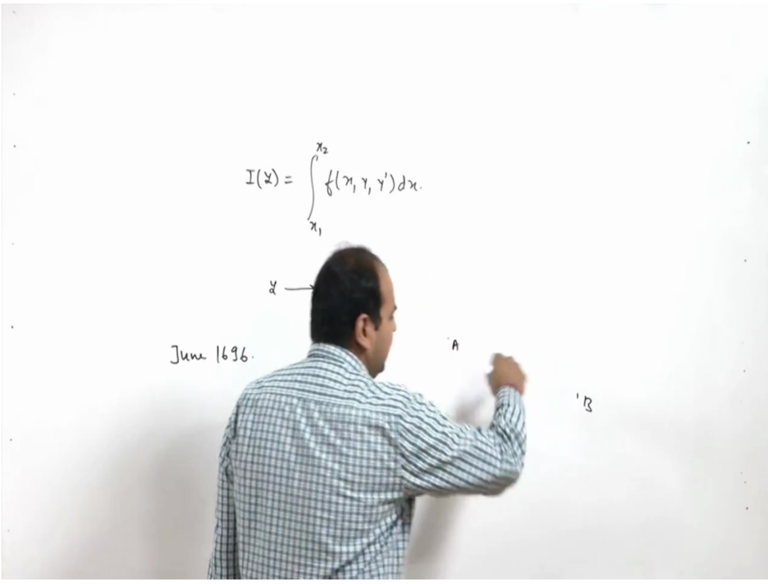

So here we simply say that we have 2 points A and B and they lie in a vertical plane not innovative line they are not either on the same level or not a vertical line and we try to find out say the curve connecting A to B, so here let me write B here, okay. We try to find out that there is no friction along this curve y, let us call this as y as a function of x here and then we take a particle on this, starting from this A and just under the influence of gravity it starts falling along this curve A, B and we need to find out say this function y equal to y of x belong which this particle takes the least time.

In fact Brachistochrone means “Brachisto” mean shortest and “chrone” means time, so it is a problem of shortest time or you can say that this is a problem of quickest descent and this problem was posed by John Bernoulli in 1696 and solved by John Bernoulli himself and his brother James Bernoulli, Newton Leibniz and L’Hospital and it turns out that the line of quickest descent is a cycloid and the word lying in the vertical plane and passing through A and B.

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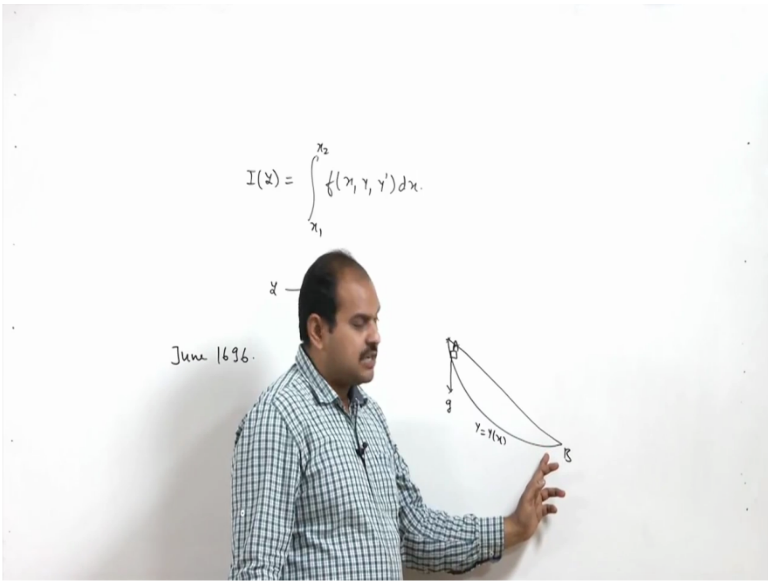

If you remember the cycloid and it is not it is not a straight line connecting this A and B. It is something like cycloid connecting these points A and B though this line straight line is a line having say least distance between A and B but this is not the curve along which this particular will take least time rather than it is the cycloid which takes least time between A to B.

So you can consider that if we have to find out a curve which is having say steepest say curve near A, so that we have a greater velocity near A and then it can reach to point B in a shortest time and it is turned out to be that it is turned out to be that this curve is your cycloid. We will discuss this problem, Brachistochrone problem in a separate lecture.

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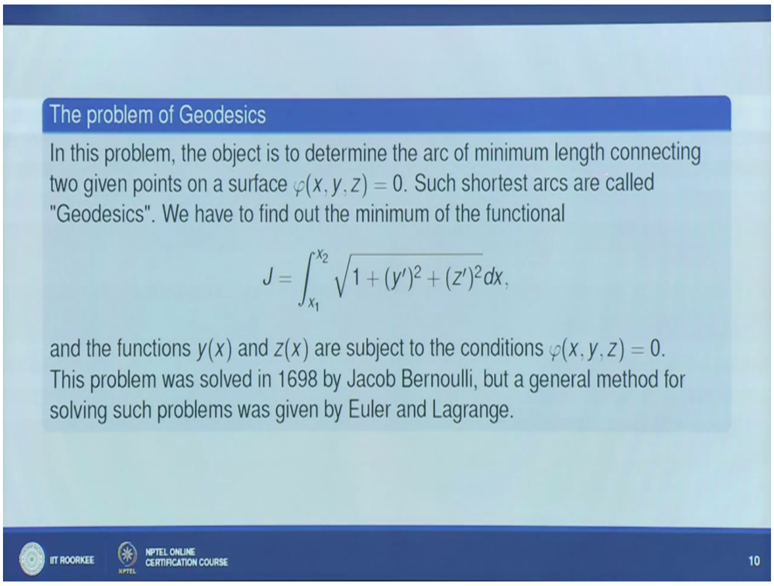

And the second important problem is the problem of Geodesics and in this problem the object is to determine the arc of minimum length connecting 2 given points on a surface y(x, y, z) equal to 0. So it means that we have a surface y, (x, y, z) equal to 0, for the time being you can consider that the earth is the surface or you can say that we have a sphere as a surface and we try to find out say the minimum distance between 2 points and they are lying on the same floor.

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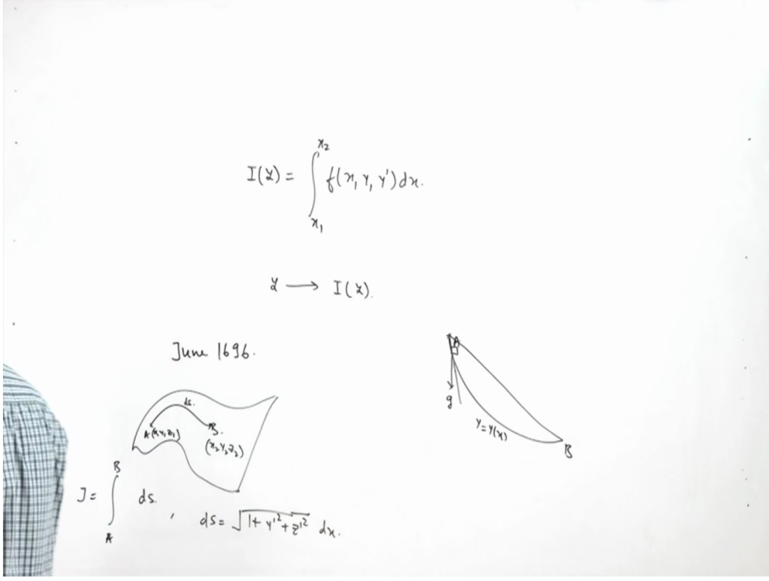

So here we have a surface and there is a 2 point on that surface and we try to find out say minimum distance between these 2 points and we can say it like this, so here we have a surface like this you can take anything and then we have a point A and B and we try to find out say minimum distance between the point A and B and in a way such that this curve will also lie on the surface.

So here we can say that we are minimising this functional which is J equal to x1 to x2 and this is nothing but surface patch between x1 to x2, we can say that if we denote this as say x1, y1, z1 and it is x2, y2 and z2. We can consider that this is your, you can find out say line integral between A to B and this is given as J equal to say point between say this A to B, so A to b d of s.

Now d of s you can find out as here under root 1 plus y dash square plus z dash square d of x. So here you can say that the we have to find out the minimal of the functional J equal to from x1 to x2 under root 1 plus y dash square plus z dash square d of x provided that this y of x and z of x are subjected to the condition that it lie on the surface, so it means that phi of x, y, z is equal to 0 and such shortest arcs are called Geodesics and this problem was solved in 1698 by Jacob Bernoulli butter general method general setup to solve these kind of problems were again given by Euler and Lagrange.

So if you remember this problem is different from the first problem, the problem of steepest descent in a way problem of quickest descent in a way such that here, not only we have to minimize this functional but also that the curve which minimises functional is also satisfying certain other foundation that is that curve has to lie on the surface phi x, y, z.

So it became say that it is kind of connective functionals or we can say that this is the problem with say certain constant, okay. So here we have to minimize this functional provided that it satisfies the certain given constant that is your x, y, z is related by phi of x, y, z equal to 0 it lays on the surface here.

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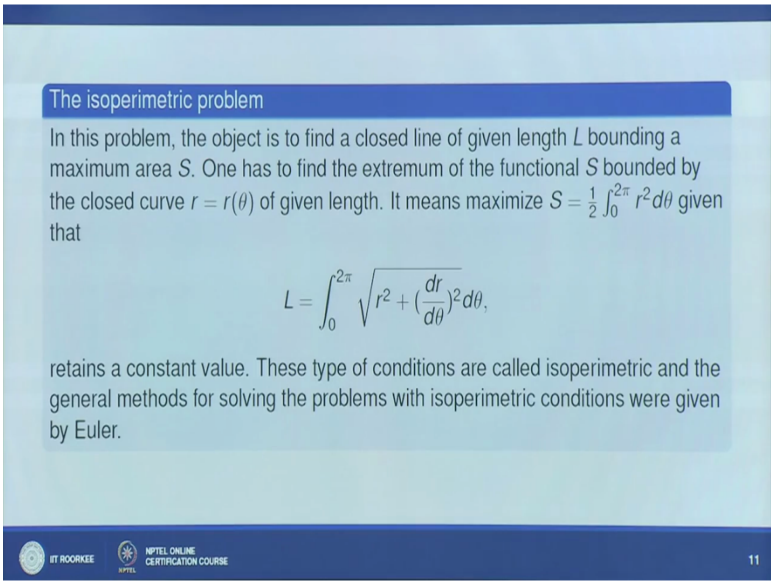

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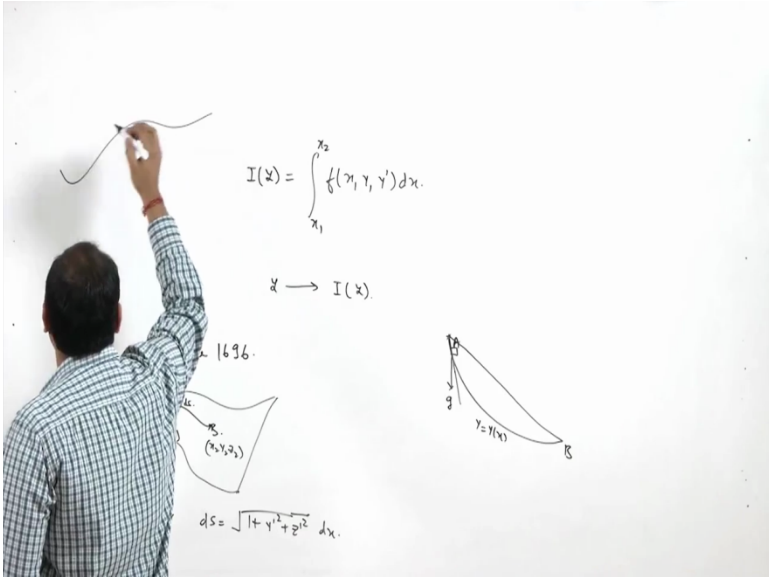

So next problem is very important problem is the isoperimetric problem. In this problem the object is to find a find a close line of given length L bounding a maximum area S. So here it means that we have a curve whose length is already given to you, so here we have a curve whose length is already given, say L is given to you and we want to find out in a way such that it encloses a maximum area.

So it means that we try to find out a closed curve bounded by this arc and we try to find out a surface having maximum area and if it is known from the ncn Greece time then such an arc is basically a circular. So circle is the solution of this whose parameter is known to us and it includes the maximum area, so we can say that let me pose this problem in a way that one has to find it find the extremum of the functional S bounded by the close curve r equal to r theta of given length.

It means that we have to find out say extremize the area functional, so this s is basically the area of the area of the series and bounded by this this r equal to curve r equal to r theta. So here this is the area 1 by 2, 0 to 2pi r square d Theta. So this is the area functional and we have to maximize provided that the length of the arc which is 0 to 2pi under root r square plus dr by d Theta is quiet d of Theta.

So keeping this L as fixed we have to maximize this and it is coming out to be that this is nothing but your cycloid. So here these types of conditions are called isoperimetric and the general method of solving the problem with isoperimetric conditions were given by Euler. So this is also similar to the second problem that we have to extremize maximize this functional provided given function is constant given functional is constant.

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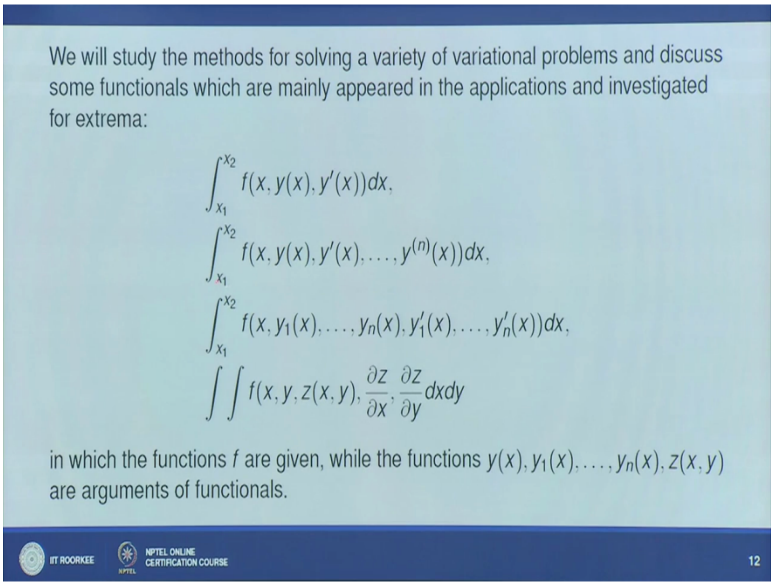

So we try to generalize in calculus of variations, we try to generalize the concept presented here in variety of ways, so one way is just now we have seen functional x1 to x2 f of x, y of x, y dash x d of x so we have seen certain example based on this but this can be generalized into these following ways. So in second we may consider problem or functional which involves not only the first derivative but the neth derivative of the function y of x.

So this another function which involved the neth derivative of the curve y and similarly so we can say that this is the functional which involved only the first-order derivative of a function y of x and this is the functional which involves the first-order derivative as well the neth order derivative of the functional and we can generalize it further we can say that this functional is not only depending only on 1 function but it is also depending on the n number of functions.

So here this is the functional defined which depends on several functions. So we can say that it is x1 to x2 f of x y1 x to y nx and its derivative here. So here this is the functional depending on several functions and we can further generalize in which this single integral is replaced by multiple integrals. So here your variable or argument is a function of x variable function of more than 1 independent variable.

So here this functional is depending on say z of xy where z is a function of x and y. So here your function you can define as this double integral f of xy z of xy and its partial derivative with respect to x and y dx dy. So these are the generalisation of the functional which we have discussed just now that is x1 to x2 f of x, y of x, y dash x, d fo x. So mainly it is the function which involved higher-order derivative of argument and it say involved several functions say y1 to y nx and this is a problem in which your argument is replaced by say function of several independent variables.

So here we want to it is given that this function f is already given while function y, y1, y nx and z xy are argument of this functional f, okay.

So now let us discuss the study let us discuss the basic concept involved integral is a variation. Here it is to be noted that this calculus of variation which is the branch of calculus of function in which we have to maximize or minimize the functional and it is closely related we will see that it is very say closely related to the analysis in which we discuss the maximize maximum and minimum values of functional.

So what we try to do here in these coming slides, we try to recall the definition from the classical calculus and then we try to see what is the relevant or what is the analogue version in calculus of variation? So first thing to discuss is the function, so function if you remember it is a correspondence between a variable an independent variable x to a given point.

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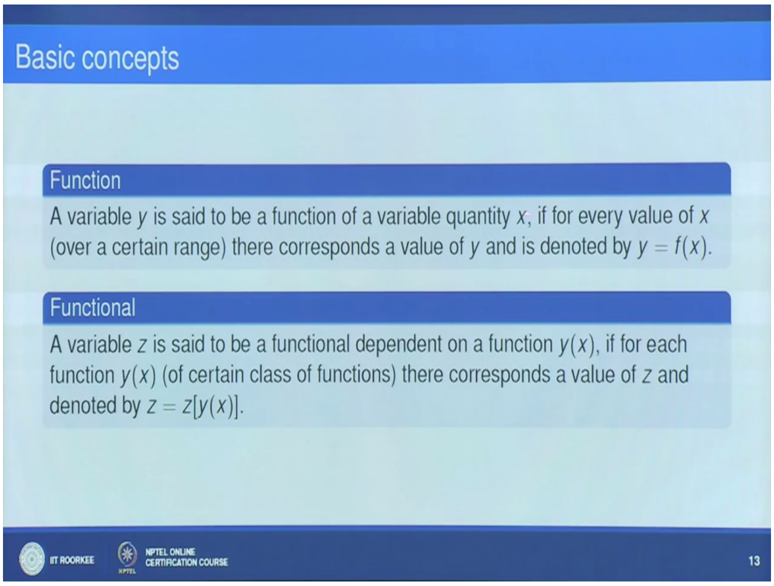

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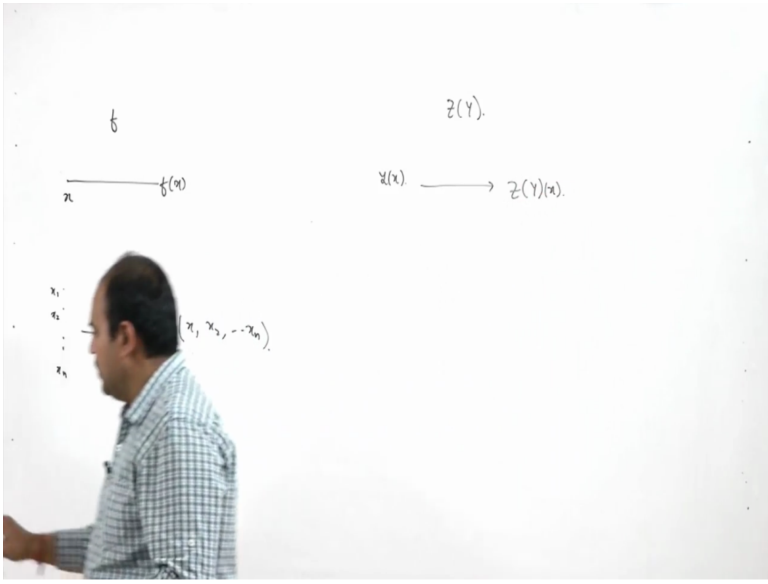

So we can say that a variable y is set to be the function of a variable quantity x, if for every value of x of course this x is ranging over a certain range where this y is defined there corresponds a value of y and is that value we can denote it as function of x. So here in function, so here this function is basically given any x you relate it to f of x, so from x we are go going up to f of x and if you look at the several variables means x1, x2 so we can write it x1, x2 and so on then we can say that this f is giving a set of values from x1 to xn if relate the value of this point to f of x1, x2 to xn.

So here we can say that function is kind of a correspondence between a point to another point which is denoted as f of x. So here in if it talk about function of several independent variable, here we can say that it is a correspondence which relate the given x1 to xn values given values of x1 to xn to a definite value f of x1 to xn. So if we define functional and in analogus version.

Here we can say that in place of x which is an independent variable now. Now we take a functional say y, y of x and we say that your functional this for the time being let me relate it to z of y here and we say that z of y is a correspondence which relate the function y of x to the value z of y as a function of x, right? So here we can say the variable z is said to be a functional dependent on a function y of x if for each function y of x, of course this function y of x is belonging to a certain class of functions on which this z is defined, they correspond a value of Z and denoted by z equals to z of y of x.

So here please recall here that here x is defined over a certain range and here in functional this argument y of x is defined on a certain class of functions. So we will discuss certain examples of certain class of functions. For example we can consider this class of functions as set of all continuous function or set of all continuously differentiable function and so on, so that we will discuss so we can say that in function it is a correspondence between a point to f of x where x lying in some kind of a range for example I can consider this x belonging to A to B and we can say that x is a correspondence for all values of x which is defined here, 2 values here.

Similarly functional is a correspondence from a given function from y to z of y, z of y of x and here we can say that y is a function, continuous function from A to B or you can consider y as a function of C1, function from A to B we will discuss what is this c and c1aba this c1 a, b denotes the functions which are whose first derivative is also continuous, okay.

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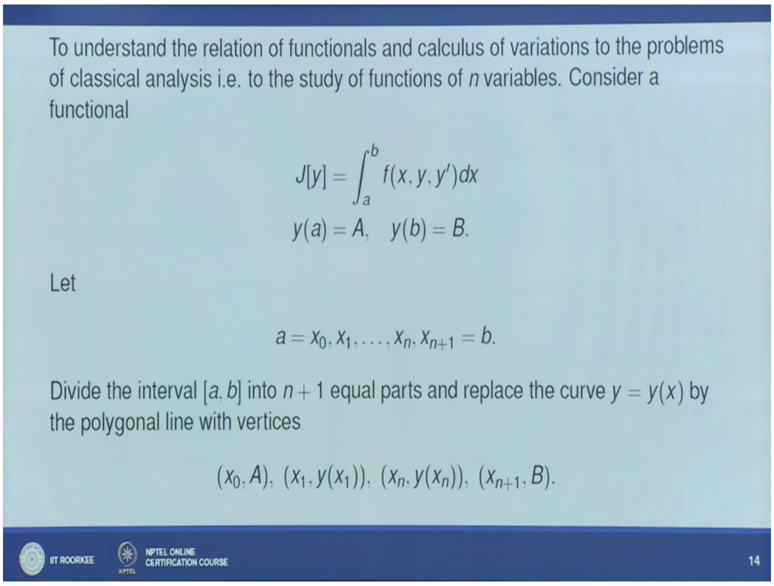

Now what we try to do here, we try to see that how this functions and this functions are related. So let us consider a particular example and we try to understand the relation of functional and calculus of variation to the problems of classical analysis. So in classical analysis we will discuss function and maximum minima of the functions, so here we will try to say how this function and calculus of variations is related to that corresponding terms.

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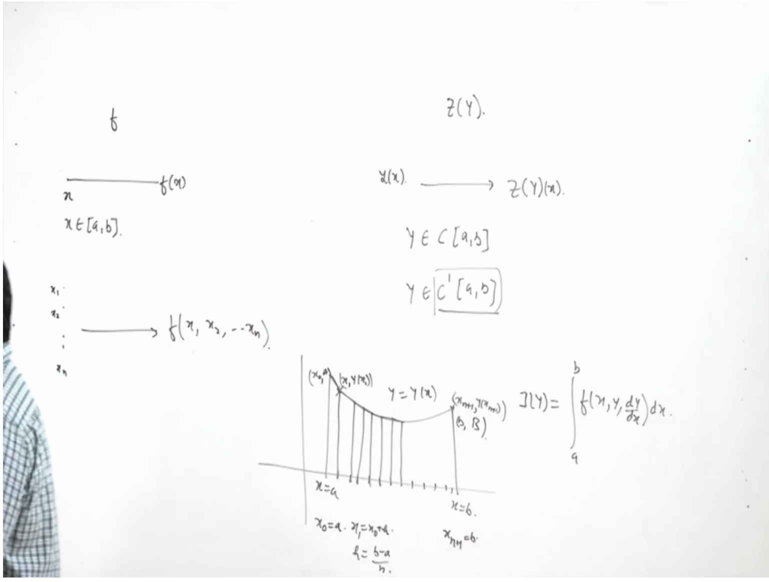

So what we try to do here? consider this functional J of y which is defined as A to B f of x y, y dash, d of x. Here y dash is dy by dx and this function y is satisfying this condition that y of a is equal to A and y of b is equal to B. So it means that, so here we have we curve say y of y of x this is y of y of x here and this is x equals to A and this is x equals to B and the corresponding functional is defined like this A to B f of x, y dy by dx and d of x and we try to find out the relation of this functional to the functions in classical analysis.

So what we try to do here, we truncate this interval a to b into n plus 1 equal parts. So here we truncate into these equal parts and so on. So here we can say that your x not is given as a and xn plus 1 is given as b and rest you can say x1 which is x not plus h is where h you can say that it is b minus a divided by n you can define the points x not, x1, xn up to xn plus 1 where x not is a and xn plus 1 is equal to b.

So we are dividing the length a, b into n plus 1 equal parts and replace the curve y of y of x by b polygonal line with the vertices. So what we try to do here? We replace this curve by these straight lines these polygonal lines, okay. So we can say we can approximate the curve by these polygonal lines and vertices is given by here vertices is given by say x not and a.

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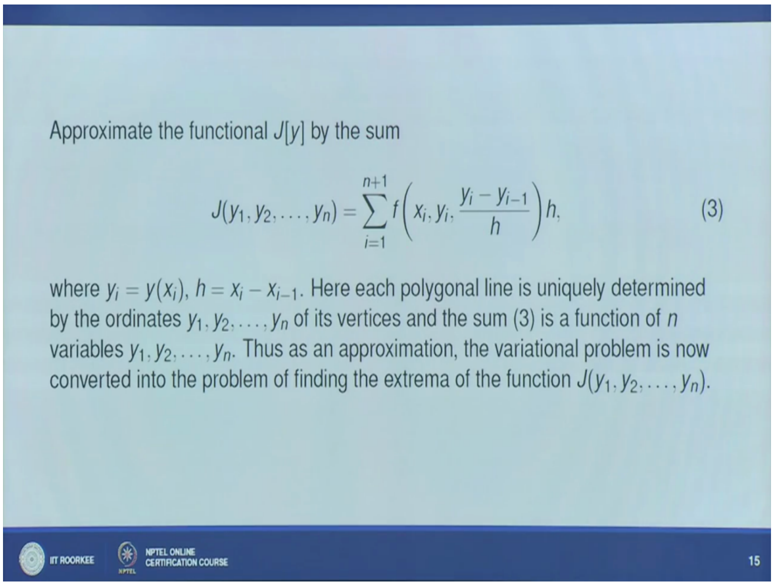

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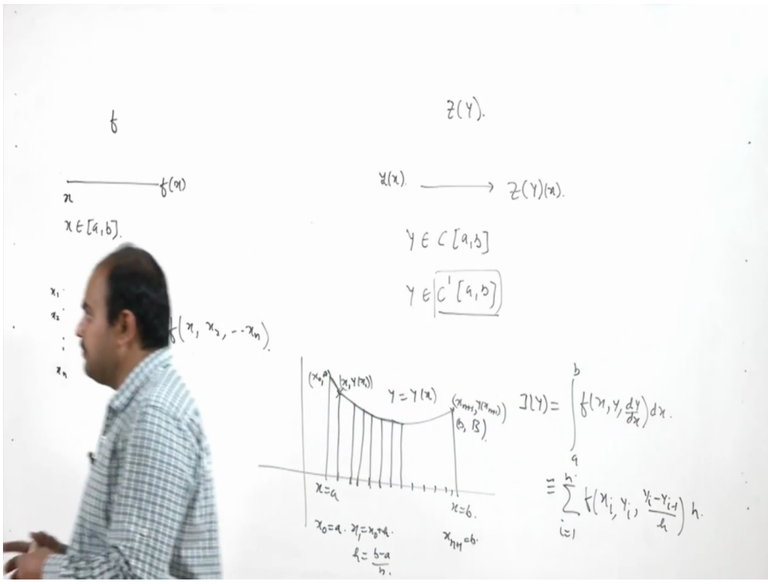

Here you're vertices, this vertices is given as x1 and the y of x1 and so on. So here your x of n plus 1 and y of xn plus 1 which is nothing but this is b and this is your yn plus 1 is given as capital B. So here we can say that, now this curve is approximated by polygonal lines and vertices are already given to us, right? So now we can say that if we do this than the approximate we can approximate the function J, y by the sum here.

So here we can approximate this as the finite sum here. Here we can simply write equal to 1 to n here f of say xi and y of xi is denoted as y of i and dy by dx can be approximated by its finite difference and we can write it yi minus yi minus 1. So here we can write yi minus yi minus 1 divided by h and you can write it like this, is it okay and into h. So here we can say that this is approximate, this integral is approximated by this finite sum.

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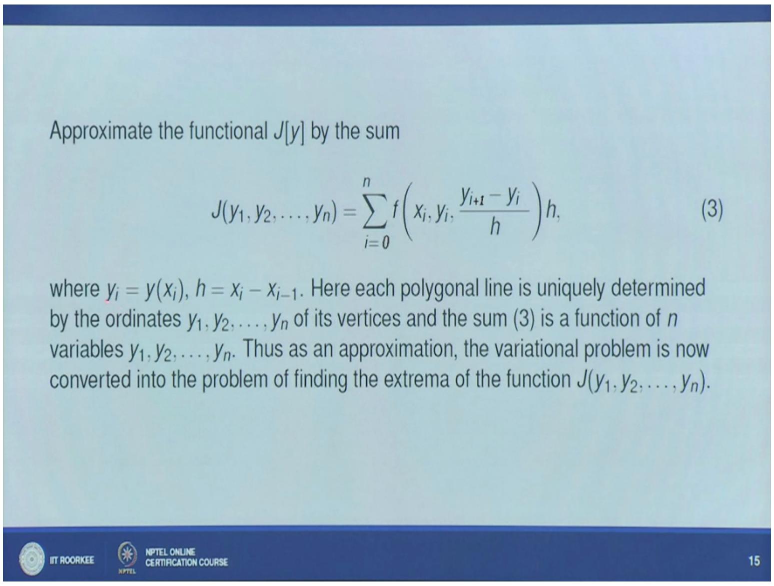

So here look at this equation number 3 this is the approximation of the functional and here we are denoting yi as y of xi and h is xi minus xi minus 1. Here each polygonal line is uniquely determined by the ordinate y1 to yn and what is y1 to yn? It is y1 is y of x1, yn is y of xn of its vertices and the sum3 is a function of n variables. Now here if y is your variable then this yi is also a variable, so we can say that here this sum is depending on say n variables that is y1 to yn.

Please remember this y not is already known to us y not is your capital A and y of b which is y of n plus 1 is also known that is given as capital B. So here this sum will depend only on say variables that is y1 to yn and we can say that this functional can be written approximated by J of y1 to yn. Here y1 to yn are variables and J can be considered as a function depending on these variables y1 to yn.

So here now we can say that with the help of these things, we can say that we can approximate the functional which is given by J of y equal to A to B fx y, y dash d of x as this (()) (25:48) i equal to 1 to n plus 1, f of xi,yi, yi minus yi minus 1 divided by h h and here we can say that as an approximation the Variational problem is now converted into the problem of finding the extrema of the function J y1 to yn.

So there this J in the original problem J is considered to be the functional which depend on this y but with the help of this finite different scheme we can say that, that can be approximated by extrema and extrema of the function J which in the end variable that is y1 to yn. So here we can say that these 2 are related to each other or I can say that the finite version of functional is a function in n variables.

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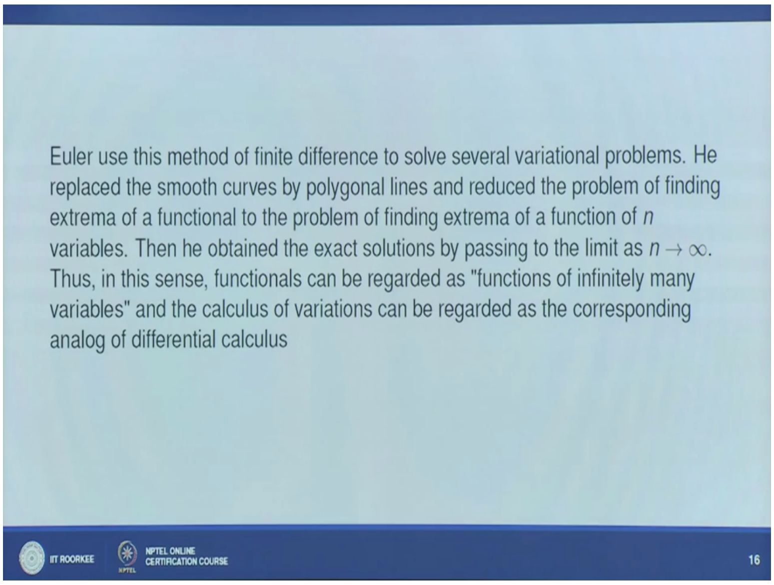

So Euler use this method to of fine difference to solve several Variational problem in a way such that the functional is truncated into a function of n independent variables and he tried to maximize and minimize the function of n independent variables and then he tried to consider the limit as n tends to infinity and then try to conclude the maximum minimum of the corresponding functional.

So here it is this that he replaces the smooth curves by polygonal lines and the reduced the problem of finding extrema of a functional to the problem of finding extrema of a function of n variables, right? And then he obtained the exact solution by by passing to the limit as n tending to infinity.

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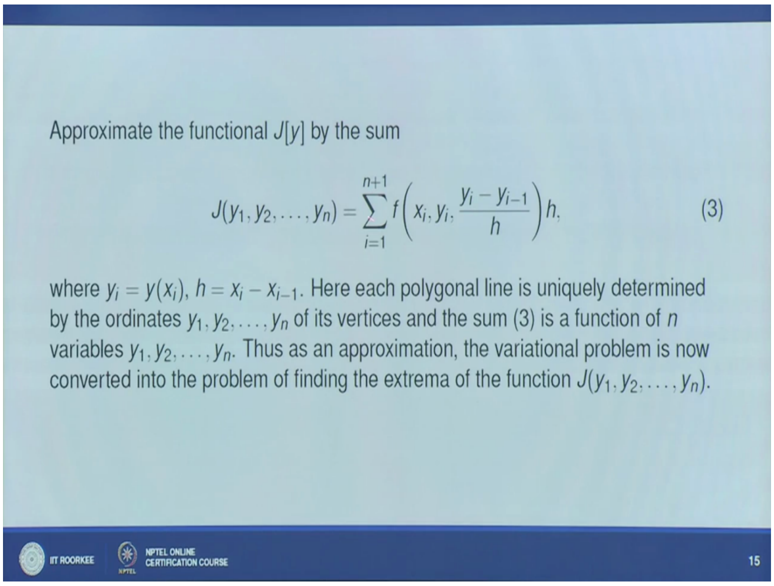

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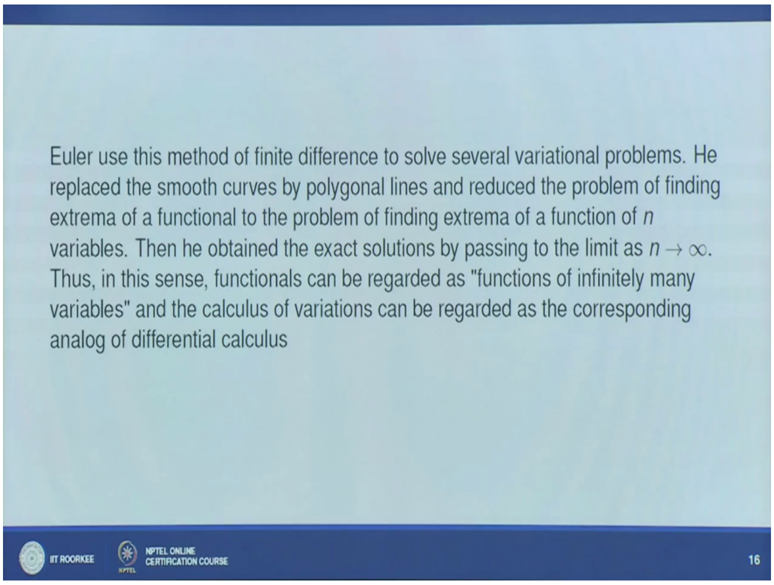

We can say that this equation number 3, if you pass on the limit as n tending to infinity then it includes the approximate this function in thus in this sense functional can be regarded as functions of infinitely many variables. So if you recall what we have done is, we have approximated the functional as the functions of n independent variable and then we can say that as n tending to infinity, we can say that this functional can be treated as a function of infinitely many independent variables.

So that is what we are writing here, that in this sense functional can be regarded as functions of infinitely many variables and the calculus of variation can be regarded as the corresponding analogue of differential calculus. Now to discuss the methods for maximising and minimising the functional we need to know certain basic definitions and these definitions are in a analogous version of the definitions available in calculus, classical calculus.

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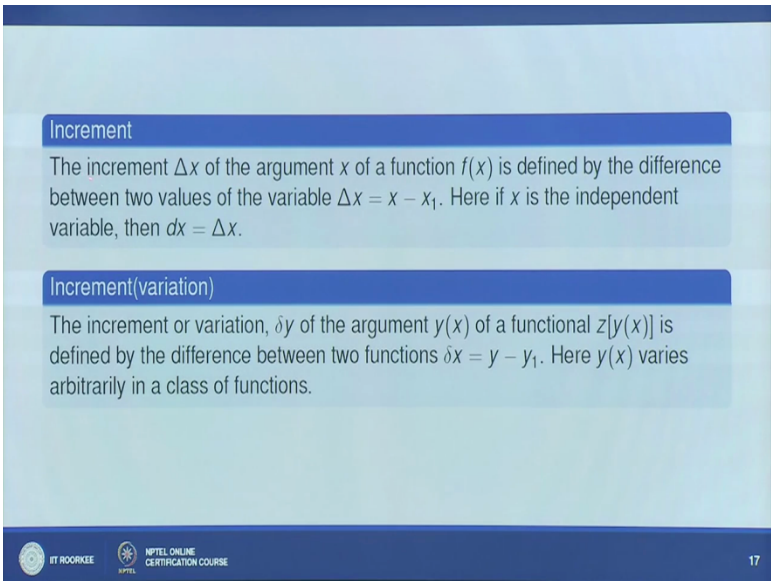

So we will discuss one by one and first we try to recall the definition given in in classical calculus and then we give the analogue version of the definition in calculus of variation. So first definition is known as increment, so in classical analysis if we talk about increment in increment in independent variable is defined as the Delta x and it can be written as the increment Delta x of the argument x of a function f of x is defined by the difference between 2 values of the variable Delta x. So it means that if we take x1 as a fixed variable fixed value then the difference between x minus x1 is denoted as increment in x and here if x is the independent variable then we may recall that differential of x is given by this increment in x.

If x is the independent variables then the differential of x is matching with the increment in x. So similarly we can define the analogues version of increment in calculus of variations, please remember here the argument of the functional is a function itself. So here we can say that the increment or it is known as the variation Delta y of the argument y of x of a functional z of y of x is defined by the difference between 2 function y and the value y1 here and here y of x arbitrarily in a class of functions on which this functional is defined.

So here in this in classical analysis it is just a difference between 2 values of x but here in variation it is the difference between 2 functions, okay. So here we need to define this variation in a proper way. Now with the help of this increment, now we can define the continuity of a function. So continuity of a function is defined in the sense that if there is a small change in argument of this function f that is a small change in x then there is a small change in the function of f of x.

And similarly in analogues version we can say that the functional z of yx is called a continuous functional if for a small change in argument, now argument is a function here y of x there correspond a small change in the functional z of y of x, so it means that if in argument y of x if we replace another value say that is y1 of x and there is a small change between y of x and y1x then we try to find out whether this difference between the functional is small or not.

If it is small we can say that your functional is continuous if it is not small we say that your functional is not continuous. So here we will discuss more thing about, how we define the closeness or how we say that this y of x and y1x are close to each other or the difference between y and y1x is small. So to discuss this, we will discuss certain more things that we will continue in the next lecture.

So here we stop and in next lecture we will define in precise manner, what you mean by continually of a function? And discuss certain properties some examples of function which are continuous functional. So here we stop and we will meet in next lecture, thank you very much.