Integral Equations, Calculus of Variations and their Applications

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Lecture 36

Solution of Integral Equations using Fourier Transform

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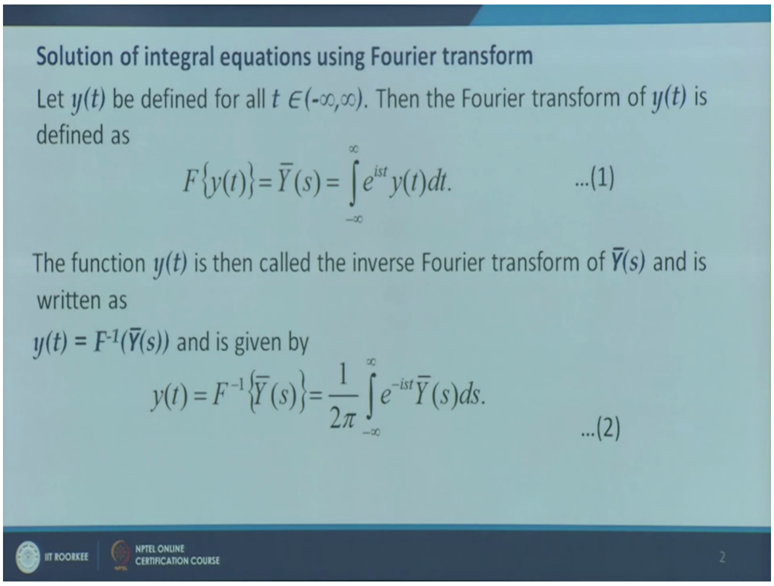

Hello friends I will come you all to my lecture on solution of integral equation using Fourier transforms. So let us first define what do we mean by Fourier transform? Let by y(t) a function defined for all T blogging to the interval minus infinity to infinity, then the Fourier transform of y(t) is defined as F y(t) equal to y bar s equal to integral over minus infinity to infinity e to the power ist y(t)dt.The function y(t) is then called the inverse Fourier transform of y bar s and is written as y(t) equal to F inverse y bar s and it is given by y(t) equal to F inverse y bar s equal to 1 over 2 pi and a lower minus infinity to infinity e to the power minus ist y bar s ds.

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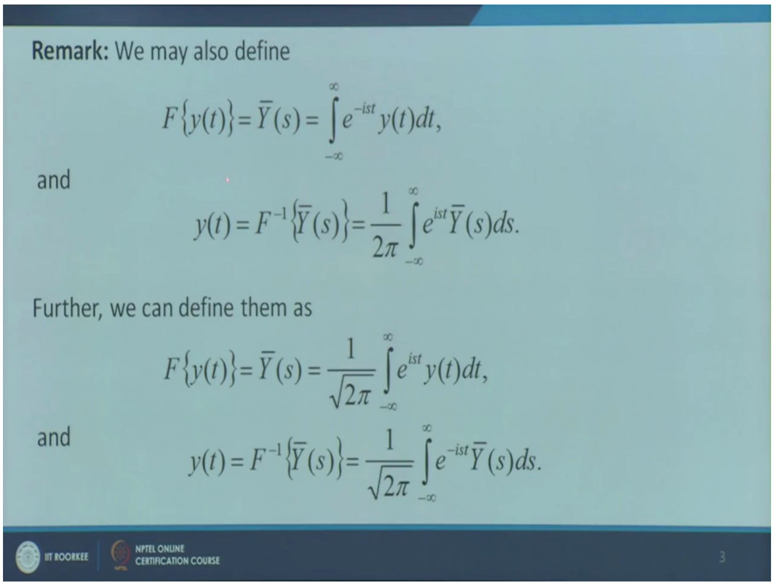

Now some youth authors define the Fourier transform of y(t) which we are denoting by y bar s as also integral over minus infinity to infinity e to the power minus ist y(t) dt but then the inverse Fourier transform will be given by F inverse y bar s equals to 1 over 2pi integral over minus infinity to infinity e to the power of ist y bar s ds. So there is no harm we can also use this formula and further some authors define the Fourier transform of y(t) which is equal to y bar s as 1 by root 2pi integral over minus infinity to infinity e to the power ist y(t) dt and the corresponding inverse Fourier transform which is y(t) is equal to 1over root 2pi minus infinity to infinity e to the power minus ist y bar s ds. So the coefficient 1 over 2 pi they distribute over the Fourier transform as well as to the inverse Fourier transform as 1 over root 2pi.

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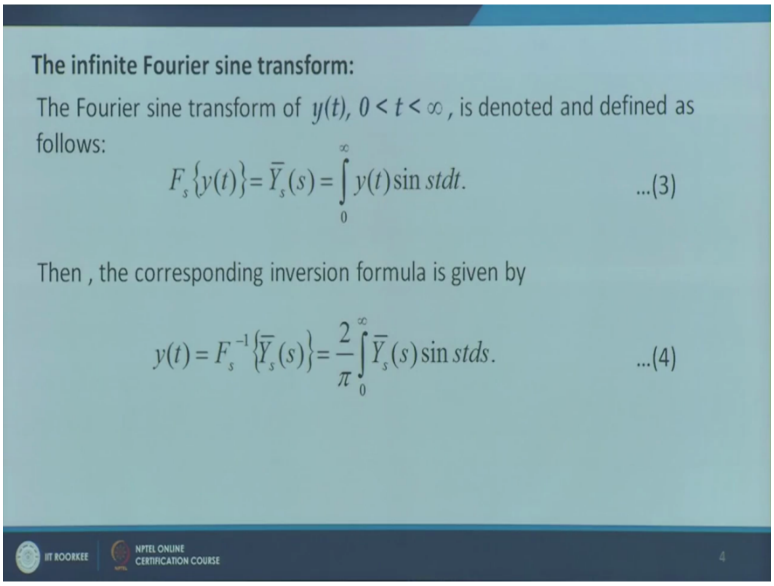

Now we next define the infinite Fourier sine transform, the Fourier sine transform of y(t) where t varies from 0 to infinity is denoted and defined as follows Fs Fs s denotes sine transform, so Fourier sine transform of y(t) is by y s bar s which is defined as 0 to infinity y(t) sine std t, the corresponding inversion formula is given by y(t) equal to Fs inverse y bar s ys bar s equal to 2 over pi 0 to infinity Ys bar S sine stds.

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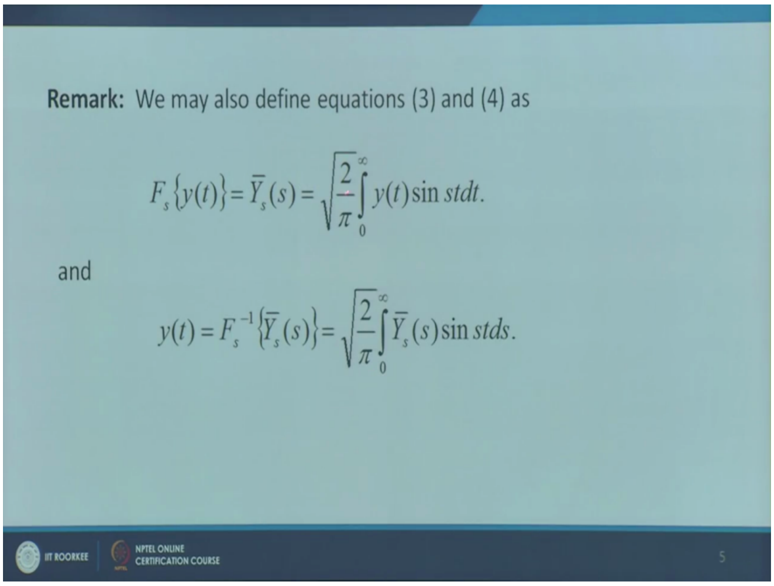

Again some authors define the Fourier sine transform of y(t) which we are denoting by Ys bar S as root 2 over pi 0 to infinity y(t) sine stdt and the corresponding inverse Fourier sine transform of Ys bar s as square root 2 over pi 0 to infinity Ys bar s sin stds. So the coefficient to over pi they distribute as root 2 over pi here and root 2 over pi here.

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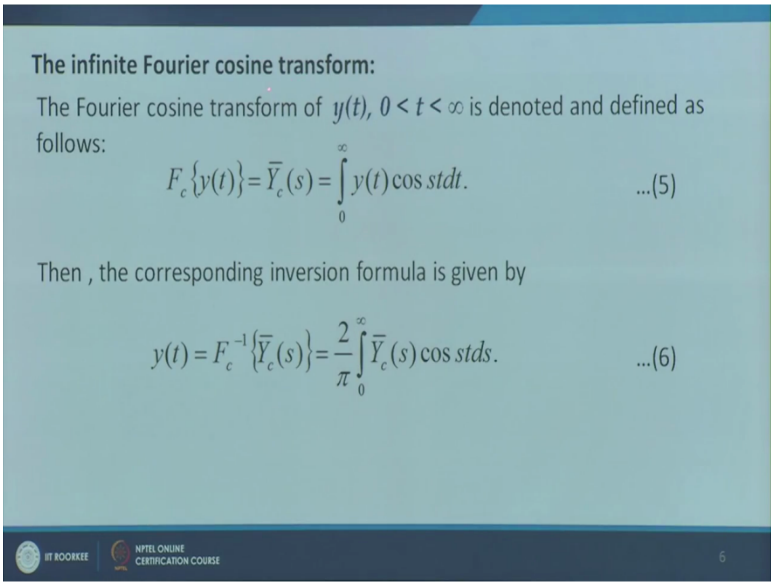

Now infinite Fourier cosine transform is defined in a similar manner, the Fourier cosine transform of y(t) where t varies from 0 to infinity is denoted and defined as F Fc, c denotes cosine transform. So Fourier cosine transform of y(t) is equal to YC bar S is equal to integral 0 to infinity y(t) cos stdt the corresponding inversion formula is given by y(t) equals to Fc inverse Yc bar S equal to 2 over pi integral over 0 to infinity Yc bar S Cos stds.

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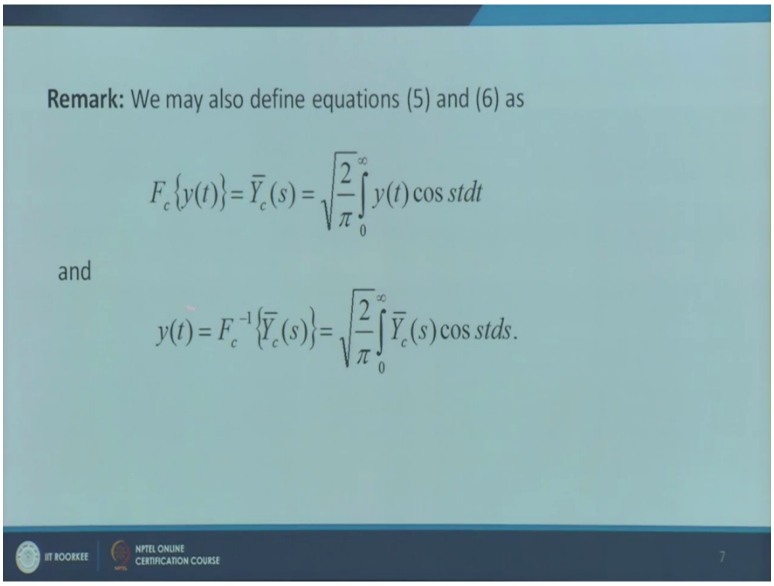

As in the case of the Fourier sine transform, some authors distribute this 2 over pi s, root 2 over pi here and root 2 over pi here. So they define Fourier cosine transform of y(t) s Yc bar S equals to into root 2 over pi integral 0 to infinity y(t) cos stdt and inverse Fourier cosine transform of Yc bar S as root 2 over pi 0 to infinity Yc bar s cos stds.

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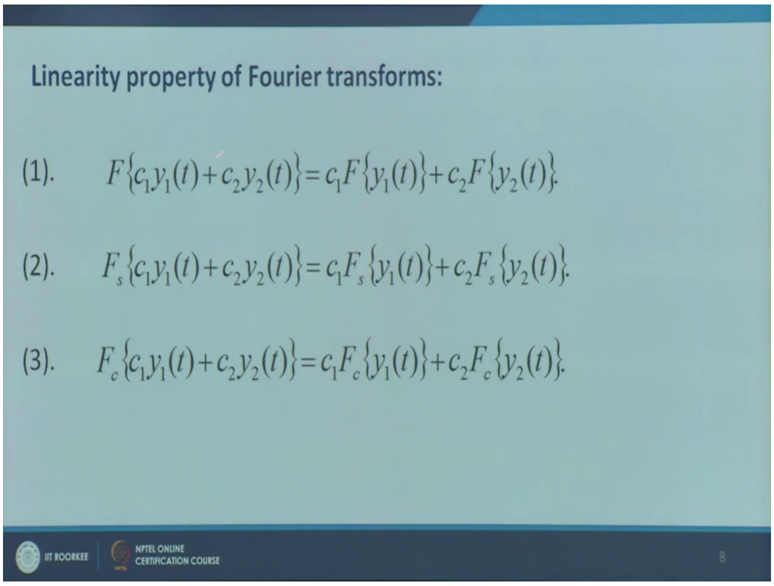

just like in the case of last class transform, the Fourier transform and Fourier sine and cosine transform also satisfy the linearity property. So Fourier transform of C1y1 and C2y2 where y1y2 are functions defined over the interval minus infinity to infinity is equal to C1F y1t plus C2F y2t and here y1 and y2 are functions defined on the interval 0 infinity than the Fourier sine transform of C1y1 plus c2y2 is C1 Fourier sine transform of C1y1 plus C2y2 equal to C1 times Fourier sine transform of y1 plus C2 times Fourier sine transform of y2, C1 C2 are constants.

Now in the third equation we are riding the linearity property for the Fourier cosine transform, so Fourier cosine transform of C1y1 plus C2y2 is equal to C1 times Fourier cosine transform of y1 plus C2 times Fourier cosine transform of y2. So these properties of the Fourier transform in Fourier sine and cosine transforms are easily verified by their definitions.

The change of scale property like in the case of Laplace transform, here also we have the change of scale property. So if y F yt equals to Y bar S then Fyat is 1 by a Y bar S by a. So and similarly Fourier sine transform of y(t) it is Ys bar s then Fourier sine transform of yat equals to 1 by a Ys bar S by a and Fourier cosine transform, if it is if it is y(t) if it is Yc bar S then Fourier cosine transform of yat is equal to 1 by a Yc bar S by a. So these properties are also verified easily by just by following their definitions.

Now let us define the convolution of 2 functions, just like in the case of Laplace transforms the convolution here of 2 functions Gx and Hx which are defined over the interval minus infinity to infinity is denoted and defined as G star H equal to integral over minus infinity to infinity Gx H t minus x dx and this is same as integral over minus infinity to infinity Gt minus x Hxdx which can be easily verified. So then the Fourier transform of the convolution of G and H this F of G star H is the product of the Fourier transforms of Gx and Hx, just like as in the case of Laplace transform. So this is how we get the Fourier transform of the convolution of the 2 functions of x.

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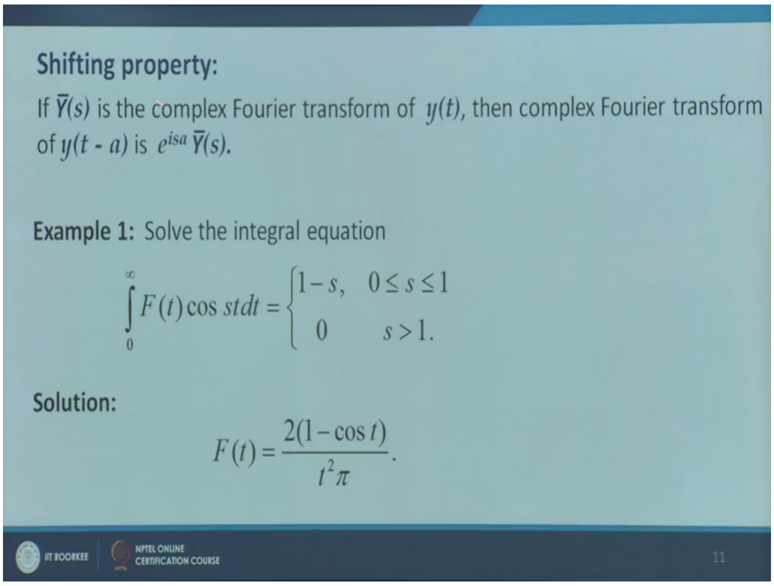

Then like in the case of Laplace transform we have shifting property here if Ys is the complex Fourier transform of y(t) then the complex Fourier transform of y(t) minus a is e to the power of isa Y bar s which can also be easily shown or proved by following the definition of the Fourier transform of y(t). Now let us see how we can solve the integral equations using the theory of Fourier transform.

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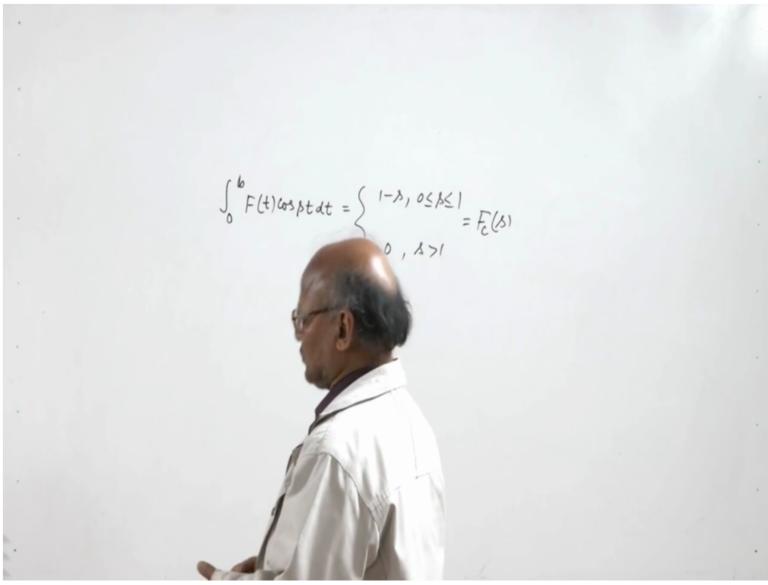

So now step this equation integral 0 to infinity Ft cos st dt equal to 1 minus s, 0 less than or equal to s less than or equal to 1, 0 when s is greater than one. So here this is nothing but you can call it as Fcs because we can identify this equation by the Fourier cosine transform of the function Ft.

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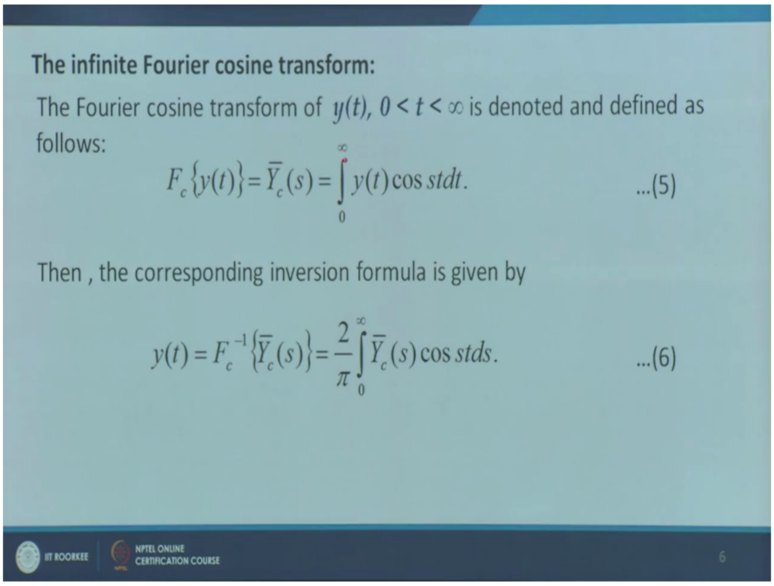

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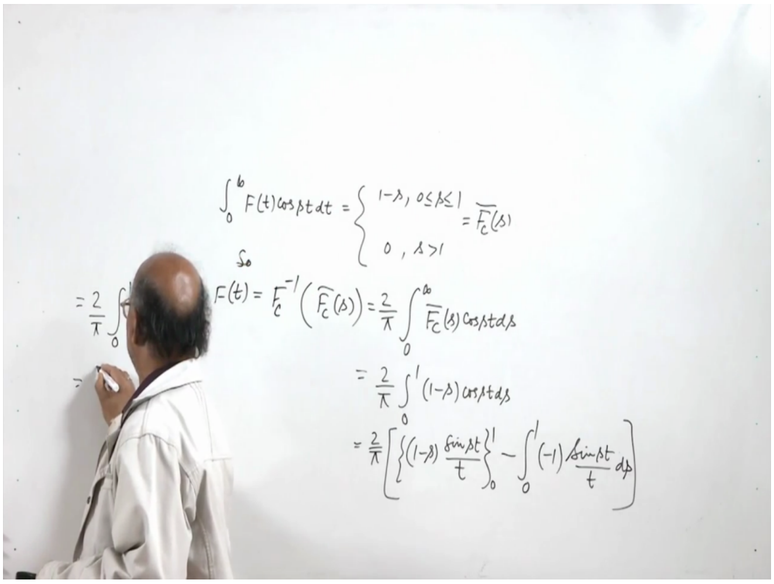

In the Fourier cosine transform, if you go to the definition of Fourier cosine transform then you will see that Fc yt which we have denoted by Yc bar S is integral 0 to infinity y(t) cos stdt. So in place of y(t) here we have Ft, so we are denoting the integral 0 to infinity Ft cos stdt by Fc bar S.

So here we are given Fc bar S and we want to undermine they are known function Ft, so we will use inversion formula for the Fourier cosine transforms. So Ft is equal to Fc inverse Fc bar s which is equal to 2 upon pi 0 to infinity Fc bar S into Cos stds. Now this is let us use the definition of Fc bar S, Fc bar S is 1 minus S over the interval 0 to 1 and elsewhere it is 0. So this integral over 0 to infinity will that can be written as integral over 0 to 1 and Fc bar is equal to 1 minus S cos stds the remaining part the integral over 1 to infinity vanishes because Fc bar S equals to 0 for all S greater than one.

Now let us integrate it by perks, so when we integrate it by perks but we have 1 minus S integral of cos st is sine st we are integrating with respect to t we are integrating with respect to s. So this will be sine st over t and we are to evaluate the value of this expression at 0 and 1 minus integral 0 to 1, derivative of 1 minus S is minus 1 and then sine st over tdt. Now when s equals to 1 this is 0 and when s equals to 0 sine st is 0, so this expression vanishes and what we have is 2 over pi integral over 0 to 1 sine st over t ds.

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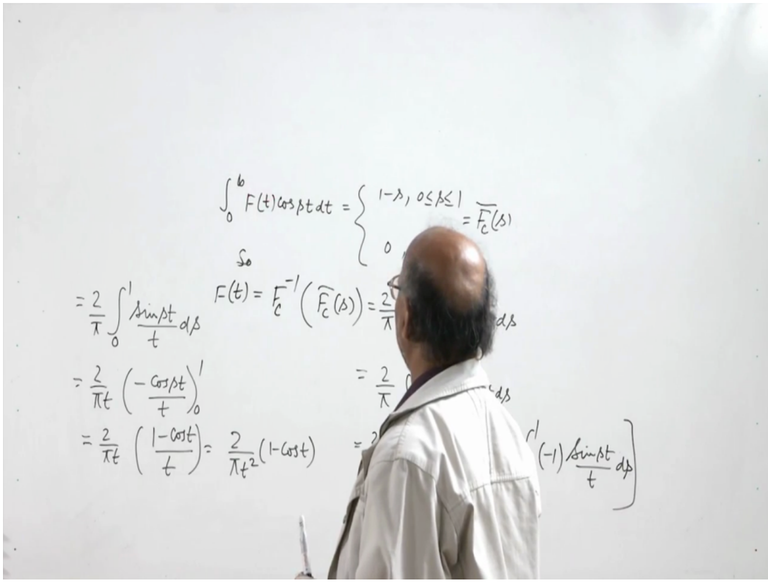

Now we can integrate it again so 2 over pi t, t we can take out and then we have minus cos st over t. So this is nothing but 2 over pi t and this is 1 minus cos t over t. So what we have is 2 over pi t square 1 minus cos t. So this is a solution of the given integral equation. We have found out the unknown function Ft, Ft is 2 over pi t square 1 minus cos t. So this is the answer of this question.

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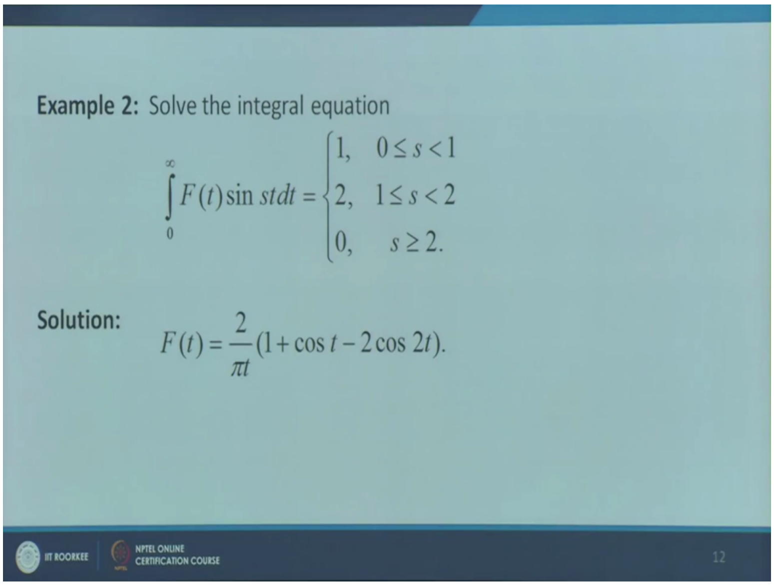

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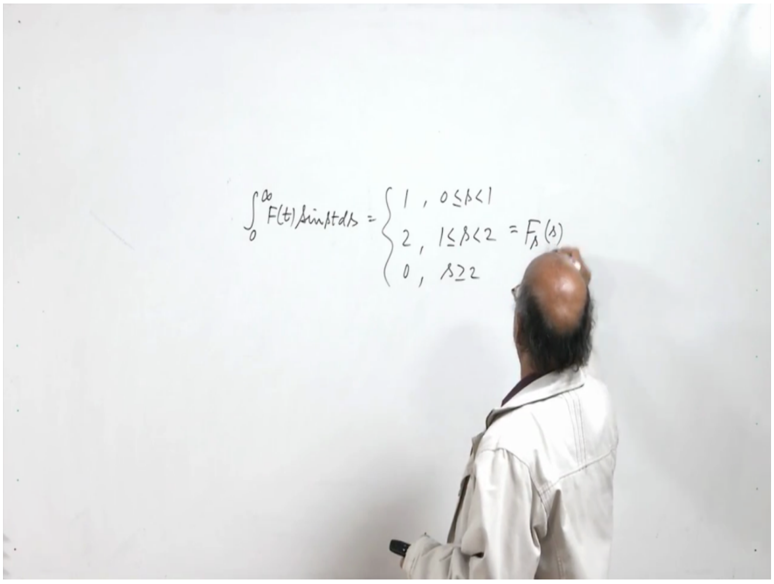

Now let's go to the second question, so here we have the integral equation 0 to infinity Ft sine st dt equal to 1 when 0 less than or equal to s less than one, 2 1, one is less than or equal to s less than 2 and 0 when s is greater than or equal to 2. So this is Fourier sine transform so this we can denote as Fss.

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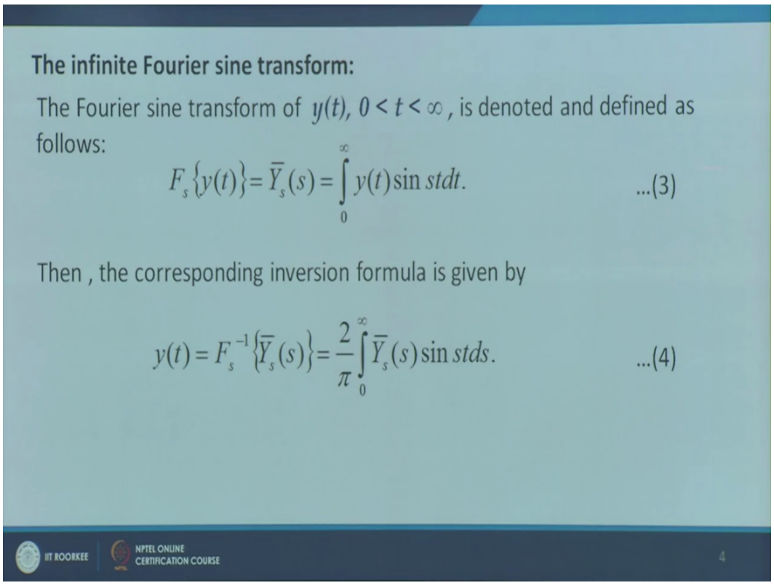

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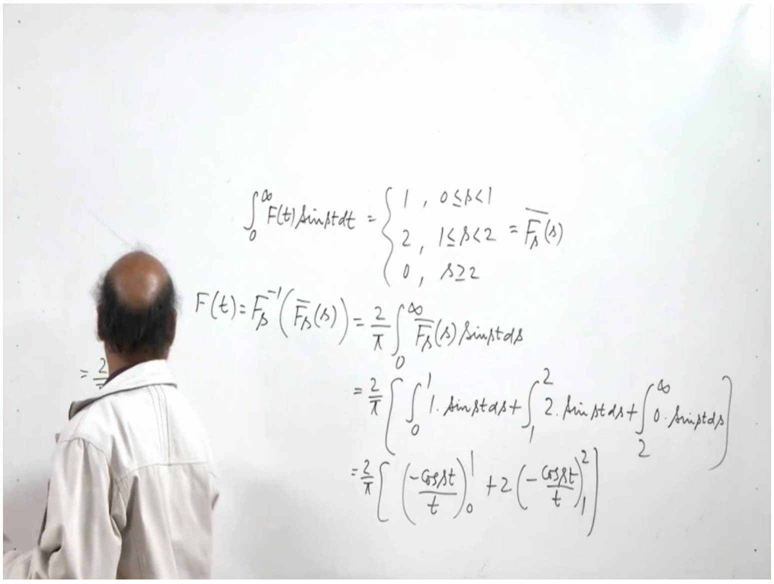

Now let us go back to the inversion formula for the Fourier sine transform. So the inversion formula for the Fourier sine transform is here. So here we have integral 0 to infinity y(t) sine stdt at Ys bar S we have integral 0 to Ft, 0 to infinity Ft sine stds. So in place of yst we have here Ft and we have to determine the known function Ft. So let us see Ft is given by Fs inverse Fs bar s and which is equal to 2 over pi integral 0 to infinity, Fs bar S into sine stds.

So this is 2 over pi integral 0 to 1 Fs bar S is equal to one, so sine stds plus integral 1 to 2, 2 sine stds and over integral 2 to infinity Fs bar S equals to 0, so 0 sine stds. So this is equal to

2 upon pi and we have minus cos st divided by t here 0 to 1 and here we have 2 times minus cos st divided by t, this is 0.

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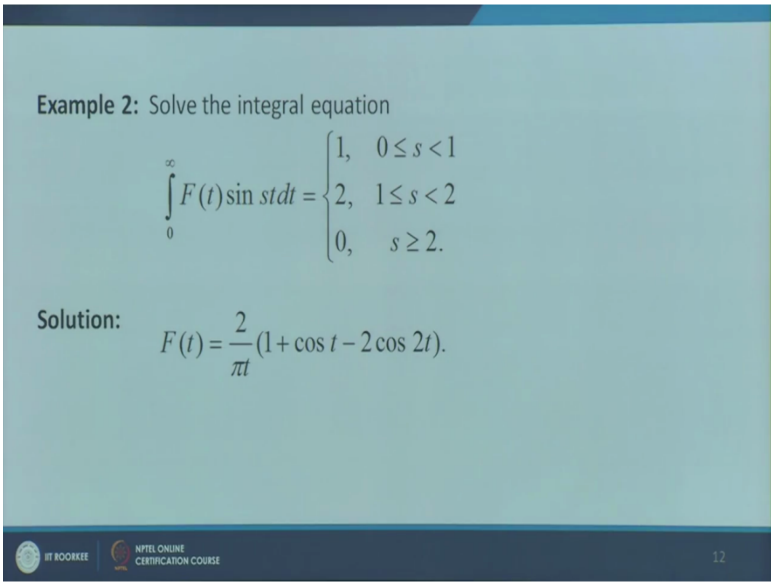

So this gives us 2 over pi into t 1 minus cos t we get 1 minus cos t here and we get 2 times when you put 2 here we get cos 2t when we put 1 here we get cost t, so cos t minus cos 2t, so this gives you 2 upon pi t 1 plus cos t minus 2 cos 2t. So this is what we have Ft equal to 2 over pi t 1 plus cos t minus 2 cos 2t.

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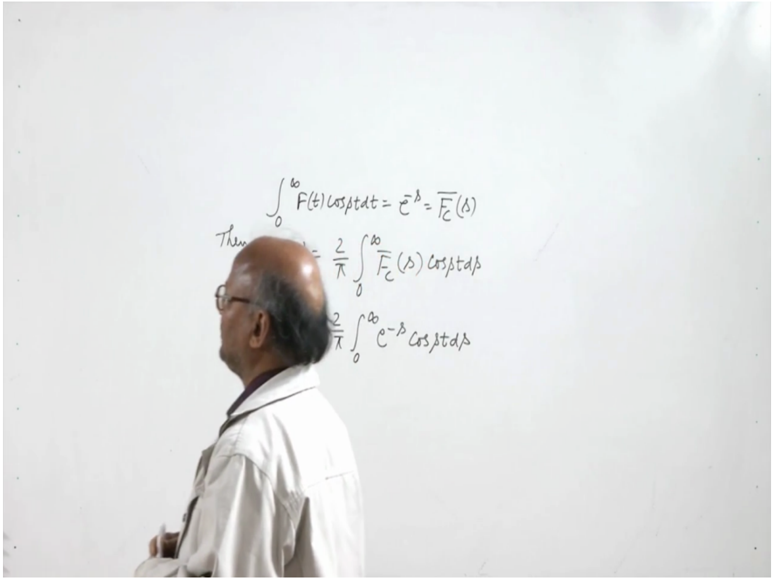

Now let us do one more question on this to get things more clear. So suppose we have to solve integral 0 to infinity Ft cos stdt equal to e to the power minus S. So again we are given the Fourier cosine transform we can denote it by Fc bar s then the unknown function Ft is given by 2 over pi, 0 to infinity Fc bar S into cos stds. So this is 2 over pi 0 to infinity e to the power minus s cos stds.

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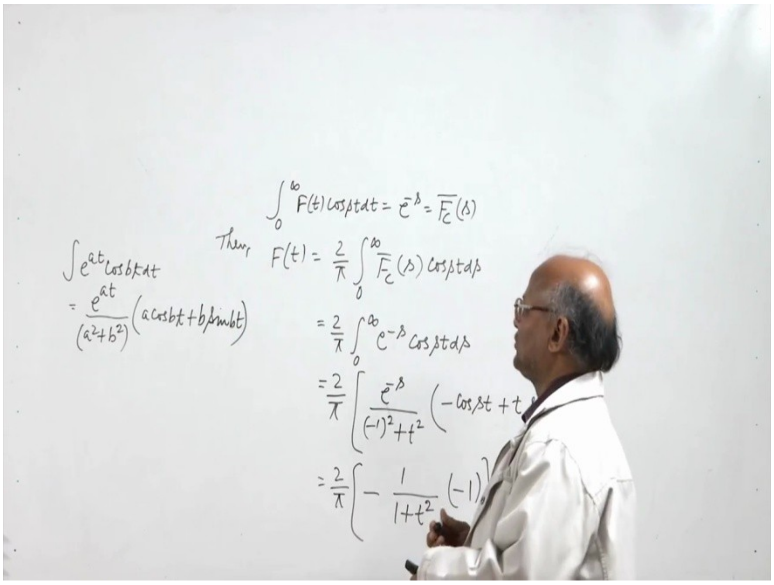

Now we can make use of the known formula, this formula is known to us let's use the formula integral over e to the power at cos btdt equal to e to power at a square plus b square a cos bt plus b sine bt. So this equal to 2 over pi when we use this formula a is equal to minus 1 instead of t we have s here. So e to the power minus s divided by a square plus b square and a is equal to minus 1. So minus 1 whole square and b is t, so we get t square b is t.

So then we have a cos bt, so a is minus 1cos b is t and t is s, so st plus t sine st, this is what we have. So when s goes to infinity when s goes to infinity because cos st is bounded by one and e to power minus s goes to 0. So this quantity goes to 0 and then e to the power minus s into sine st when s goes to infinity also goes to 0 because sine st is bounded and need for minus s goes to 0.

So this goes to 0, now we have 2 upon pi and then we put the lower limit, so minus e to the power 0 is 1. So 1 upon 1 plus t square and then here we put t equals to 0, sorry s equals to 0 we get minus 1 and then they put s equals to 0 this becomes 0. So we get 2 upon pi into 1 plus t square. So we get Ft equal to 2 upon pi into 1 plus t square. So this is how we solve the integral equations using Fourier transforms.

Now in our next lecture we will see how we can solve Cauchy integral, how we can solve integral equations using Hilbert transforms, with this I would like to conclude my lecture, thank you very much for your attention.