Integral equations, calculus of variations and their applications
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Lecture 36
Hilbert Transforms-2

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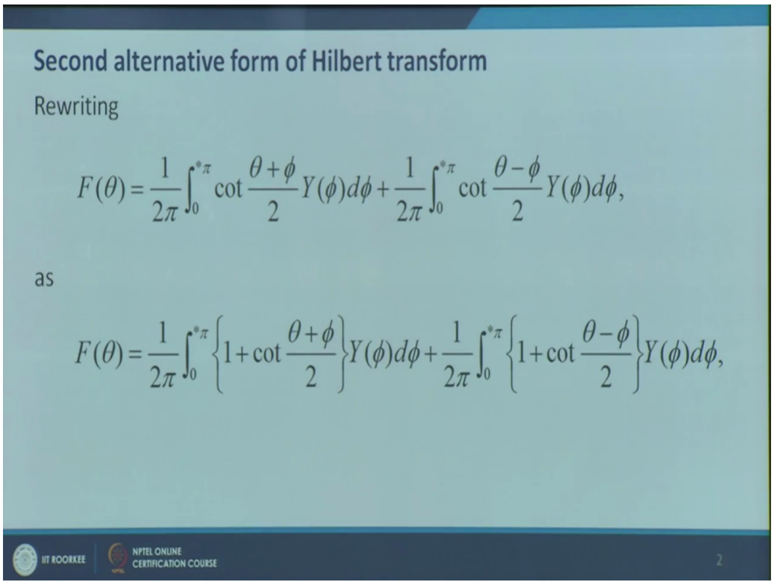

Hello friends I welcome you to my second lecture on Hilbert transforms. We are going to now discuss second alternative form of Hilbert transform pair. let us recall that we had the equation in the first alternative form we had capital F theta written in this form, capital F theta we had written in this form and then we had combined these 2 forms and we got the first alternative form of the Hilbert transform pair by replacing phi by minus phi in the first integral.

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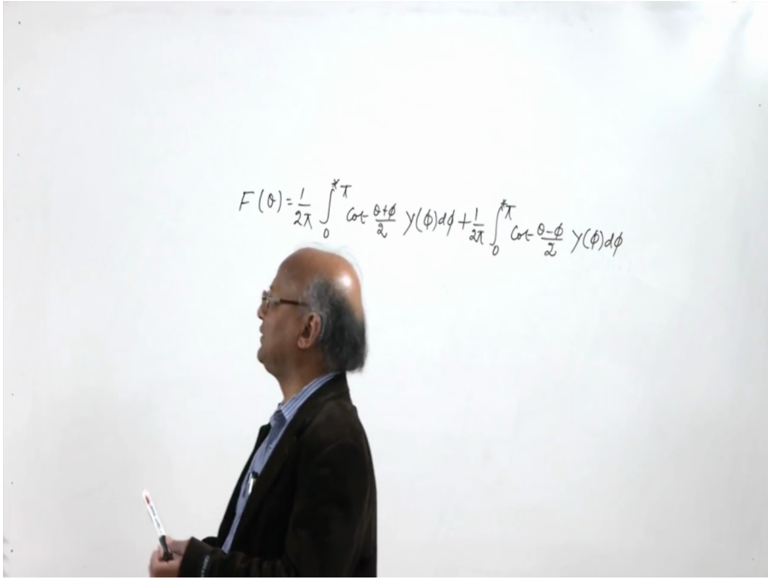

So this was our first integral and this integral now let us see we can write like this, so F theta equal to over 2 pi integral over 0 to pi cot theta plus Phi by 2 Y phi d phi plus 1over 2 pi cot theta minus phi by 2 Y phi d phi. In the previous lecture when we were finding the first alternative form so is while finding the first alternative form of the equation 1 we encountered this equation.

So this equation F theta equal to over 2 pi integral over 0 to pi cot theta plus Phi by 2 Y phi d phi plus this can be expressed as 1 over phi F theta equal to one over 2 pi 0 to pi one plus cot theta plus I buy 2 Y phi d phi plus one over 2 pi, 0 to pi 1plus cot theta minus phi by 2 Y phi d phi, how we can do this?

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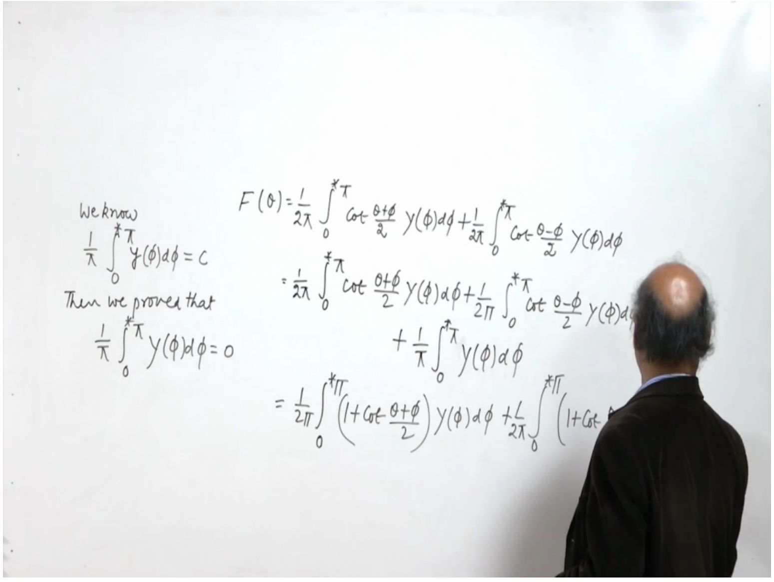

If you can see we can write it as , if you recall in the last lecture we have seen that 1 over pi, integral 0 to pi Y phi d phi this we had denoted by c and this we denoted by c, 1 over pi, integral 0 to pi Y phi d phi we denoted by c and then we proved that 1 over pi, integral 0 to pi by phi d phi equals to 0, what we had done was, we replaced Y phi equal to small Y phi minus c and then we separated the 2 integrals and we got that this value is equal to 0.

So this can be if we add it here we are actually adding 0, so 1 over pi 0 to pi Y phi d phi, if we add this here actually we are adding 0, so there is no change so I can write it 1 over 2 pi and then again we can write it half , half of this with this and half of this with this and so I can get integral 0 to pi one plus cot theta plus Phi by 2 by phi d phi plus one by 2 pi 0 to pi 1 plus cot theta minus phi by 2 Y phi d phi. So this is how we arrive here by distributing half of this with this integral half of this with this integral, so this is how we come to this place and then we can write it as, now let us combine the 2 integrals, okay.

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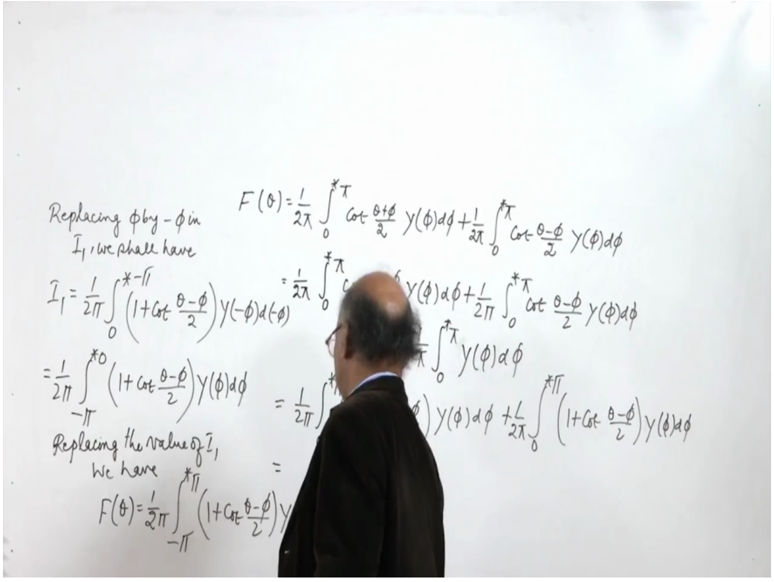

So what we will do again? Let us call them as say again I1 and I2, let also plays phi by minus phi I1, replacing phi by minus phi in I1 we shall have I1 equal to 1 over 2pi minus pi 1 plus cot that minus Phi by 2 Y minus Phi d minus Phi, okay. D minus Phi is minus d phi, so I can write it as 1 over 2 pi minus pi 2 0 1 plus cot theta minus phi by 2, Y minus Phi is equal to Yphi, so we get this.

So I1 is 1 over 2 pi integral over minus pi to 0 1 plus cot theta minus phi by 2, Y phi d phi, so this integral can be replaced by that and then we can combine, so replacing the value of I1 we have F theta equal to 1 over 2 pi integral over minus pi 2 pi 1 plus cot theta minus phi by 2 Y phi d phi, so we get this F theta equal to 1 over 2 pi integral over minus pi 2 pi 1 plus cot theta minus phi by 2 Y phi d phi.

 from the first alternative pair we recall that the second equation there is Y theta equal to 1 over 2 pi minus pi 2 pi coy phi by 2 F phi d phi and also recall that F theta is an even function of theta, so I can write this equation also as Y theta equal to one over 2 pi minus pi to pi one plus cot Phi minus theta by 2 F phi d phi because integral over minus pi to pi F phi d phi is equal to 0 because Fphi is an odd function, so this integral is same as this integral, so corresponding to this form we have the, corresponding to this form of the integral equation we have this as a solution of the integral equation. So the relation is 1 and 2 together give us the second alternative form of the finite Hilbert transform pair.

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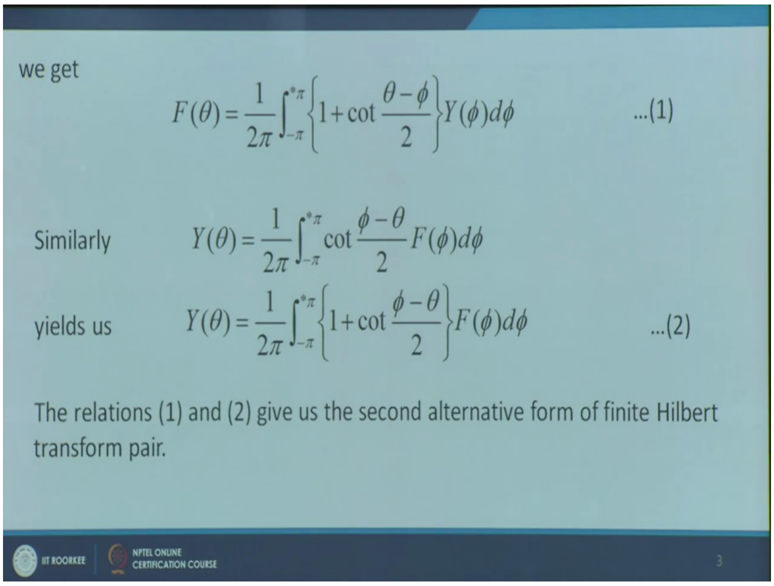

There is one more alternative form which we call third alternative form of Hilbert transform pair, let us recall that. Finite Hilbert transformer of a function Yphi is defined as this F theta equal to one over pi integral 0 to pi sine Theta over cost theta minus cos pi y phi d phi where the inverse is given by y theta equal to one over pi 0 to pi sine phi over cos phi minus cos theta F phi d phi plus 1 over pi 0 to pi y phi d phi. So this is the Hilbert transform pair.

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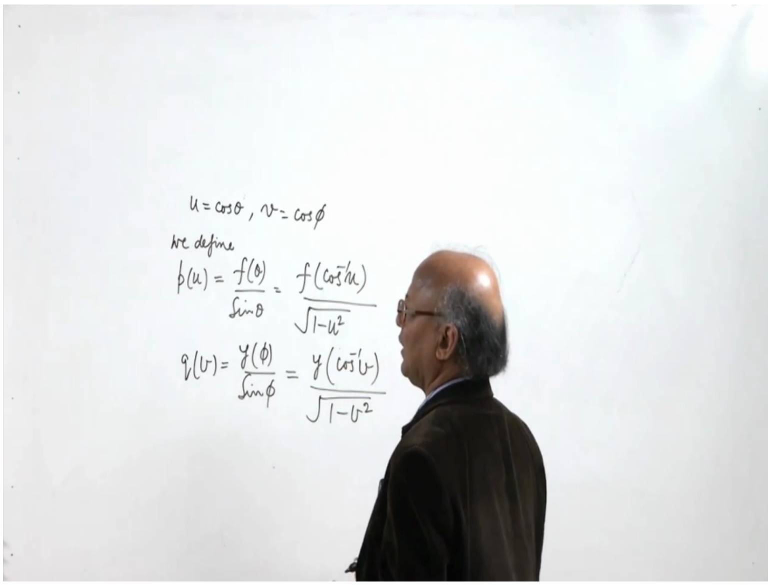

Let us now derive the third alternative form of this. What we do is, we introduce the new variables u by equations u equal to cos theta and v equal to cos phi and also we define the functions pu equal to F theta over sine theta, so when u is cos theta, theta will be cos inverse u, so f of cos inverse u we have and then sine theta will be equal to under root 1 minus u square.

Here we have this integral; in this integral we will make the change of variables, so we are defining pu and qv and qv is equal to we define as y phi over sine phi, so y phi over sine phi is y cosine inverse u cosine inverse v divided by under root 1 minus v square now with this change of variables let us say what we get from the first equation? this equation number 3.

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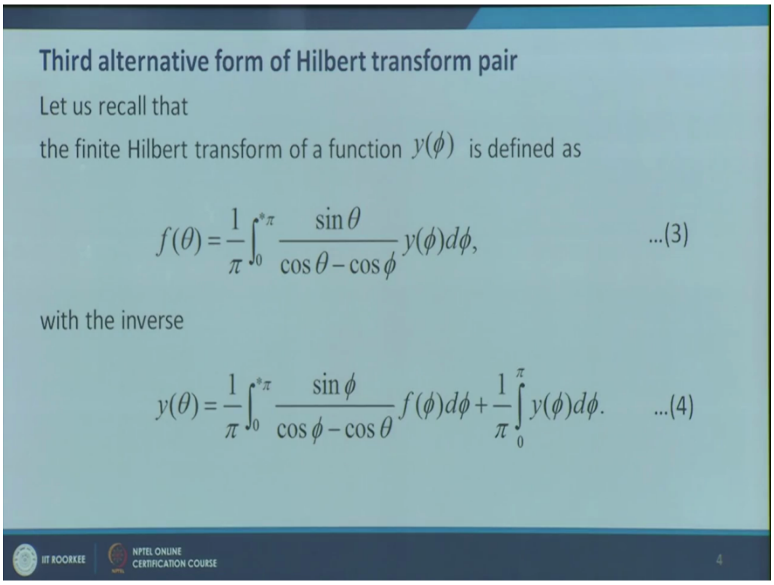

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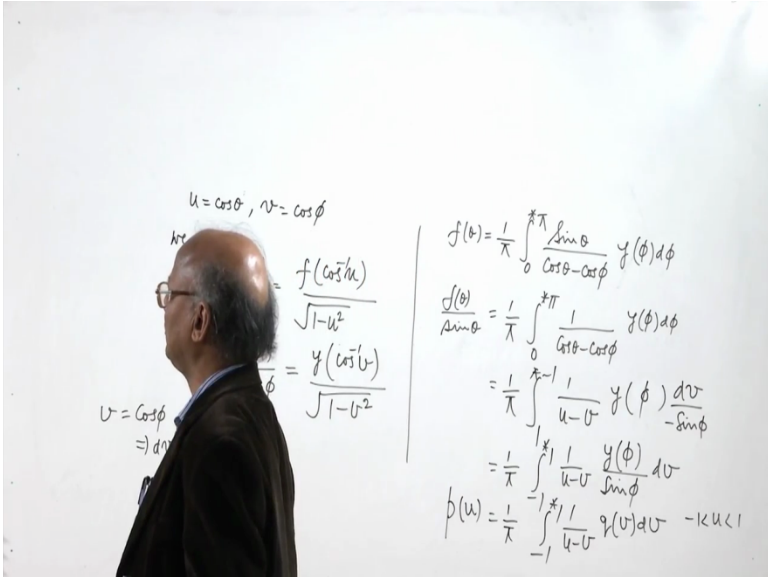

So F theta is equal to 1 over pi, 0 to pi sine Theta over cos theta minus cos phi y phi d phi. theta is independent of phi, so I can divide this equation by F sine theta so I get F theta upon sine theta equal to 1 upon pi integral over 0 to pi 1 over cos theta minus cos phi y phi d phi be equal to costs phi gives us dv equal to minus sine phi d phi, so let us replace the value of d phi here, so this will be equal to 1 over pi the limits of integration will become cos theta is u, cos phi is b y phi is y cos inverse b and d phi is equal to dv upon minus sine phi and here the limits of integration will become 1 to minus 1.

So we can write it as 1 over pi minus 1 to 1, 1 over u minus b, y and sine phi is equal to under root 1 minus v square this will be y phi, okay. I can write it as y b I can write it as yb over or I can say y phi over, phi is cos inverse b so y phi over sine phi I can write dv, these are same things, okay. So this is nothing but phi you can keep it as Phi.

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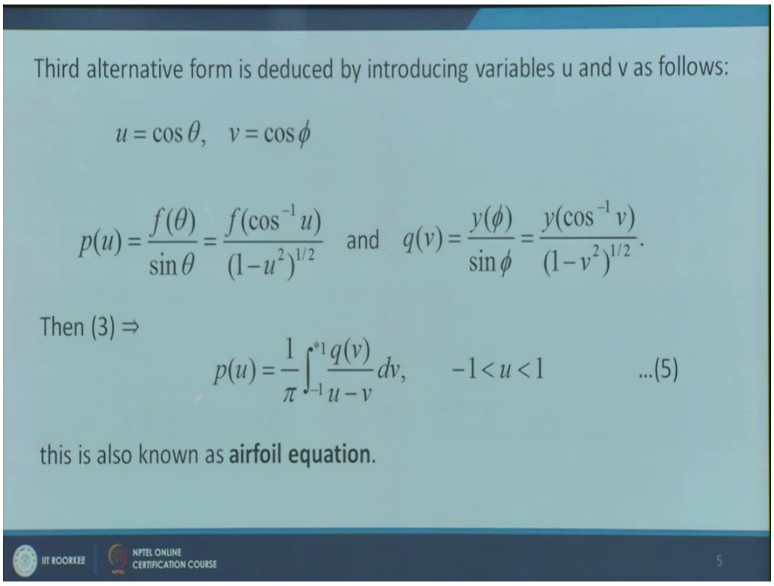

Let me write it phi, okay. So y phi over sine phi we can write as qv, so 1 over pi minus 1 to 1, 1 over u minus b and then qvdv and u varies from minus 1 to 1 because theta varies from 0 to pi, so minus one less then u less than one. so pu this first equation, first equation of F theta equal to this can be rewritten as this F theta upon sine theta is pu, so pu equal to one over pi integral over minus 1 to 1 qv over u minus dv, this equation is also known as airfoil equation.

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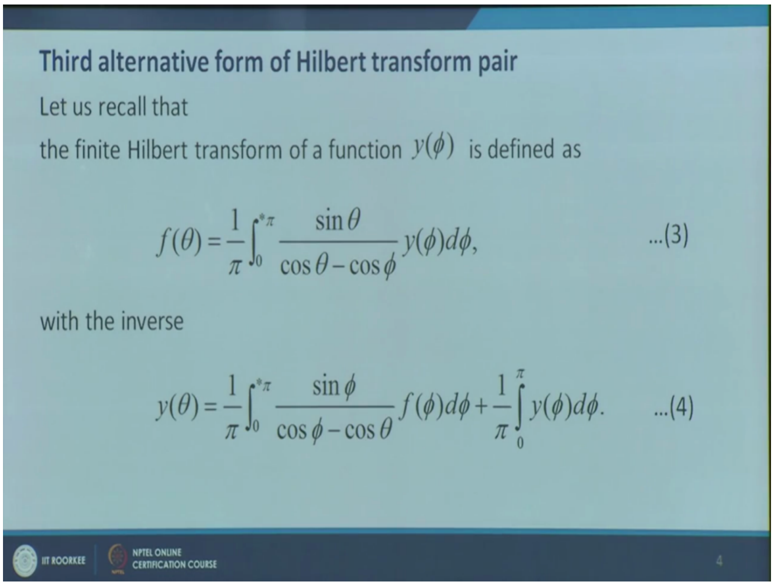

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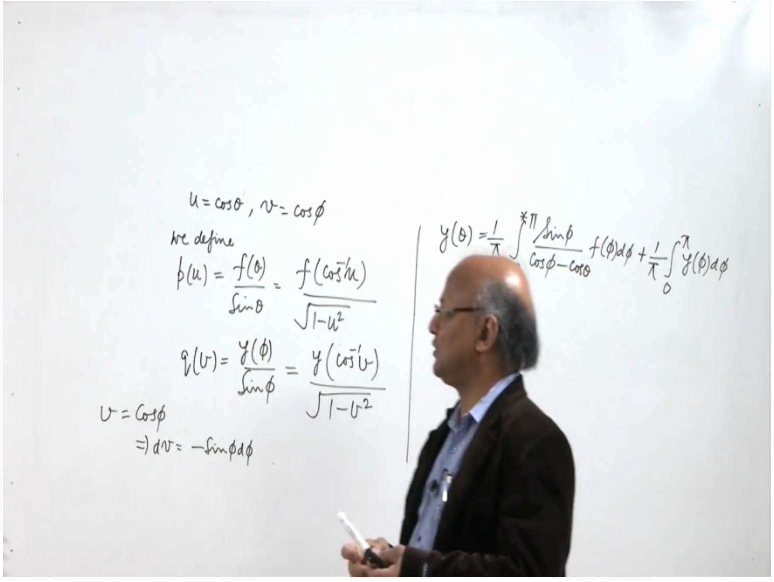

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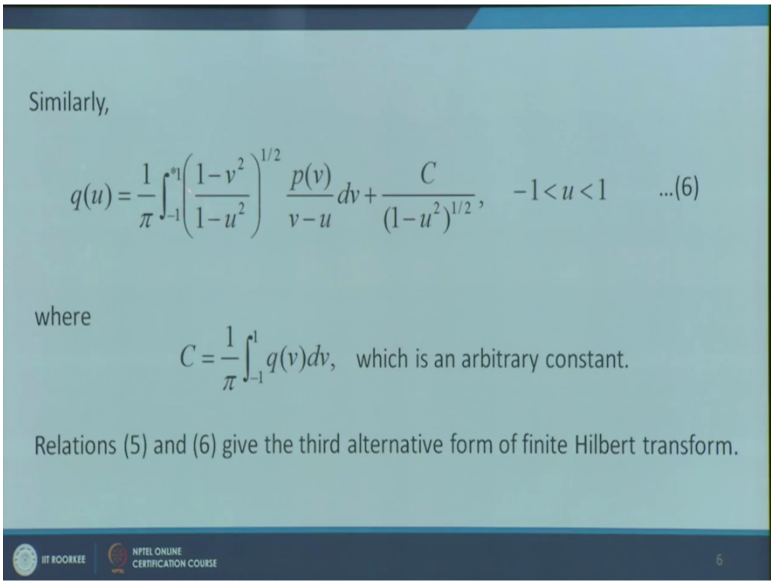

Now similarly let us go to the second equation. this equation y theta equal to one over pi 0 to pi sine phi over cos phi minus cos theta f phi d phi 1 over pi 0 to pi y phi d phi, so let us write the form of this, now making use of this change of variables and so what we will get is qu equal to this. Let us see how we get this qu equal to 1 over pi integral over minus 1 to 1, like this.

So what we do is, we again divide by sine theta so y theta upon sine theta equal to one over pi integral over 0 to star u pi sine pi upon cos Phi minus cos theta 1 over sine theta F phi d phi plus 1 upon pi 0 to pi y phi upon sine Theta d phi. Now qv is 1 upon y phi over cosine Phi, okay. So when we have y theta over sine theta we shall write qu because pu, u is a function of theta, so y theta over sine theta we shall write as qu.

So this is qu equal to now here 1 over pi d phi, what do we get? d phi is equal to sine phi d phi is equal to dv, so sine phi and d phi we write as minus dv and this is f of phi, okay. So we can write it as minus dv for sine phi d phi and here we will write cos phi minus cos theta cos phi is equal to b minus u, what we are left with? f phi is equal to pv over pv into sine phi because v is a function of phi.

So pv is equal to f phi divided by sine phi, so fphi is pv sine phi the limits of integration will be cos phi is equal to be, so there will be 1 to minus 1 and we shall have sine theta here plus 1 over pi, here also y phi, y phi is equal to qv sine phi, so we have phi the limits of integration for phi are 1 to minus 1 and y phi is equal to qv sine phi, over sine theta d phi sine phi d phi si minus dv, so this will minus dv so this will minus dv we will get and this will change to minus 1 to 1 and we will have this as 1 over pi minus 1 to 1 this will be sine phi is under root 1 minus v square, sine theta is square root 1 minus u square and what we will get then?

Pv upon v minus u square root 1 minus v square upon 1 minus square into dv, this is for the first integral and second integral will be 1 over pi minus 1 to 1 qv dv over under root 1 minus u square sine theta is under root 1 minus u square, so let us see what we have? Okay, so now what we will do is, because this u is independent of v, so we can also write it as qu equal to 1 over pi minus 1 to 1 pv upon v minus u under root 1 minus v square upon 1 minus v square dv plus c upon under root 1 minus u square, where c is the integral, 1 over pi integral minus 1 to 1 qvdv.

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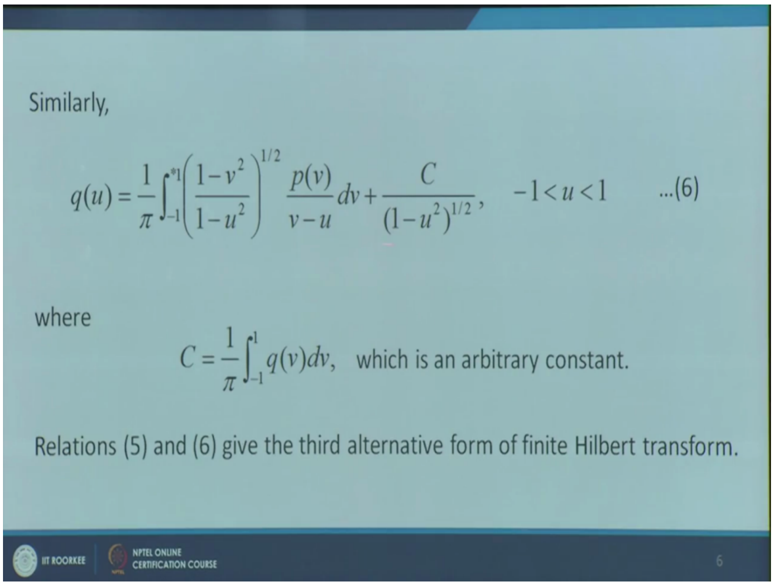

So qu is equal to 1over pi minus 1 to 1 under root 1 minus v square upon 1 minus u square pv upon d minus udv and then c upon under root 1 minus u square c is given by this expression which is an arbitrary constant. Now this equation which gives us the solution of the integral equation this together makes a Hilbert transform pair third alternative form. So this equation is the equation where we are looking for the solution that is a function q and that solution is given by this equation. So the equation 5and 6 together give us the third alternative form of the finite Hilbert transform.

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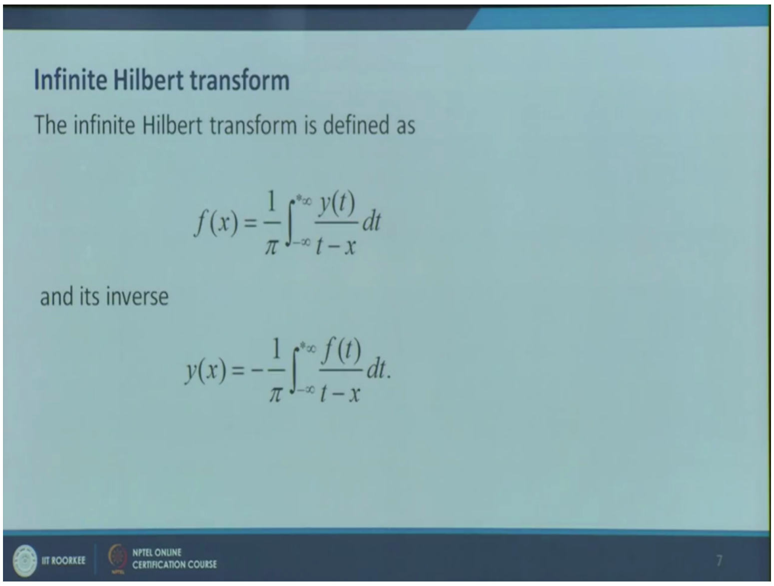

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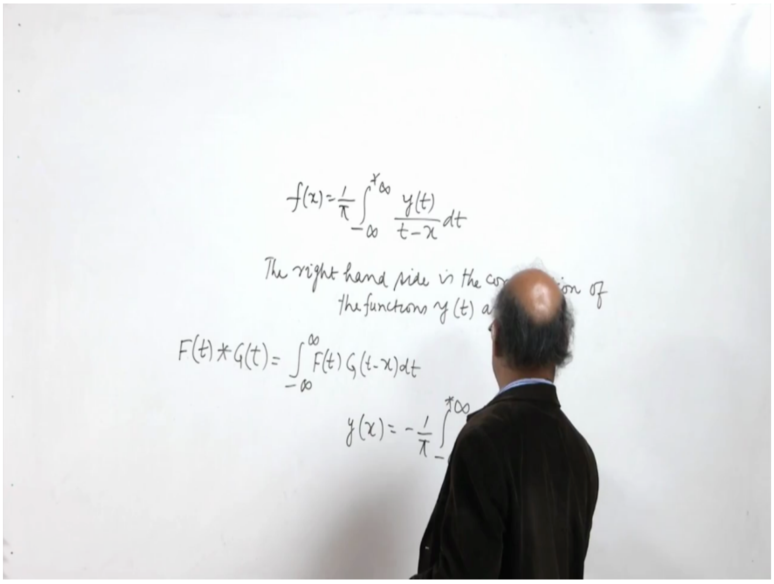

Now let us go to the definition of in finite Hilbert transform. In finite Hilbert transform is defined as fx equal to 1 upon pi minus infinity to infinity y(t) over t minus x dt. You can see here that fx equal to one over pi minus infinity to infinity y(t) over t minus x dt, this is nothing but the right-hand side is the convolution of the function y(t) with the function 1 over pi t.

So this right-hand side is a convolution of the functions y(t) and 1 over pi t. If you recall how we defined the convolution of 2 functions ft and gt convolution of 2 functions Ft and Gt is defined as Ft star Gt equal to integral over minus infinity to infinity Ft Gt minus x dt, so right-hand side is the convolution of y(t) and the function is 1 over pi t and its inverse that is the solution of this integral equation is given by yx equal to minus 1 over pi. So solution is yx equals to minus 1 over pi integral over minus infinity to infinity Ft over t minus x dt. So the unknown function in the integral equation y(t) is determined by the the question giving is the inverse that is yx equals to minus 1 over pi integral over minus infinity to infinity ft over t minus x dt.

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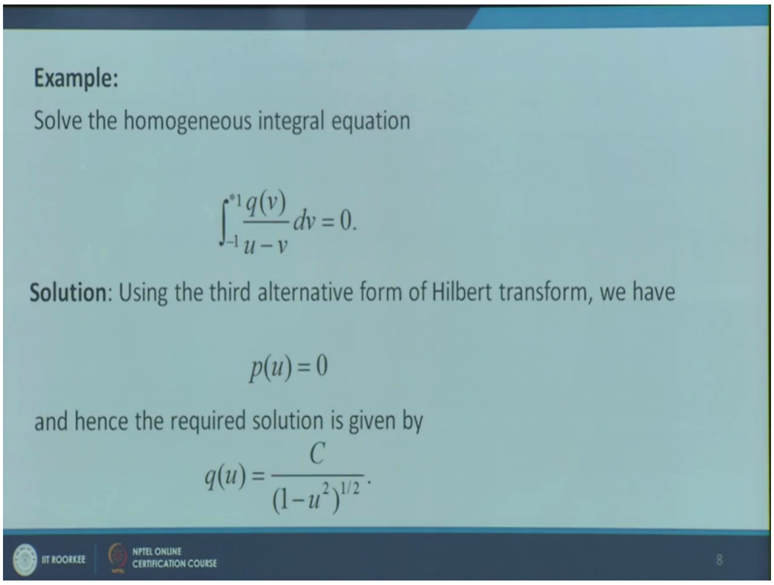

We will take an equation and see how we will solve this, let us look at this homogenous integral equation because here the function fx is a 0. So let us go to a third alternative form Hilbert transform pair, this one. Third alternative form of the Hilbert transform pair, so pu actually is taken as 0 here in this example, this is the case of finite Hilbert transform not of in finite Hilbert transform.

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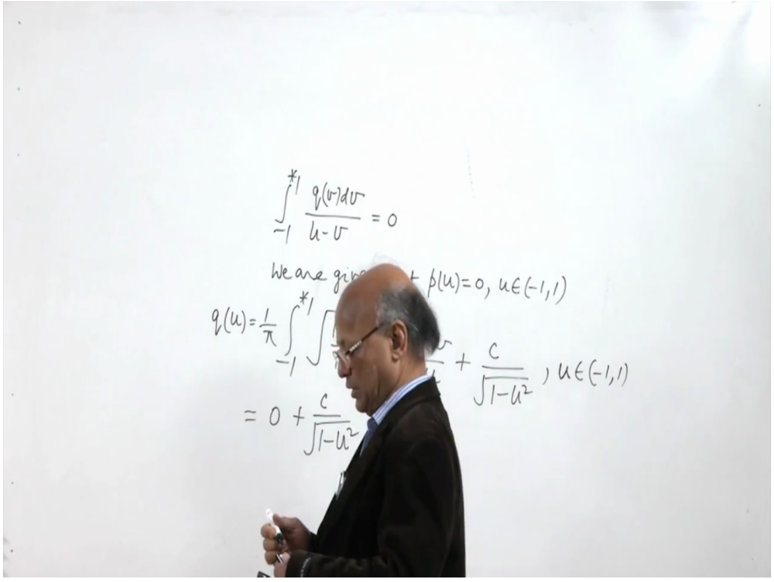

So in this equation we are given that integral over minus 1 to 1 principal value of the integral qvdv over u minus v is equal to 0. Now let us compare it with the third alternative form of the finite Hilbert transform, so this is the equation pu equal to one over pi integral over minus 1 to 1qvdv upon u minus v. So if you compare the given equation with this equation we have pu equal to 0.

Now let us see the the solution of the integral equation is given by qu is equal to one over pi integral over minus 1 to 1 under root 1 minus v square upon 1 minus u square pvdv divided by v minus u plus c upon under root 1 minus u square, u belongs to minus 1,1. Now we know that from this equation we know that pu equals to 0 over the interval minus 1 to 1. We are even that pu equals to 0 for the integral minus 1, 1.

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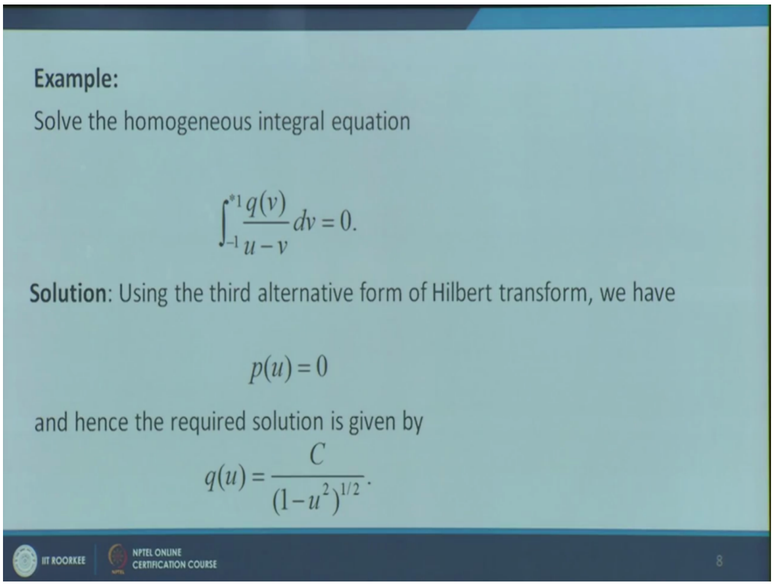

So let us put it in this equation, here we put pu equals to 0 pv vary from minus 1 to 1. So the first expression becomes 0 and what we get? c upon under root 1 minus u square, so where c is the c is given by c is an arbitrary constant, 1 over pi integral over minus 1 to 1 qvdv, so here the solution of this equation the homogenous integral equation is given by qu equals to see upon under root 1 minus u square. So this is how we solve this example by making use of the third alternative form of this Hilbert transform pair.

With this I would like to conclude my lecture, thank you very much for your attention.