Integral Equaltions, Calculus of Variations and their Applications
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Lecture 04
Conversion of integral equations into differential equations

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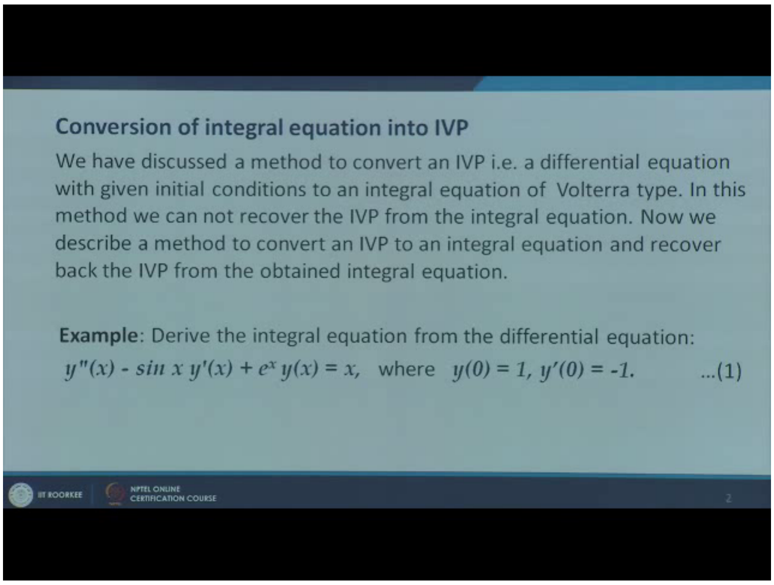

Hello Friends! I welcome to my lecture on conversion of integral equations into differential equations what we will do is first we will begin with a differential equation initial value problem. And we shall convert it into a integral equation as we have already seen in previous lectures that when we consider an initial value problem and convert it to an integral equation we obtain a Volterra integral equation of the second kind.

Now what we are going to do is now we shall be converting this Volterra integral equation into the initial value problem. So one method which we had discussed earlier where we assumed the highest derivative occurring in the differential equation as a unknown function by 5x there if we cannot apply that method to recover back the initial value problem. So there is another method which can be used to convert a given initial value problem into the Volterra integral equation of the second kind and then recover back the initial value problem from the Volterra integral equation.

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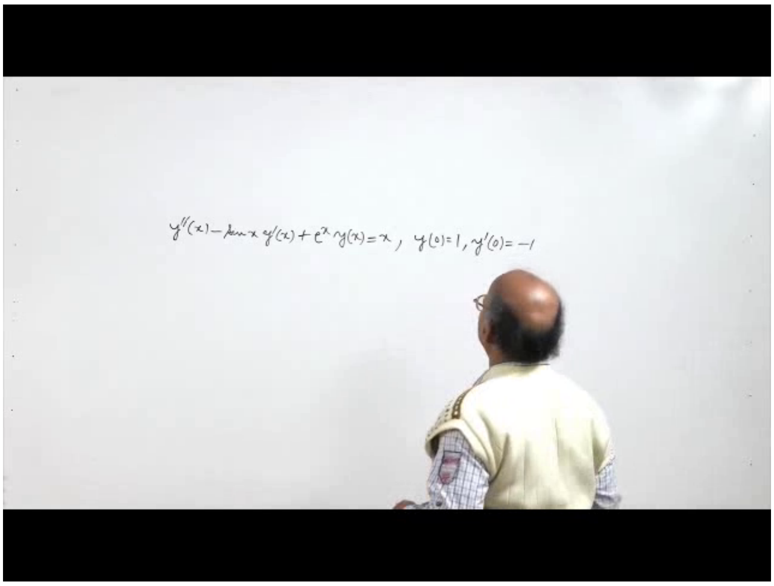

So this is what the initial value problem here y double dash (x) minus sin (x) y dash(x) y double dash(x) minus sin(x) y dash(x) plus e to the power x into y(x) equal to x. And we are given the initial conditions y(0) equal to 1 y dash(0) equal to minus 1. So what we want to do is we will first get the integral equation associated with this initial value problem and then we shall recover this initial value problem from the integral equation resulting integral equation.

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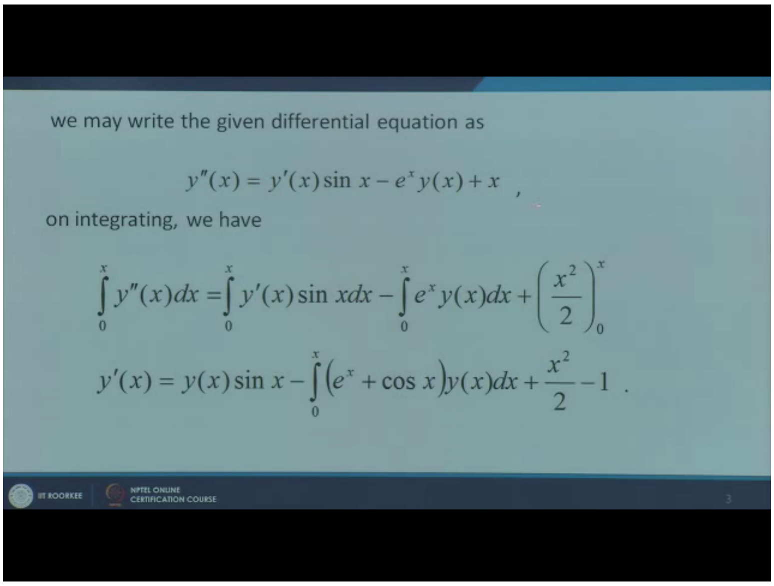

So what we do is let us write this differential equation as y double dash(x) equal to y dash(x) into sin(x) minus e to the power x y(x) plus x. Then we integrate this equation with respect to x then we integrate it with respect to x from 0 to x what do we get 0 to x y double dash(x) into dx equal tto integral 0 to x y dash(x) into sin (x) dx minus integral 0 to x e to the power x y(x) dx plus x square by 2 which is the integral of x and evaluated over the limits 0 and x.

So here when we integrate y double dash (x) we get y dash(x) the value of y dash(x) at x is y dash(x) minus y dash(0) and y dash(0) is given to be equal to minus 1.

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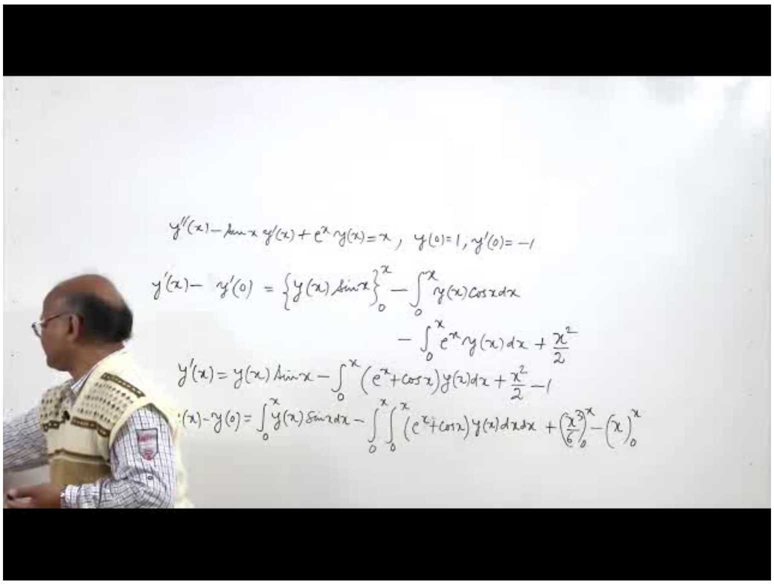

So left hand side will be y dash(x) minus y dash y dash(0) equal to y(x) 0 to x y dash(x) into sin(x) we can integrate that by parts so the integral of y dash(x) is y(x) into sin(x) 0 to x and then we are integrating by parts.

So integral 0 to x and we have y(x) here into the derivative of sin (x) derivative of sin(x) is cos (x) and then we have minus integral 0 to x e to the power x y(x) dx. So we will get integral 0 to x e to the power x y(x) dx. And when we have x square by 2 because its value at 0 is 0. So x square by 2 and y dash(0) is equal to minus 1 so we will have y dash(x) plus 1 here.

And that can be taken to the other sides so we have y dash(x) equal to here y(x) into sin(x) at x it is y(x) sin(x) but at 0 sin(x) is 0 so we shall have y(x) into sin (x) minus we can combine these two integrals their limits are of integration are same. So we shall write minus 0 to x e to the power x plus cos x into y(x) dx plus x square by 2 plus y dash(0), y dash(0) is minus 1.

So we have y dash(x) equal to y(x) sin(x) minus 0 to x e to the power x plus cos x into y(x) dx plus (x square by 2) minus 1. Now let us again integrate this equation when we integrate it again what we will have, y(x) 0 to x and then integral of y(x) into sin(x) over the interval 0 to x and then minus 0 to x.

So here we shall have double integral this is y(x) minus y(0) equal to integral 0 to x y(x) sin(x) minus double integral e to the power x plus cos x into y(x) dx dx and then plus integral of x square by 2 will become x cube by 6 and minus.

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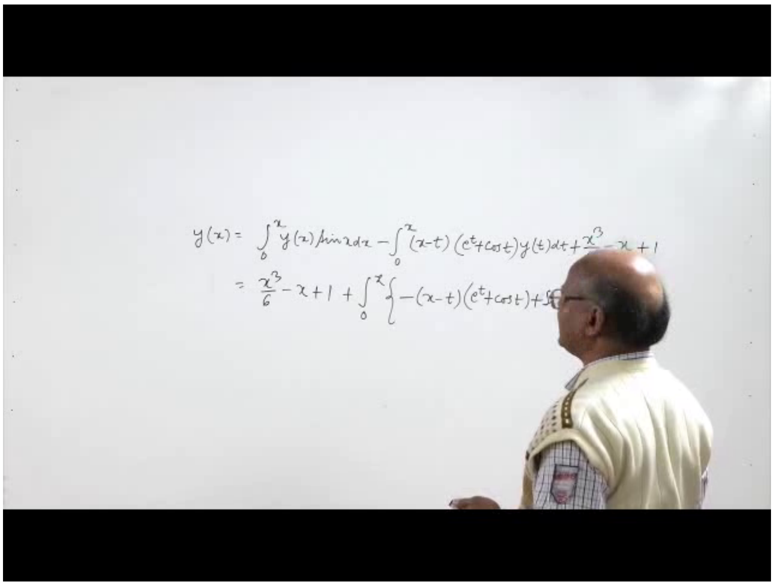

So y(0), y(0) is equal to 1 we can put the value y(0) as 1 here and what do we get here y(x) equal to integral 0 to x y(x) sin(x) this double integral can be converted into a single integral.

So minus 0 to x x minus t into e to the power t plus cos t into y(t) dt plus x cube by 6 minus x and y(0), y(0) equal to once we get plus 1 here. So we get x cube by 6 y(x) equal to x cube by 6 minus x plus 1 integral 0 to x then we have this minus sign can be taken inside the integral so minus (x minus t) into (e to the power t plus cos t) plus sin (t) into y(t) dt.

So I have sin(t) here because we are combining this integral and this integral, ok so we have combined this so I can write it as x cube by 6 minus x plus 1 we can combine the two integrals and write it as 0 to x minus (x minus t) (e to the power t plus cos t sin t0 into y(t) dt. So this is what we have and it is a as we can see it is a Volterra integral equation of the second kind. Because the unknown function y(t) across inside the integral here where the limits of integration are 0 and x. So both the limits are not constant. So this is a Volterra integral equation of the second kind.

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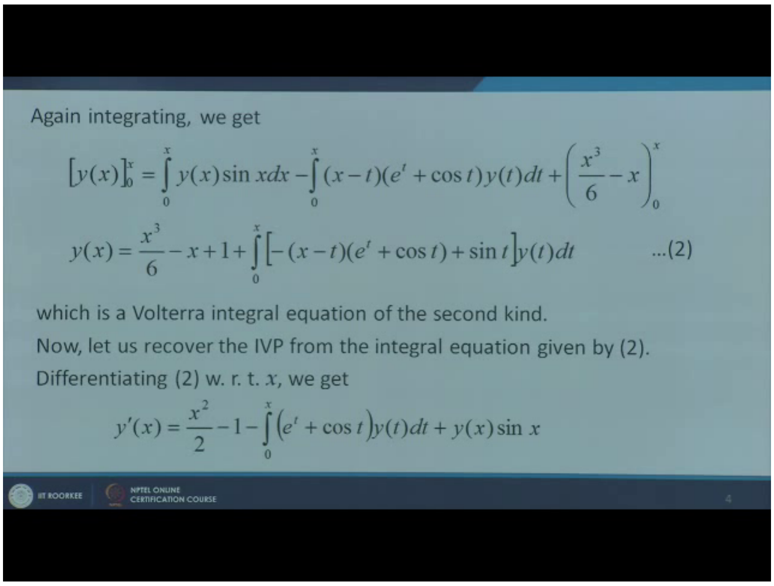

Now let us recover the initial value problem from this integral equation. So we shall begin with the equation here this one y(x) equal to x cube by 6 minus (x plus 1) plus integral 0 to x minus (x minus t) into e to the power t plus cos t plus sin t into y(t) dt. We shall begin with this integral equation and we shall obtain the initial value problem with which this started and got this integral equation.

So what we will do let us differentiate this with respect to x. When you differentiate it with respect to x what you get is, y(x) becomes y dash(x) and x cube by 6 when you differentiate you get 3 x square by 6 you get x square by 2 and minus x becomes minus 1, the derivative of 1 is 0. Then you have to differentiate this integral with respect to x.

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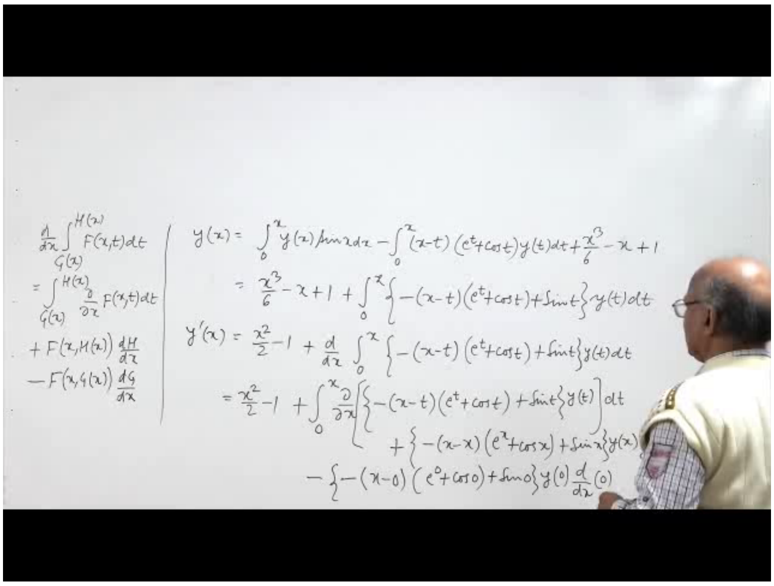

So let us see how we get this so when you differentiate what you are getting is y dash(x) equal to 3 x square by 6 so x square by 2 minus 1 and then d over dx of this integral. So here we are differentiating the integral where the limits of integral integration are not constants 1 is 0 which is a constant the other one is a variable x. So let us recall the formula for differentiation under the sign of integration which is the Leibnitz rule.

And so let us apply the Leibnitz rule d over dx of integral dx to hx f(x, t) dt this is nothing but integral gx to hx partial derivative of f(x, t) with respect to x dt and then plus the value of the integrant f(x, t) at the upper limit which is f (x) hx. Then derivative of the upper limit with respect to x, so dh by dx minus the value of the integrant and the lower limit so f(x) gx and then the differentiation at derivative of g lower limit with respect to x, so dg by dx.

So let us apply this formula to differentiate this, so this is x square by 2 minus 1 plus integral 0 to x delta over delta x minus x minus t e to the power t plus cos t plus sin t into y(t) dt cos instead of t we put here x so we get minus x minus x e to the power x plus cos x plus sin x into y(x) and then d over dx of x minus we now put in the integrant t equal to 0. So we get minus x minus 0 into e power 0 plus cos 0 plus sin 0 int y(0) into d over dx of 0, ok.

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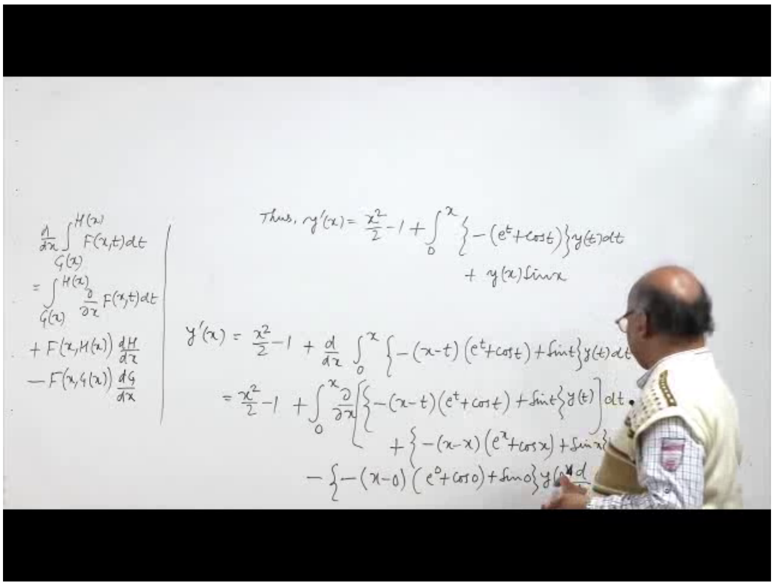

Now what we get is so thus y dash(x) is equal to x square by 2 minus 1 and when we differentiate here with respect to x what we get plus integral 0 to x, so this is minus x plus t, ok into e to the power t plus cos t when we differentiate with respect to x what we will get is minus 1 here, so minus 1 into e to the power t plus cos t derivative of sin t with respect to x will be 0, so we have into y(t) dt.

And here x minus x is 0, so we are multiplying this thing by 0 and then here we have sin x into y(x) derivative of x with respect to x is 1 so we get plus y(x) into sin(x). Here what we are doing this expression is multiplied by 0 into d 0 by dx, d0 at dx is 0 so this whole thing becomes 0. So we get the following.

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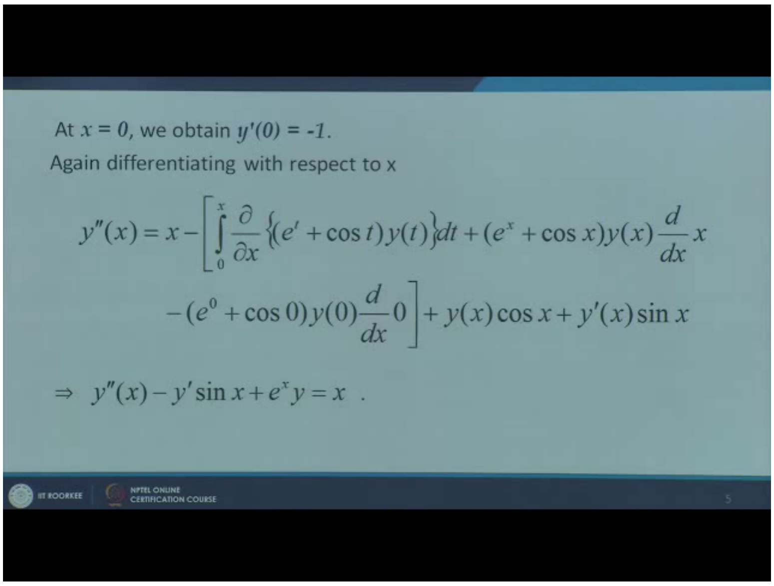

So you can see we get y dash(x) equal to x square by 2 minus and then minus sin from the integral can be taken outside so we have minus 0 to x e to the power t plus cos t into y(t) dt plus y(x) into sin x.

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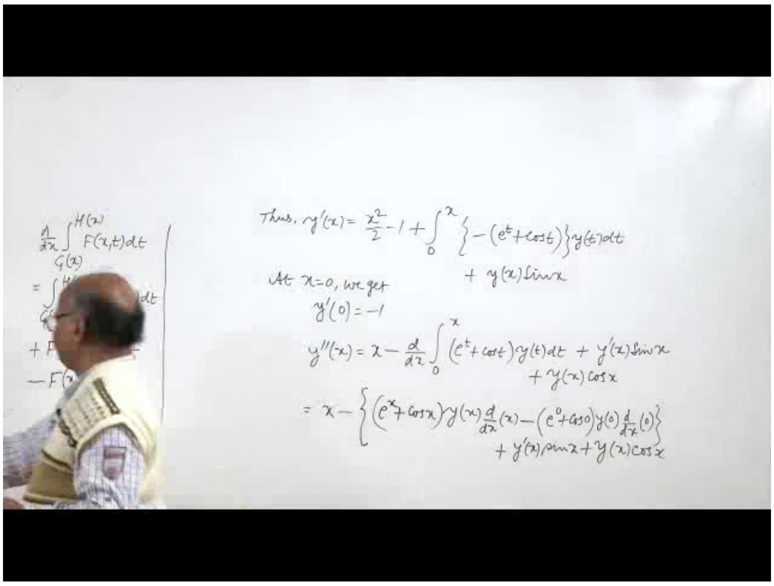

Now let us at x equal to 0 let us see here when we put x equal to 0, let us see when we put x equal to 0 here we get d as 0 minus 1 integral limits are 0 0 so the value of the integral is 0, so y dash(0) equal to minus 1.

Now let us differentiate it again with respect to x, so now we differentiate this equation with respect to x so we get y double dash(x) equal to 2x by 2 which is x derivative of minus 1 is 0 minus d over dx plus d over dx 0 to x minus with this minus I can write here outside.So minus d over dx and e to the power of t plus cos t into y(t) dt plus derivative of y(x) into sin x with respect to x. So we can apply here this summation of the derivative of all the products of two functions.

So we shall have y dash(x) into sin x plus y(x) into derivative of sin x is cos x. Now let us again apply the Leibnitz here so you can see here the integrant does not depend on x therefore when we take the derivative of the integrant that is we take this first term we evaluate so partial derivative of this e to the power t plus cos t y(t) with respect to x will be 0. So this part will become 0 and therefore we shall have to write x minus we evaluate the value of the integrant at the upper limit.

So we get e to the power x plus cos x into y(x) d over d x of x and then at the lower limit is 0, so minus we evaluate the integrant at the lower limit. So e to the power 0 plus cos 0 y(0) and then we differentiate 0 with respect to x. So we have y dash(x) sin x plus y(x) into cos x this is 0 because derivative at 0 is 0, so what we have is x minus y double dash x equal to x minus e to the power x plus cos x into y(x) derivative of x with respect to x is 1. So we have the following.

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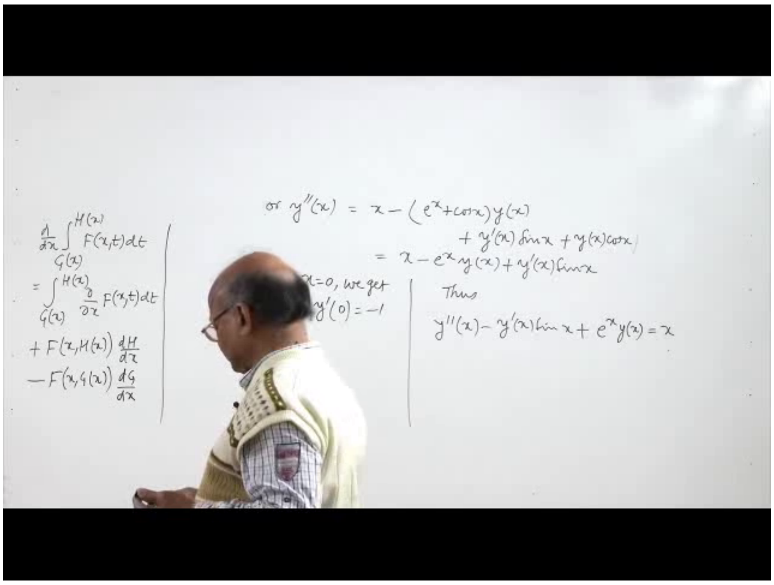

So we can write r y double dash(x) will be equal to x minus e to the power x plus cos x into y(x) plus y dash (x) into sin x plus y(x) cos x we can see that when we multiply y(x) here and simplify so this is further equal to e to the power x sorry cos x y (x) into cos x this term will cancel and we shall have x minus e to the power x into y(x) plus y dash(x) into sin x, or we can write it as thus y double dash(x) minus y dash(x) into sin x plus e to the power x into y(x) equal to x.

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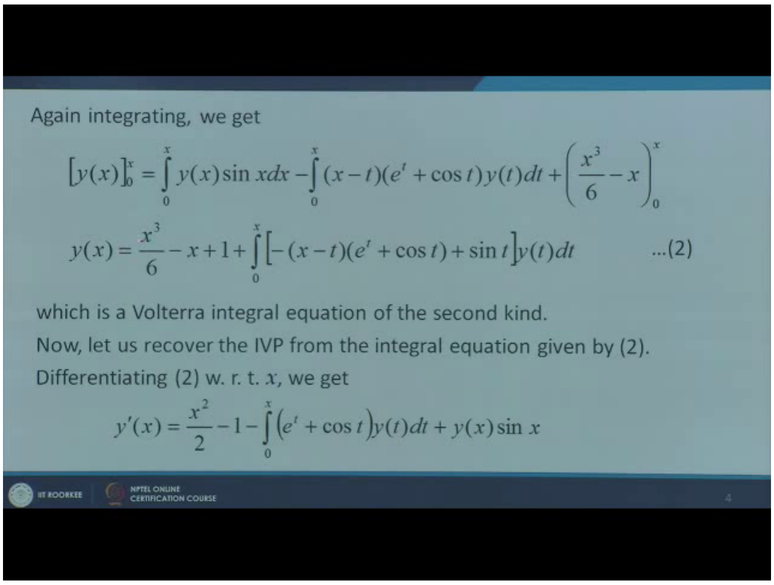

So we have got the differential equation back, there were two initial conditions one initial condition was the value of y at 0, the value of y at 0 we needed so that value at y at 0 we can see from the Volterra integral equation with which we started here you put x equal to 0 so this is 0 this term is also 0 and the integral will be 0 because the limits of integration will be both 0.

So y(0) equal to 1 we can see from the given Volterra integral equation of the second kind and y dash(0) equal to 1 we got when we differentiated this Volterra integral equation of the second kind with respect to x. So we have got the second order differential equation with the two initial conditions and therefore we have got back the initial value problem. So this is how we recover back the initial value problem from the integral equation. Now let us this is what we have.

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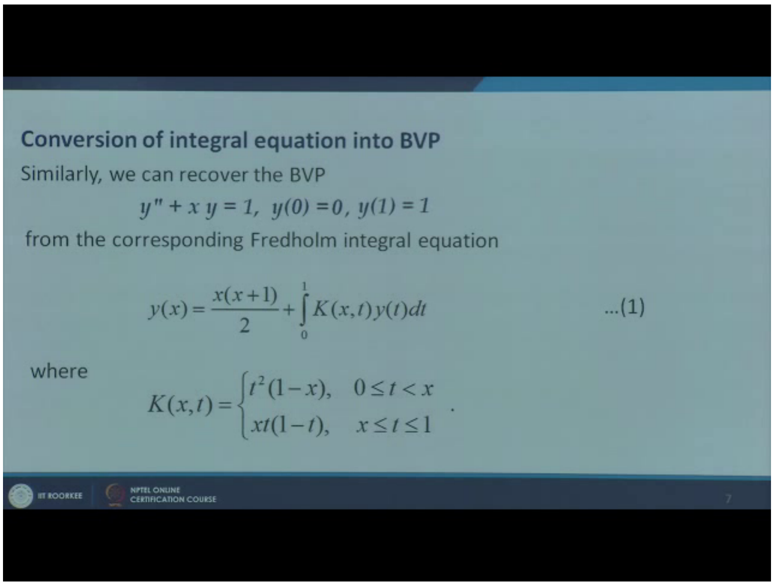

Now let us go to the case of a boundary value problem we shall see how we can get the boundary value problem back from the integral equation. Now we have earlier seen in previous lecture that the boundary value problem can be converted into a integral equation and the integral equation that we get is certain integral equation of second kind. So if you consider this boundary value problem y double dash plus xy equal to 1.

Then the two boundary conditions are given at x equal to 0 and x equal to 1 y(0) is given as 0 y(1) is equal to 0 then in the previous lecture we have seen that this boundary value problem when it is converted into an integral equation, the integral equation that we get is this Fredholm integral equation of the second kind y(x) equal to x into x plus 1 by 2 plus integral 0 to 1 k(x, t) into y(t) dt where we had obtained k(x, t) equal to t square into 1 minus x 0 less than equal to t less than equal to x and x t into 1 minus t when t lies between x and 1, so x is less than or equal to t less than or equal to 1.

Now let us see how we can we from this integral equation from this Fredholm integral equation of the second kind the boundary value problem, y double dash plus x y equal to 1 where y(0) and y(1) are both 0 at x equal to 0 and x equal to 1 so let us see how we get this?

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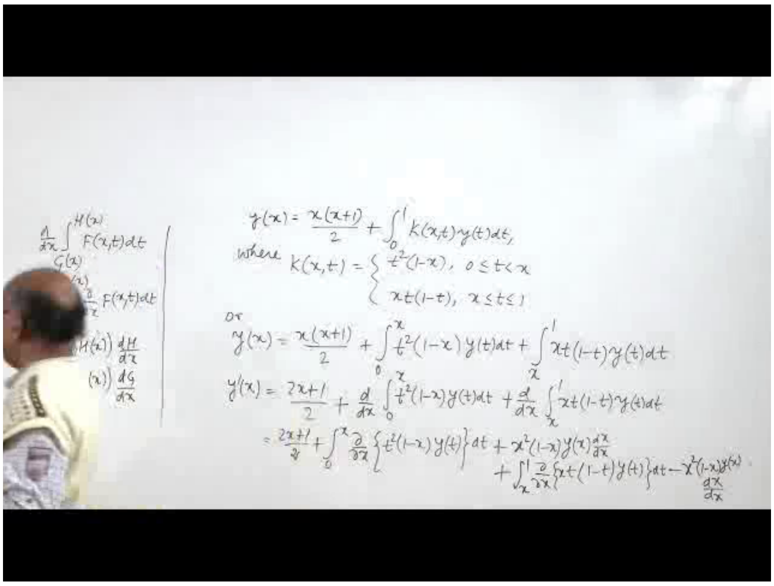

Ok! So we shall begin with the Fredholm integral equation y(x) equal to x into x plus 1 by 2 plus integral 0 to 1 k(x, t) y(t) dt where k(x, t) is t square into 1 minus x 0 less than or equal to t less than x and x into 1 minus t xt into 1 minus t when x is less than or equal to t less than or equal to 1, ok.

So the integral equation can be written as let us substitute the value of k (x, t) x into x plus 1 by 2 plus we break the interval 0 to 1 into 2 parts 0 to x and x to 1, so we get 0 to x over 0 to x interval the value of k(x ,t) is t square into (1 minus x) into y(t) dt plus the integral the value of the k(x, t) is xt into 1 minus t over the interval x to 1. So x to 1 xt into 1 minus t into y(t) dt, ok.

Now we shall have to differentiate this in order to get the boundary value problem. So let us differentiate it with respect to x. So y dash(x) equal to this is x square plus x so 2x plus 1 by 2 when we differentiate now here d over dx of integral 0 to x t square into 1 minus x into y(t) dt plus d over dx of x to 1 xt into (1 minus t) into y(t) dt. So for the differentiation we have to apply the Leibnitz rule of differentiation under the integral sin so this is 2x plus 1 by 2 plus ok.

So this is here f(x, t) this is f(x, t) when you take when you differentiate this integral so we have to take the partial derivative of this f(x, t) so delta over delta x of t square into 1 minus x into y(t) dt into y(t) and then we have dt. Then we put the t equal to x so we have x square into 1 minus x into y(x) into d x yx minus we put t equal to 0 in this integrant and then differentiate 0 with respect to x.

So that term become 0 so let us not write that then we have integral x to 1 we differentiate this with respect to x, so x t into (1 minus t) into y(t)plus now we put x t equal to 1 when you put t equal to 1 in the integrant that is the upper limit and differentiate 1 with respect to x derivative of 1 with respect to 0,

So that term will become 0 and then we put the lower limit in this integrant and differentiate with respect to x and put it negative sign.So we write here minus sign so we put t equal to x here so x square into 1 minus x into y(x) into dx by dx.

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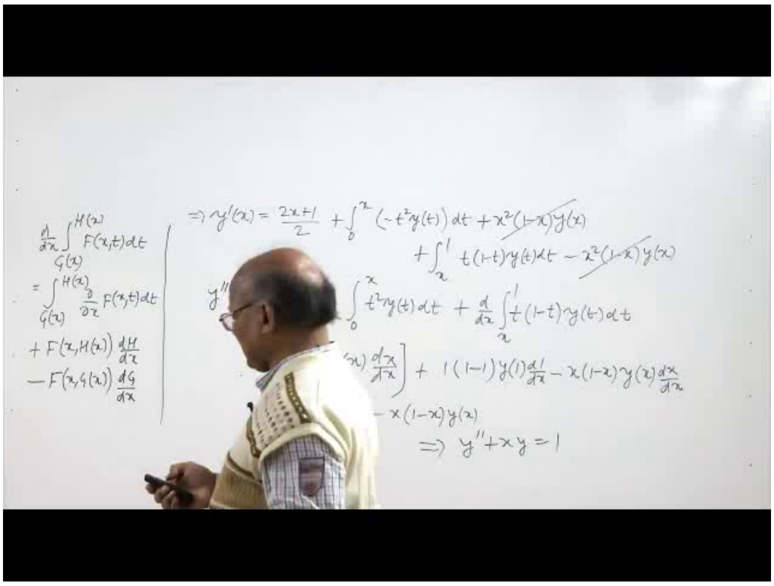

So if you simplify this what we get is this. So this will give you y dash(x) equal to 2x plus 1 by 2 and then derivative here let us write derivative with respect to x, now this is t square by t minus t square x by t. So we will get minus t square by t here. This is x square times (1 minus x) into y(x) derivative of x with respect to x is 1 and then x to 1 when we differentiate here with respect to x we get t into 1 minus t into y(t) dt and then we get minus x square into (1 minus x) into y(x).

So y dash(x) let us see how much we are getting 1 plus 2x by 2 , 1 plus 2x by 2 minus 0 to x ok t square y(t) dt and you can see here this term and this term cancel so we get plus integral x to 1 t into 1 minus t y(t) dt. So we have got the same value here. Now we have got to second derivative.

So when you differentiate it once more y double dash (x) becomes 2 by 2 that is 1 plus third derivative with respect to x we can put a negative sign here. Now again we use the Leibnitz formula this is minus then d over dx is a derivative partial derivative of t square y(t) will be 0, so first term will be 0 we put here t as x so x square y(x) then dx over dx we put t as 0 then differentiate 0 with respect to x so derivative is 0 so we have this and then here again when we take partial derivative of this integrant with respect to x it is dependant only on t so it is partial derivative will be 0 so this term will be 0.

So let us see if we can write it as x into 1 minus x y(x) dx over dx and then this will be when we put upper limit we get 1 into 1 minus 1 the upper limit is 1 y 1 into d 1 over d x minus x into 1 minus x into y(x) into dx over dx.. So what we get is 1 minus x square y (x) and this is 0 ok we have minus x into 1 minus x y(x). We can open this so we get 1 minus x square y(x) term will cancel. So minus x into y(x). So we get y double dash(x) plus x y(x) equal to 1.

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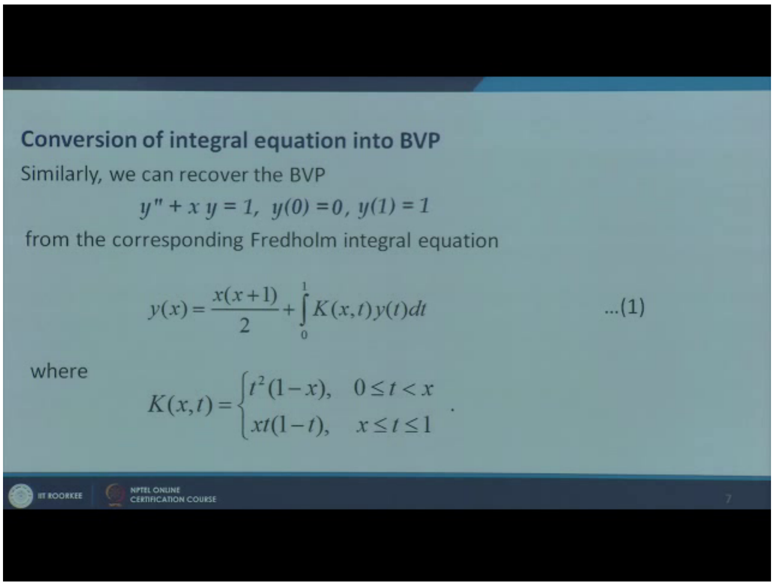

Now from the integral equation with which we started yeah we can see here when you put x equal to 0 what you notice ok because we need to find y (0) at 0 y(0) is equal to 0 y(1) equal to 0 .

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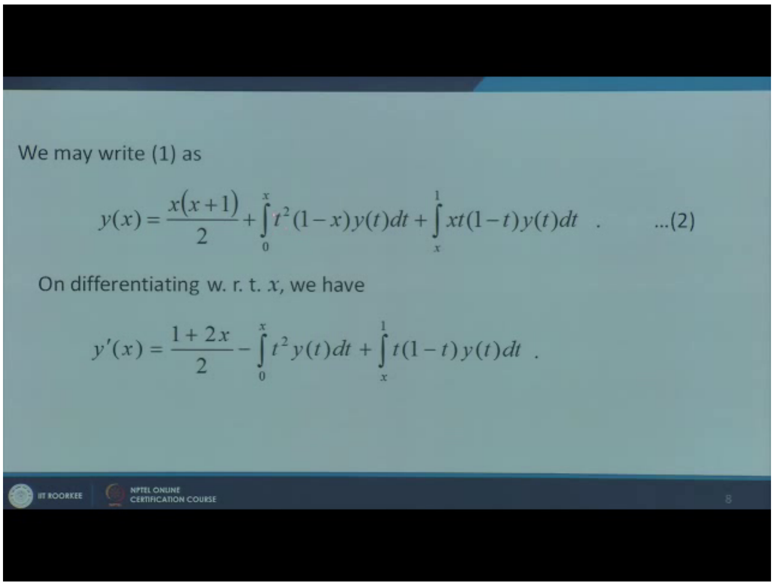

So when you do see that what we get let us see the this is the integral equation when you put x equal to 0 this is 0, ok and here this integral is also 0 because both the limits are 0 0 and here when you put x equal to 0 integrant becomes 0 so integral become 0.

So y(0) equal to 0 we get from this integral equation and when you put x equal to 1 when we put x equal to what we get is this is 0 at x equal to 1 and at x equal to 1 this is also 0 because the integration limit for 1 1 so we get that.

And at x equal to 1 this is y (1) equal to 1. So y(1) equal to 1 we are getting. This is what I have to do discuss in this lecture I thank you very much for your attention.