Integral equations, calculus of variations and their applications

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Lecture 39

Calculus of Variations: Introduction

Hello friends, welcome to today's lecture. In today's lecture we will discuss the concept of calculus variation; in fact this is a first lecture of calculus of variation. So here in this lecture we will try to discuss what is calculus of variation? And what we tried to do in this calculus of variation and how it is related to your usual classical analysis?

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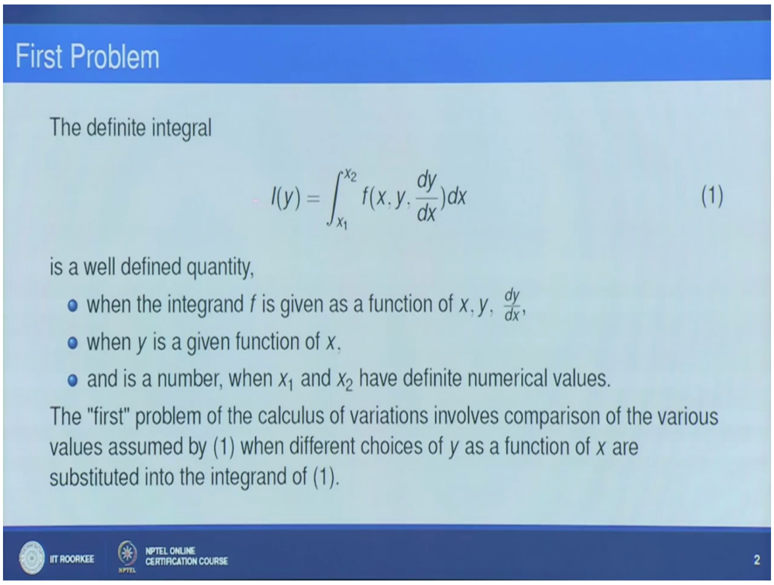

So here to start with let us consider this definite integral the Iy and this is a definite integral from x1 to x2 f of xy dy by dx and this integral is with respect to d of x and we know that this is a well-defined quantity provided this integrand f is a given function of x,y and dy by dx and here y is given function of x and this will give you a number provided this x1 and x2 are having some kind of definitely numerical value.

So we say that in calculus of variation the first problem system find out say the value is corresponding to different functions y and we compare the values of these definite integral corresponding to the given function y. So we can say that the first problem of the calculus of variation involves comparison of the various values assumed by this integral denoted by 1 when different choices of y as a function of x substituted into the integrand of 1.

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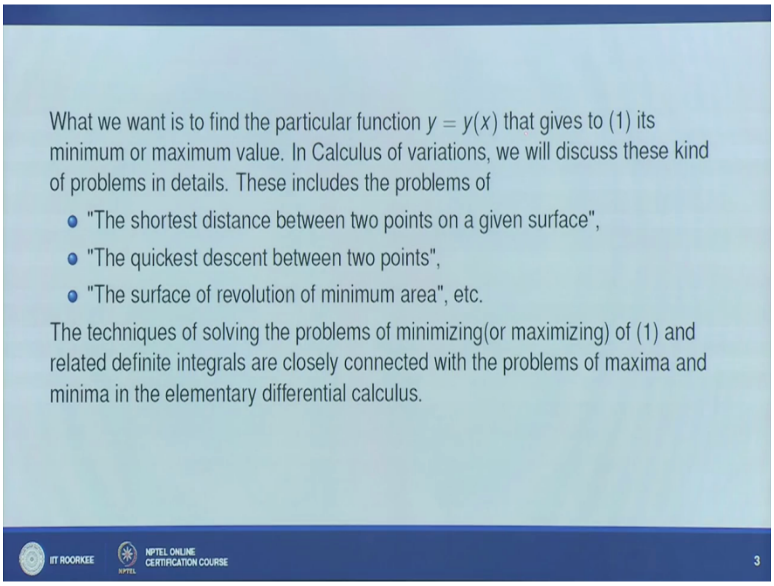

It means that we have a value, if y is replaced by some y1 and then if we replace this y by some other function say y2 then we may have 2 different values of this integrand and in calculus of variation we try to compare these 2 values and try to find out a function for which the value of this integral is either maximum or say minimum or you can say that here we want to find out a particular function y which is a function of x that gives value of this different integral as its minimum or maximum value.

So it means that once definite integral s defined in terms of say functions of this another function y of x. We try to find out say a function of y in a way such that this will achieve its maximum minimum value, right? So in calculus of variation we will discuss this kind of problems in detail. So integral is of variation with try to consider problems which can be transferred into this kind of definite integrals.

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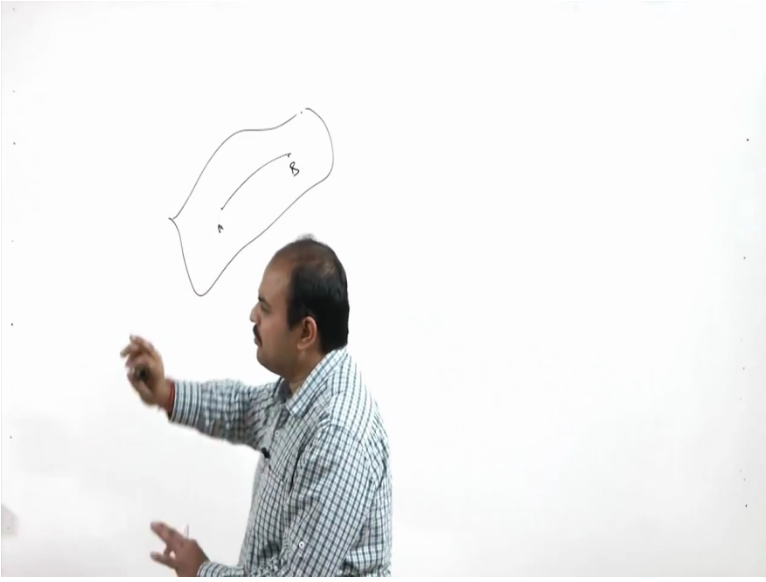

So here there are certain problems and some problems includes the problems of shortest distance between 2 points on a given surface what it means that suppose we have a surface it is like a you have a surface here and then you try to find out that there are 2 points A and B and then we try to see that what is the shortest distance between these 2 points? Provided that these 2 points lie on given surface.

So this is one of the problems discussed in calculus of variation and we will discuss this problem in detail as the problems known by Geodesics. So such curves which gives the shortest distance between A and B are known as geodesics that will discuss in due course of time and the second problem very important problem is the quickest descent between 2 points.

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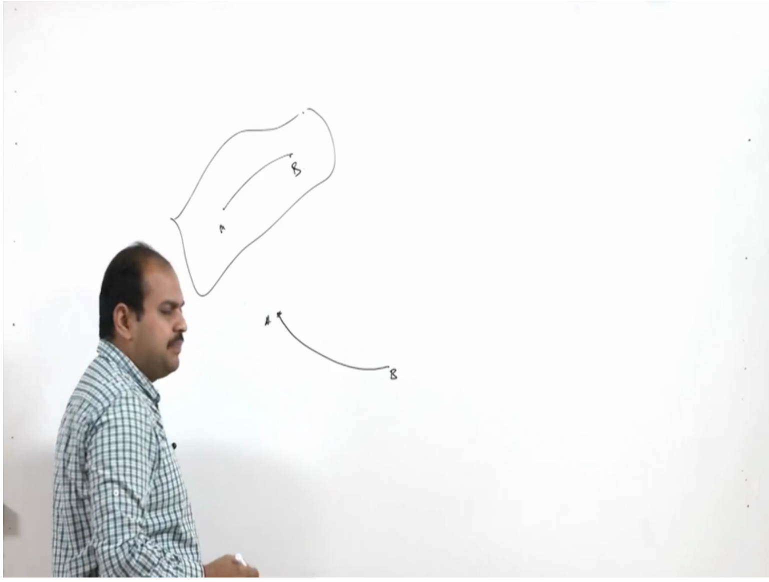

So it means how we consider this problem? That suppose in a vertical plane we have 2 points A and B, of course they are not on a same height and then we try to find out the curve connecting these 2 points A and B in a way such that the, if we take any particular say with start from the point A and it falls under the influence of say gravity from A to B then it will take least time from reaching A to B.

So it means that the problem of quickest descent is the problem of finding the curves. Such that under which if we particle it slides from A to B under the influence of gravity will take the least time, okay. So that also we will discuss in detail and that is a problem known as Brachistocrone problem and this is the first problem is which is supposed to be the beginning of calculus of variation, we will discuss this problem also.

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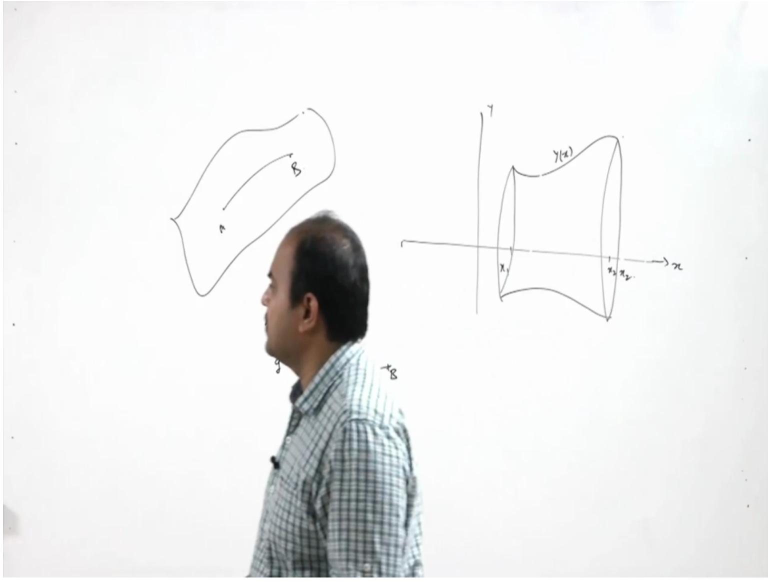

And the one more important problem related to calculus of variation discussed in calculus of variation is the surface of the revolution of minimum area. It means that suppose we have a say one function say y which is a given function of y of x then we try to between say 2 points say x1 and x2 and then what we try to do here, we simply revolve this curve around this x axis with the angle 2 pi then it will generate a kind of, right? Yes it is like this, okay. So here it is x2 and then we try to check the surface area of the revolving surface and we try to find out the curve y of x such that the surface area of the figure, so obtained has a least surface area that we try to discuss in this.

So here we try to see that we will discuss these problems in detail that all these problems and the reduced in finding the maxima minima of the definite integral posts like this one, so all these problems can be termed as or can be written in terms of some kind of definite integral which depends on the say some function y and we try to find out say minimum and maximum value of these definite integrals and these details or these methodology is closely related to the problem of maximum and minima in the elementary differential calculus or I can say that techniques of solving the problems of minimising or maximising of different definite integrals even in 1 and related definite integrals are closely connected with the problems of maxima and minima in the elementary differential calculus, we try to see how it is?

So now these definite integrals are termed as a name which is known as functional. So what is functional? A functional is a mapping which assign a definite number to each functions belonging to the same class is called functional, so if you remember we have written down our definite integral like this, it is from x1 to x2 f x,y and y dash d of x. So it means that if you take a value y here then the corresponding value is given by I of y axis.

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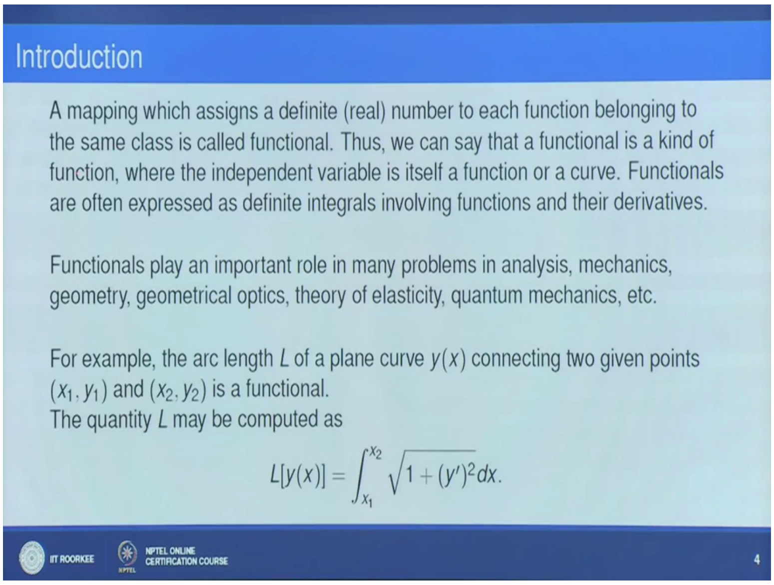

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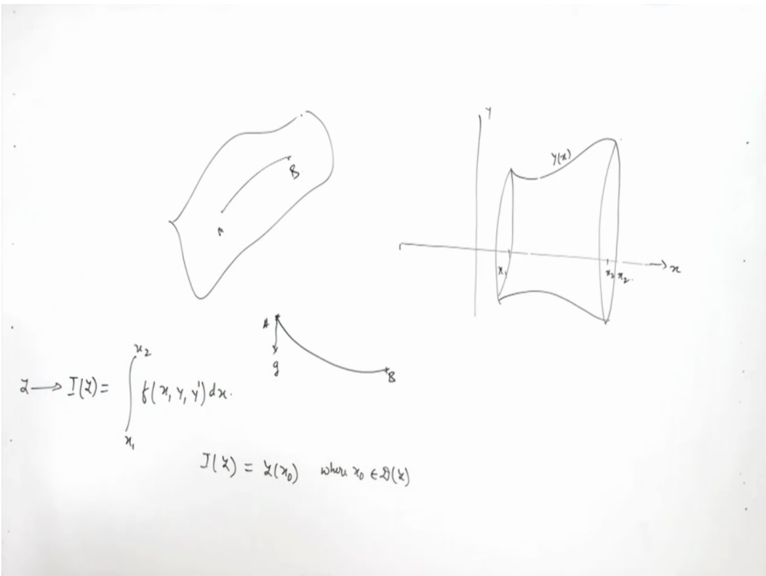

So it means that for any given function y there is a value given by I of y. So basically it is a kind of a correspondence between a function to a value, so it means that it is some kind of generalisation of a usual function, so we can say that a functional is a kind of function which say correspond between y to f of y to I of y. So here we can say that that a functional is a kind of function where the independent variable is itself a function or a given curve.

So here functional can be generalized as a generalized version of function. So it is also a functional the only thing is that independent variable is already a function of some function of x. So functionals are often expressed as definite integrals involving functions and their derivatives so in general you can define any correspondence between functions y to the value I of y.

And so there are certain ways you can define I of y as say this will go to say you can simply define it like this also y of x not where x not is some point in domain of say y, so here also this also represent your functional but here in calculus of variation with try to consider the functional which are termed as or which can be written as these kinds of forms, okay. So here we will discuss that functional are often expressed as definite integrals involving functions and their derivatives. So in general we try not to discuss this kind of functionals, okay.

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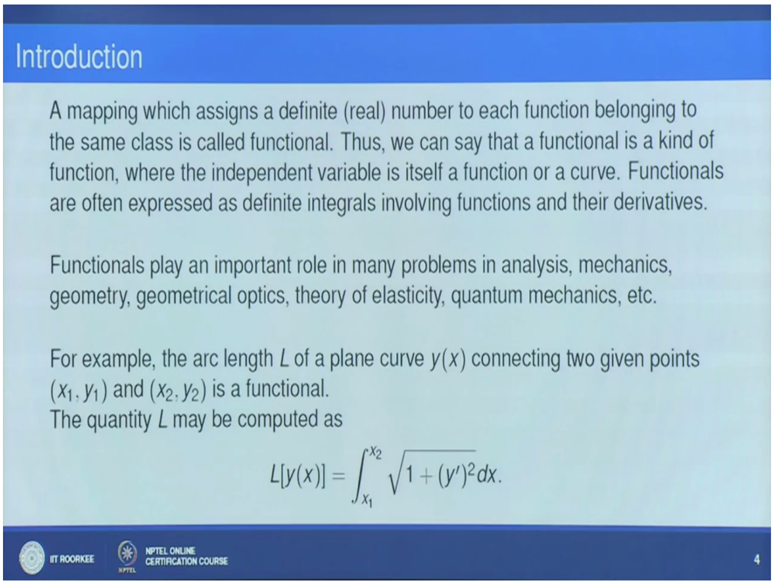

So and the study of functional plays a very important role in many problems and analysis, mechanics, geometry, geometrical optics, theory of elasticity, quantum mechanics etc. so that we see as an application of calculus of variation. We will consider a certain problem of these fields as an application of calculus of variation that will shortly we will discuss.

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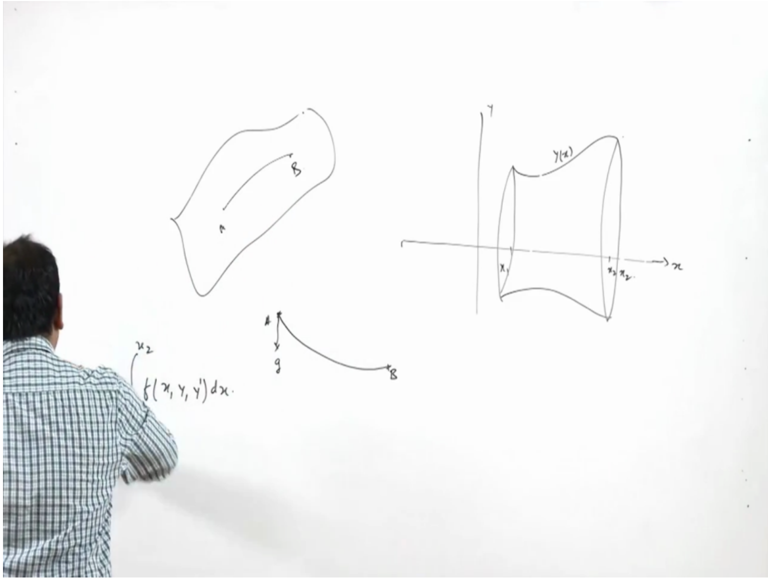

For the beginning we can say that the arc length l of a plane curve connecting2 given points is also functional, what is this? So initially if you remember if we have surface, if we have a surface and we have 2 points A and B and try to find out say minimum length between minimum length of this curve connecting these 2 points A and B. So let us say that these the surface is a planar surface or you can say that these 2 points lie on a say x,y plane or y,z plane or something like that and we try to find out say minimum distance of the curve.

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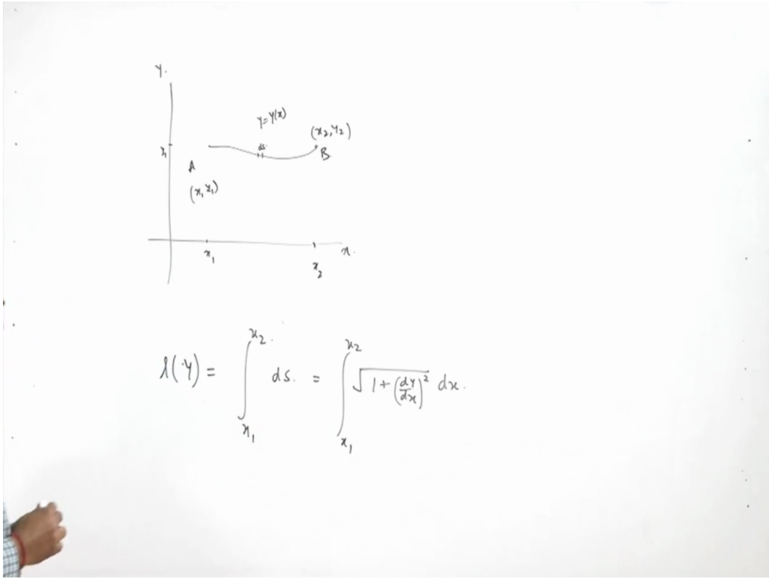

So here what we have? Let us say that we have a x,y plain. So here we have x,y plane and then we have a point A and we have a point B. A is given as x1, y1. So here this is your x1 and the corresponding is your y1 and here we have say another point say x2, y2. So here we have x2, y2 and then we try to find we can draw many curves connecting this A and B and we try to find out that curve which minimises the length of the curve connecting this A and B.

So here I can say that if you look at the length of y it is given by say x1 and x2 and if we take this as d of s then we can simply write d of s. So it means that length of y you can define as the integral between x1 to x2 of this element. So here if we represent this y curve as y is function of y of x then you can say that d of s is given by x1 to x2 and this is 1 plus dy by d of x whole square d of x where x is lying between x1 to x2.

So we can say that the length of the curve connecting A and B, a is given by this functional and this can be simplified into this x1 to x2 under root 1 plus dy by dx whole square d of x and then this is one of the functional we are discussing. Now if we in case of planer reason, if we consider these 2 point lying on a surface then it means that in place of A and B lying any plane.

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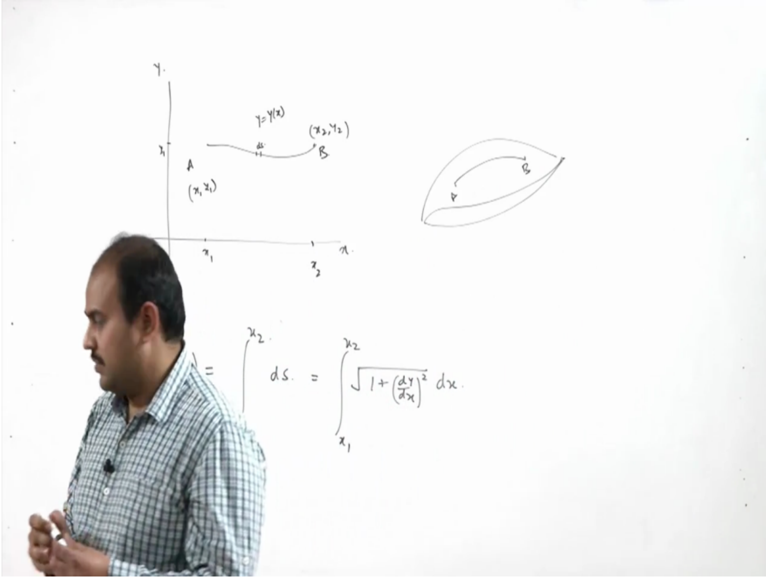

If we say that it is lying on some kind of say some kind of a surface then not only we have to minimize the we have to consider the length of y but also we have to put a restriction that these 2 points A and B lying on a given surface.

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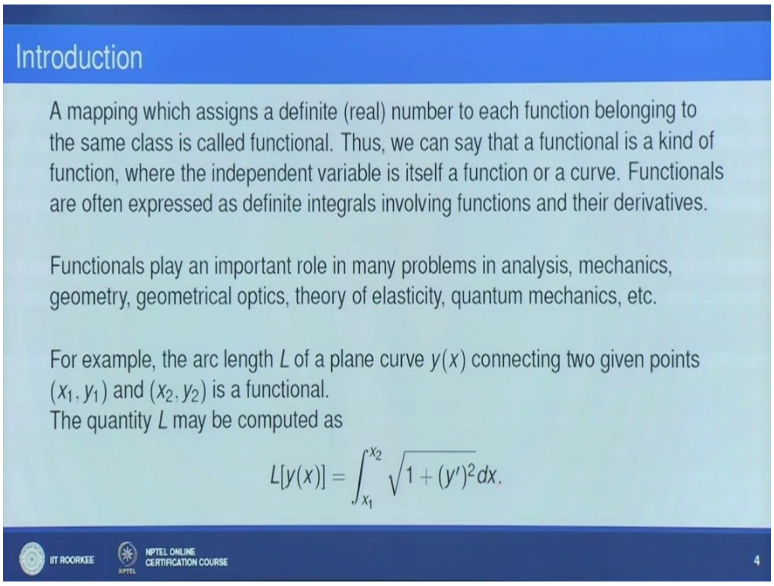

So the length of the curve connecting 2 points is termed as a function of here. So here we can say that this L yx is given and x1 to x2 under root 1 plus y dash square d of x. So here if you use one function U will have a 1 value of this indefinite integral, if we use another function yx then we have another value corresponding to that function or you can say that given 1 curve there is a different length.

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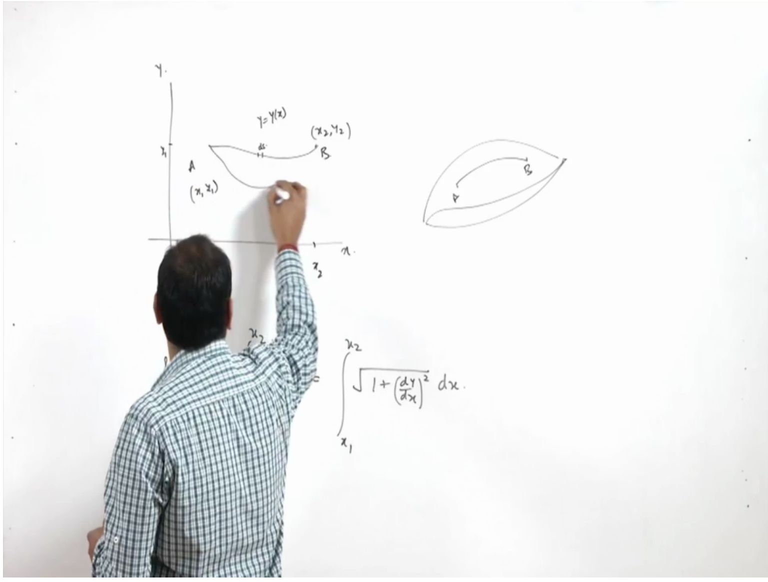

If we use another colour then we have a different length, so if we can use this then this is y this is y1. So we can say here we have a value or you can say length of the curve connecting through this curve and we have another length if we use this y1 as a curve. So we can say that this ly is a function of your variable y which is a function of x also. So we can say that this is a functional.

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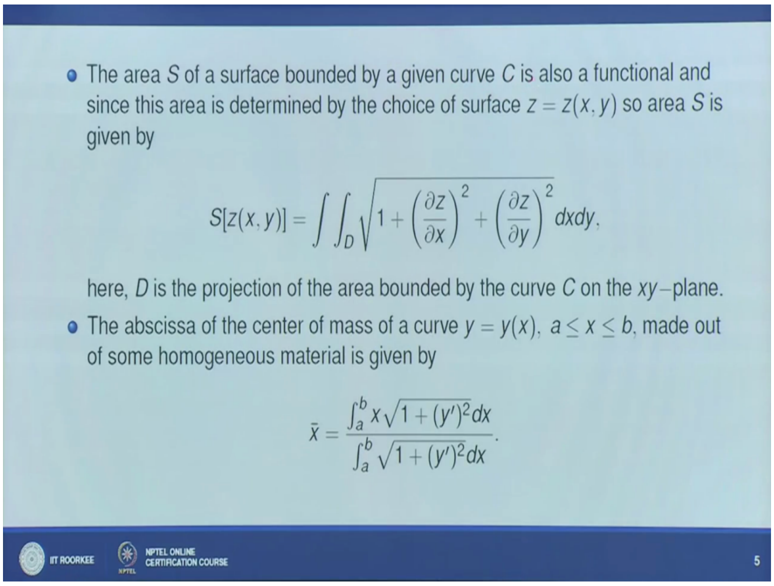

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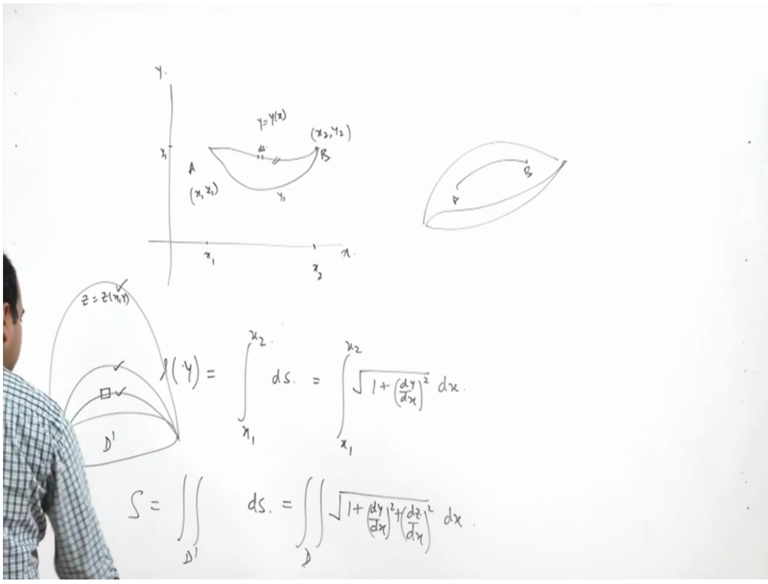

Similarly the area S of a surface founded by a given curve is also a functional and since this area is determined by the choice of the surface. So we have a curve c, so let us say that we have a curve c. So this is say the, this is a curve now here this curve A is bounded, so we have a surface, so we can say we have a surface like this and any you can consider like that.

So we have a surface like this and this is C is the curve bounds this surface in this area. So here we want to find out the surface area of these surfaces which are bounded by the curve c, is it okay and we can say that if we choose this surface then surface area is different. If we choose this surface then the surface area is different and similarly whatever surface you choose your surface area is quite different.

So we can say that here your surface area S which is given by sat d of s over this reason we can call it say D dash. So we can find out the surface area based on this D of s where D of s is the say surface patch on this surface D of s. So if I take say 1 surface z equal to z of xy, right? And then this can be written as double integral over d and this is the surface area element can be written as 1 plus y dash means dy by d of x whole square plus dz by d of x Whole Square and d of x.

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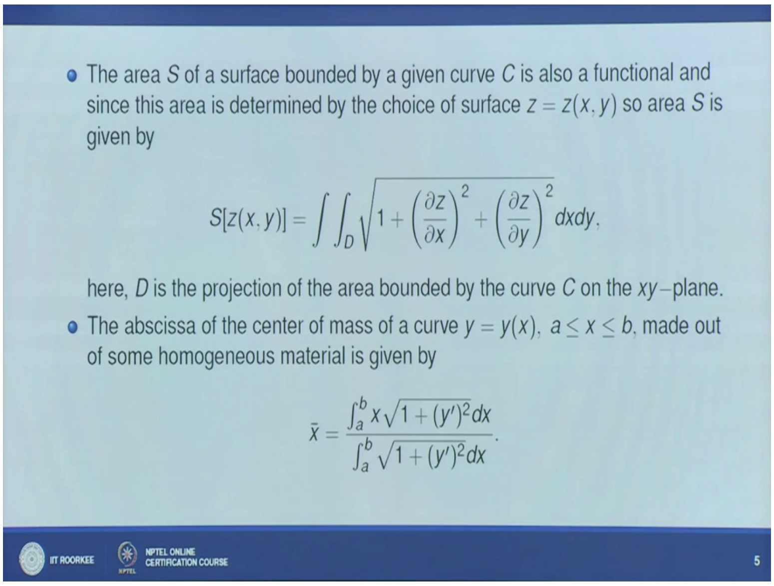

And here D is the projection of this on xy plane, so here D is denoted as the projection of your surface bounded by this scene on the xy plane, so sorry here it is dxdy. So here we can say that the area S of a surface bounded by a given curve c is also a functional because if we use different surface your surface area is different and since this area is determined by the choice of surface z which is given as z of xy.

So area S is given by this, so here you’re functional related to surface area is given by D integral over D under root 1 plus z as square plus zy square double z by double y square dx to a,y. Oh! So there is a small problem here, here it is here it is, so there is a wrong thing so here it is dz by dx whole square plus dz by dy whole square, okay. I’m just correcting it.

So here we can say that the surface area corresponding to the surface Szx,y is given by this double integral D and the surface element can be written as under root 1 plus double z by double x whole square plus double z by double y whole square dx dy and here is this D is the projection of the area bounded by the curve c on the xy plane, so this is also an example of a functional.

Now we may also define a functional in the sense of centre of mass, so if we represent that this function y equal to yx is made of some kind of material then we try to find out the centre of mass of this material. So this y of yx will give you some kind of a curve which is made by some material and we try to find out say centre of mass here. So we can we have already seen that this can be written as x bar equal to ratio of 2 functional A to B, x under root 1 plus y dash square d of x divided by A to B under root 1 plus y dash square d of x.

So this is the ex-component of the centre of mass, similarly you can find out the y component of the centre of mass. So in that case your y bar is defined by you simply replaced x by y. So this is also the functional, so it means that if you change the y of x then your centre of mass is going to change.

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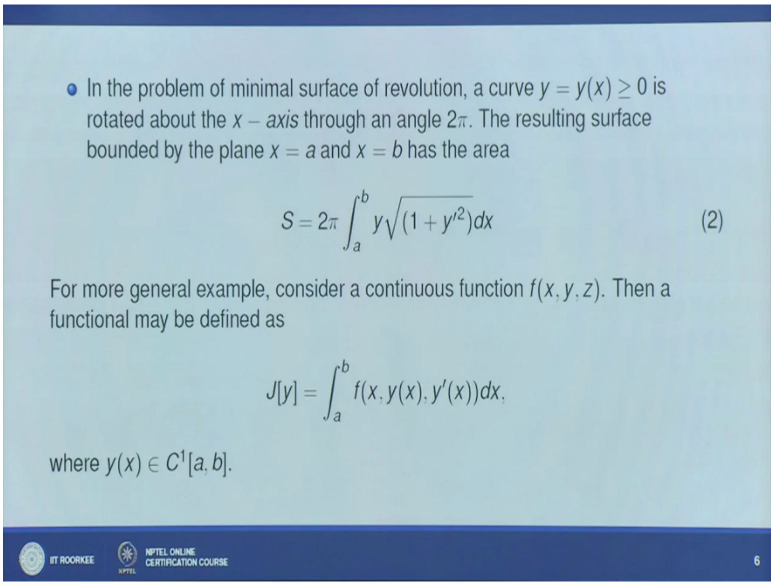

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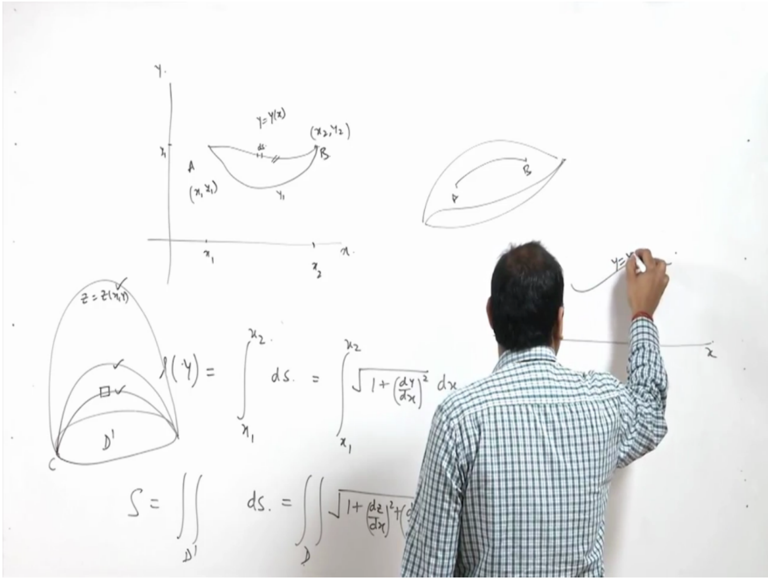

So it means that if you change this curve y, it means if you are changing the shape of the shape of the material shape of the say graph then its centre of mass will also change. So next thing is the that the problem of minimal surface of revolution, this we have discussed that if we have a curve y equal to y of x and is rotated about the X axis through an angle to buy then the resulting surface bounded by the plane x equal to a and x equal to be as this area.

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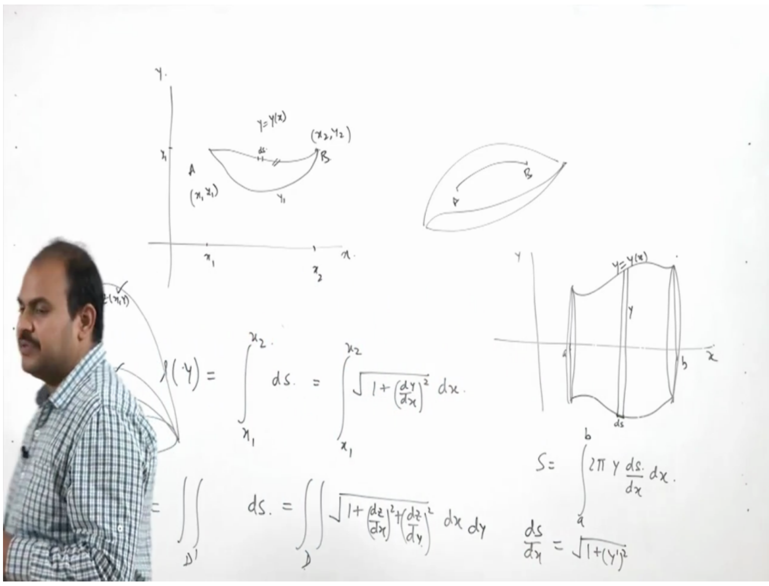

So this we can write it like this, this we have in fact drawn but let me write it here, so here we have this is x, this is y and this is the graph y equal to y of x and this is your A and this is B and this is rotated through x axis of say, okay. So it is this thing, so now we try to find out say surface area of this, so if you look at here this we can patch it these 2 entire shape can be truncated into this kind of patch and we can call this as patch as this is the surface patch like d of s and we can say that y is the say radius of this particular batch and you can say that the surface area is given by 2 pi y and d of s, right?

Now here since this is only line integral, so we are considering only say single, so we can write it here, 2 pi y d of s d of x d of x. So and this is we are considering only between A to B, so we can write it A to B. So we can say that the surface area of this generated shape is given by A to B 2 pi y ds by dx, d of x. So you can find out say ds of dx as here you can write d of dsf, ds by dx is equal to under root 1 plus y dash square.

So here y dash represents d y by d of x, so here you can say that the resulting surface bounded by the plane x equal to A and x equal to B has this area S equal to 2 pi a to b y 1 plus y dash square d of x and we so this is also one example of functional and if you look at in general we can define say general example of functional as this with the help of a continuous function f which is continuous in terms of its arguments.

So it means that function f is continuous with respect to x, y and z then you can define functional defined as this J of y is equal to a to b f of x, y of x, y dash x, d of x. So here if you look at them by taking particular values of this f you can say that this first problem that is the length of the plane curve and the surface area and the surface area is also can be considered as a particular case of this and here, we will consider one property that is this kind of functional is having a localisation property.

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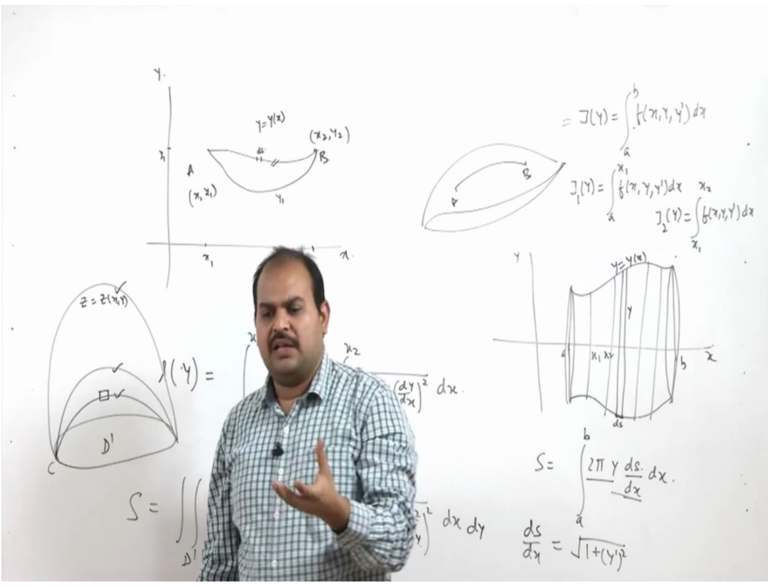

What is a localisation property? Here we can truncate the region of y of x, for example this particular problem. Here we consider this y of y of x and then we can truncate the reason between a to b into smaller parts and for each smaller part for example we have this a to say x1 and then x1 to x2 and so on. So here we can divide this entire range a to b into several sub patches and we can find out say corresponding functional and find out the value of its functional.

So it means that you can say that for example here we have J of y is equal to a to b and f of x, y,y dash d of x. So here in for this particular problem here f x, y, y dash is given by 2 pi y dsx ds by dx. Now I'm saying that if we want to write it say J1y, what is J1y? Here you write it a to x1 and f of x, y, y dash d of x. So you can say that J1 is nothing but this sorry.

So J1 is between a to x1, similarly you can define J2 y is equal to your x1 to x2, f of x, y, y dash d of x. So similarly you can find out patches, find out say functional defined in this sub patches and then if you sum all these values then you will get the values defined by this original linear functional original function defined by JY equal to a to b fx, y, y dash d of x, so we can say that this has a localisation property.

But if we consider this part the functional defined here this is not having this property, here we cannot do (()) (26:26), so we call these kind of function as nonlocal functionals and in calculus of variations we generally deal with the functional with the localisation property. So in calculus of variations we generally deal with local functional not this nor-local function.

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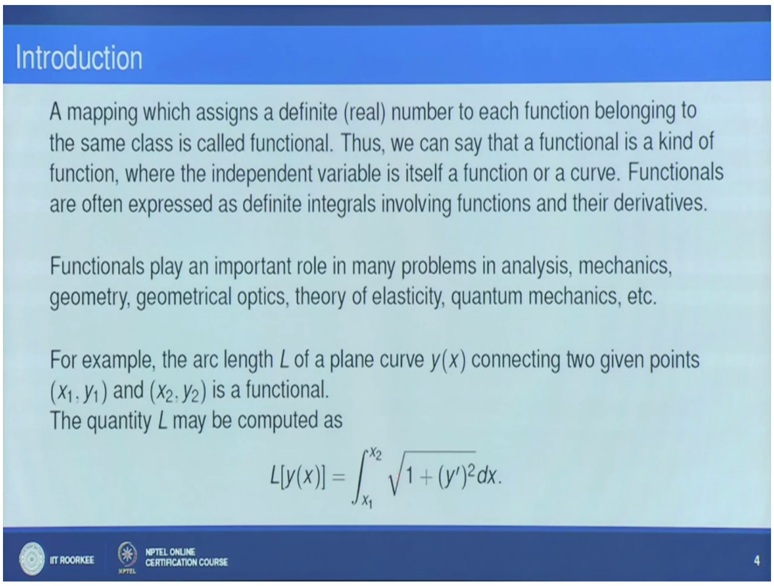

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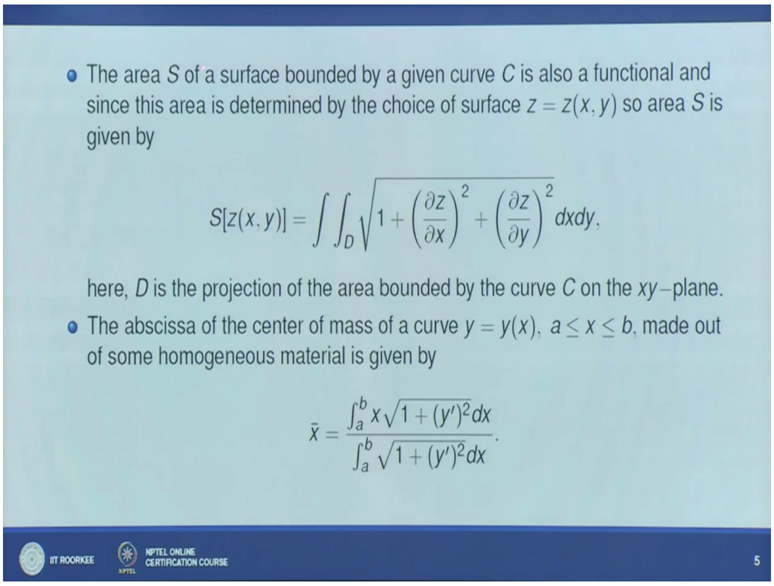

There are certain functions available, so methods available to deal this kind of products they are for more general examples we can consider a continuous function fx, y, z which is a continuous function with respect to its argument then a function may be defined as J of y equal to a to b f of x, y of x, y dash x d of x and here by taking a particular examples of f consider that this problem which is the arc length of a plane curve and the surface area of the surface bounded by say a given curve c and the problem of minimal surface of revolution given by this equation number 2 is a particular case of this, right?

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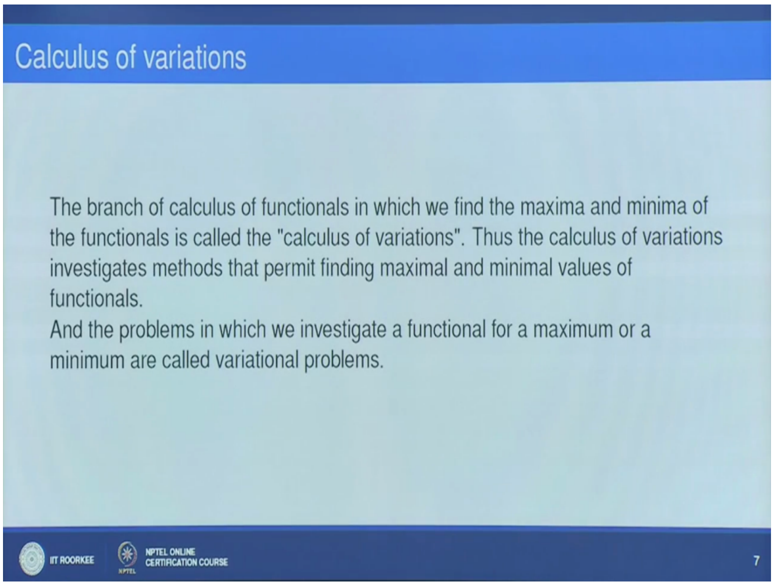

So here summarizing all this we can say that the calculus of variations is a branch of calculus of functional is in which we find the maximum and minimum of the functionals and we can say that thus the calculus of variations in this branch we discuss the methods that permits finding maximal and minimal values of functions and the problems which we investigate R functional for a maximum or minimum are called variational problems.

In fact there are certain principle rules in mechanics in physics that can be related to functional that can be related as the maximising or minimising some kind of functional in that process and they can that that rules can be converted as a maximising and minimising of that functional, so that kind of problems are also known as that kind of rules are also known as variation principle.

So that also we will discuss in calculus of variations, so for today's class we will wind up things here. In next class we will discuss the concepts and basic definitions of calculus of variation, thank you for listening us. Thank you.