Integral equations, calculus of variations and their applications

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Lecture 33

Cauchy type integral equations-3

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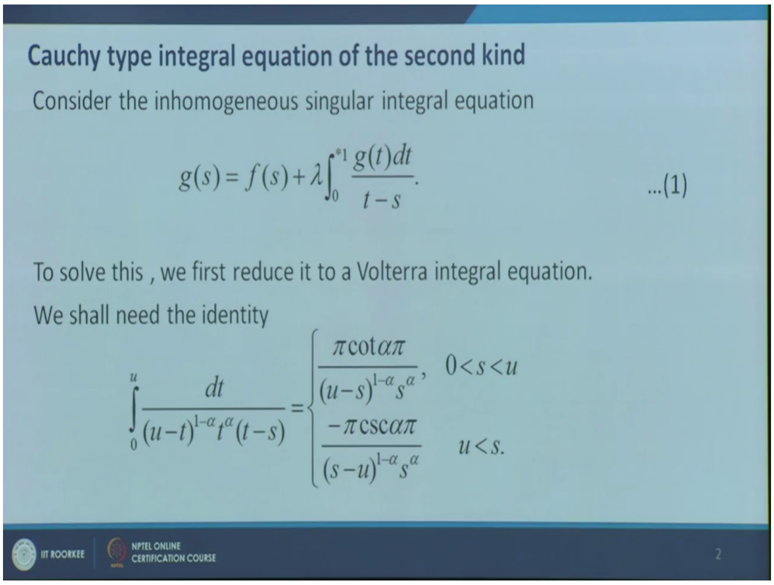

Hello friends, I will commute to my third lecture on Cauchy type integral equations. We in the last lecture of, we considered Cauchy type integral equation of the first kind now we shall try to solve the Cauchy type integral equations of the second kind. So let us consider the inhomogeneous singular integral equation, in homogeneous or we can also call it known homogeneous singular integral equation.

So consider the non-singular integral equation gs equal to fs plus lambda times integral 0 to 1 gtdt t minus s where this star indicates the principal value of the Cauchy principal value of the integral. So to solve this Volterra, to solve this singular integral equation, the first reduce it to a Volterra type integral equation and then the way we solve Volterra type integral equation, we shall solve it.

Now to do this we will need this identity integral 0 to u dt upon u minus t raise to the power 1 minus Alpha t to the power Alpha into t minus s equal to Phi Cot Alpha pi over u minus s to power 1 minus Alpha s to the power Alpha when 0 is less than s than u minus Pi cosec Alpha by over s minus u to the power 1 minus Alpha s to the power alpha whenever u is less than s.

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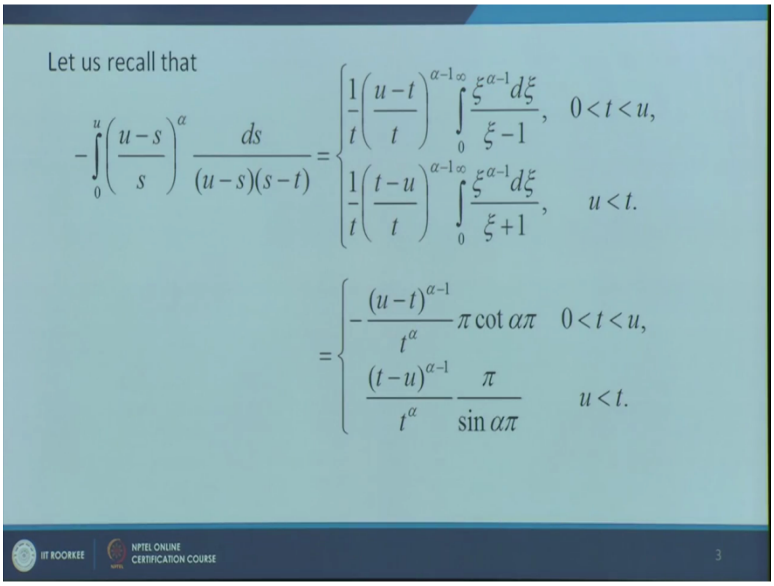

So let us recall that in the first lecture on Cauchy type integral equations we had shown that minus integral 0 to u, u minus over s to the power Alpha ds over u minus s into s minus s is equal to this expression, when 0 is less then t less than u and it is equal to this when u is less then t and there I had shown that integral 0 to infinity Xi to the power Alpha minus 1 dxi over Xi plus 1 is equal to Pi over sine Alpha pi by putting Xi equal to t upon 1 minus t.

Now here but we had not than this integral 0 to infinity xi to the power Alpha minus 1 dxi over xi minus1, so I will show you how we can calculate the integral 0 to infinity? xi the power Alpha minus 1 dxi over xi minus 1. Now we have to show that if you recall if you compare the 2 values you can see that integral 0 to infinity site the power Alpha minus 1 dxi over xi minus1 we have to prove that it is equal to minus pi cot Alpha pi.

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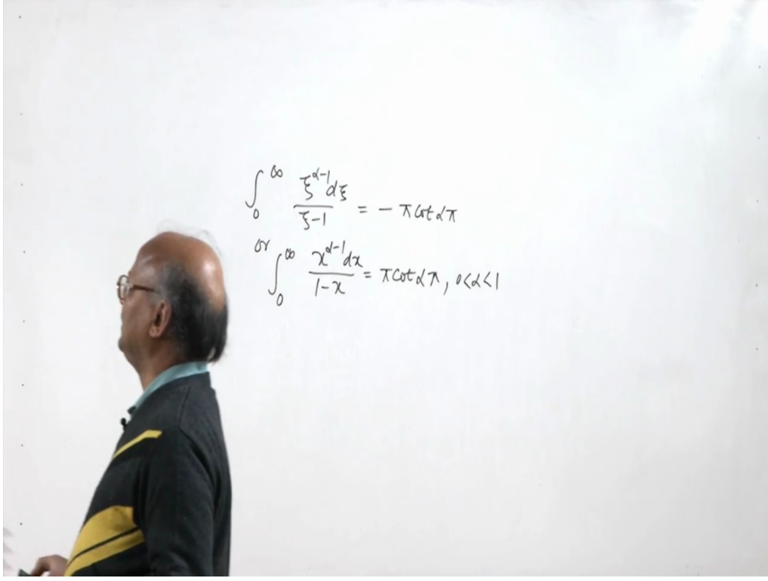

So we have to show this integral 0 to infinity, xi to the power Alpha minus 1 dxi over xi minus 1 this is equal to minus pi cot Alpha pi with our we can say that we have to prove that integral 0 to infinity for convenience we can write in place of xi we can write x, so x to the power alpha minus1, dx divided by 1 minus x equal to pi cot Alpha pi 0 less than alpha less than 1.

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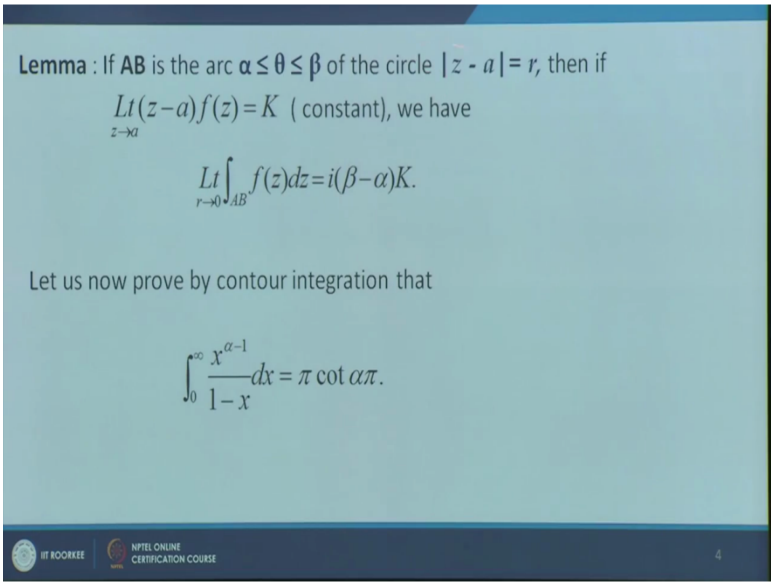

Now we shall find this integral, this real integral values while using the contour integration, so what we will do is , so first we shall prove this lemma if ab is the arc Alpha less than or equal to Theta less than or equal to beta of the circle mod of z minus a equal to r. We know that in the complex plane mod of z minus a equal to r represents a circle whose centre is at the complex number a and whose radius is r.

So then if limit z tends to a z minus a into fz equals to K where K is a constant that we have limit are tends to 0 integral AB integral over AB fz dz equal to i times B minus alpha over K, this result we will use when we evaluate this integral y contour integration. So let us see how we prove this. So this is the centre and r is the radius, so this is mod of z minus a equals to r. So AB is the arc of the circle mod of z minus a equals to r in the arc AB of the circle lies between the argument theta that is alpha less than or equal to theta less than or equal to beta.

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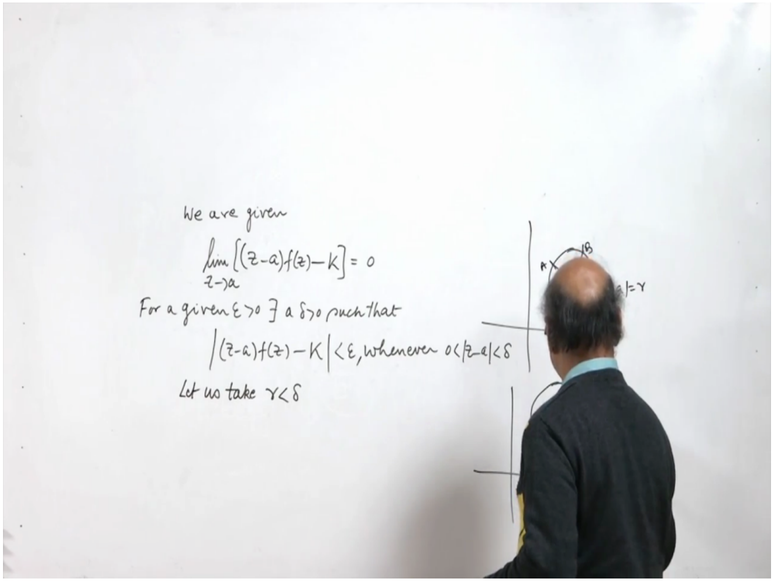

So this is alpha angle, this is alpha and this is beta angle. Now we are given that limit z tends to 0 we are given that limit z tends to a, z minus a into fz is equal to K, r we can say limit z tends to a, z minus a into fz minus K is equal to 0. So why the definition of limit, for a given Epsilon greater than 0 there exist a Delta greater than 0 such that mod of z minus a into fz minus k is less than Epsilon whenever 0 is less than mod of z minus a less than Delta, now we have to show that as r goes to 0, integral over AB fzdz equal to i times beta minus Alpha into K.

So when r goes to 0 the can take our arc b smaller when the number delta, so let us choose r, so let us take r to be less than delta then we have this inequality whenever 0 is less than mod of z minus a plus then r and we are taking r to be less than delta, so mod of z minus a equal to r mod of z minus a equal to r circle will lie inside the disc this is mod of z minus a equal to r, it will lie inside this, this is Delta radius this is r radius, so this is a centre.

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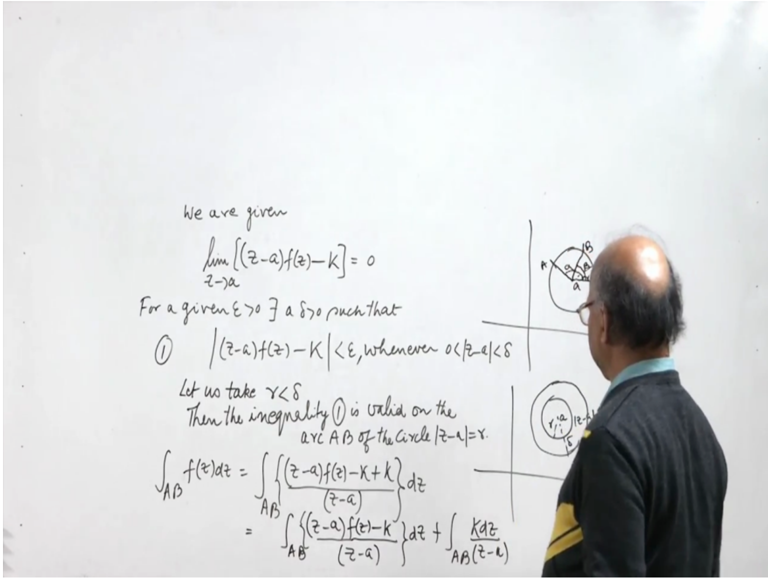

So this inequality, so the inequality 1 will be valid that the inequality 1 will be valid on the arc, AB of the circle mod of z minus a equal to r. Now let us let us write integral over ab fzdz equal to integral over AB z minus a into fz minus k plus k divided by z minus a into dz. Now this integral can be expressed further as, we can break it into 2 parts z minus a into fz minus k upon z minus a into dz plus integral over AB k dz upon z minus a and let us denote them by I1 and I2 the 2 integrals.

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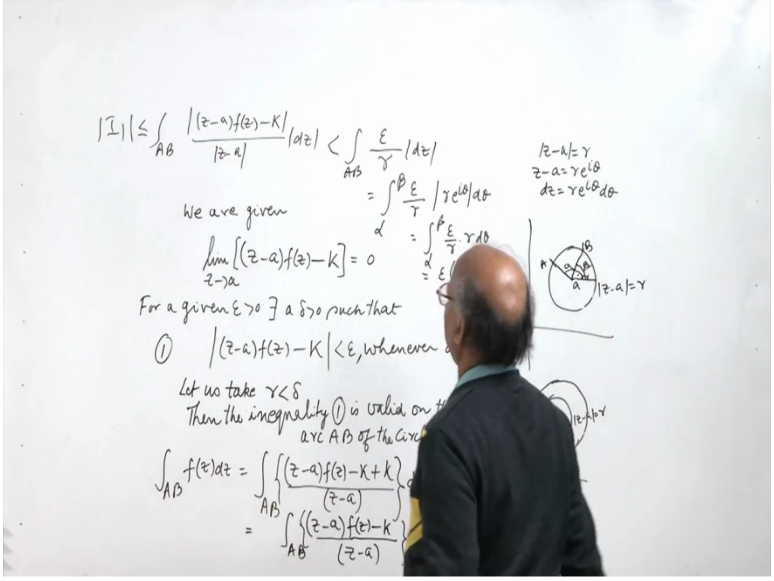

So first let us estimate the value of the integral I1, okay. So let us see mod of I1 is less than or equal to integral over AB mod of z minus into fz minus K upon mod of z minus a into mod of dz on the arc AB the inequality 1 is valid. So mod of z minus a into fz minus K can be made less than Epsilon, so less than Epsilon and AB arc lies on the circle mod of z minus a equal to r. So we have r here into mod of dz.

Now on the arc AB mod of z minus a equal to r or I can right in the parametric form z minus a equal to re to the power i theta, so dz equal to re to the power i theta into d Theta. So this is equal to and for the arc AB Theta varies from alpha to beta this is given to us, so integral over alpha to beta, Epsilon over r into mod of re to the power i theta into d theta, so this is integral alpha to beta, Epsilon over r into r d theta which is equal to Epsilon times beta minus Alpha.

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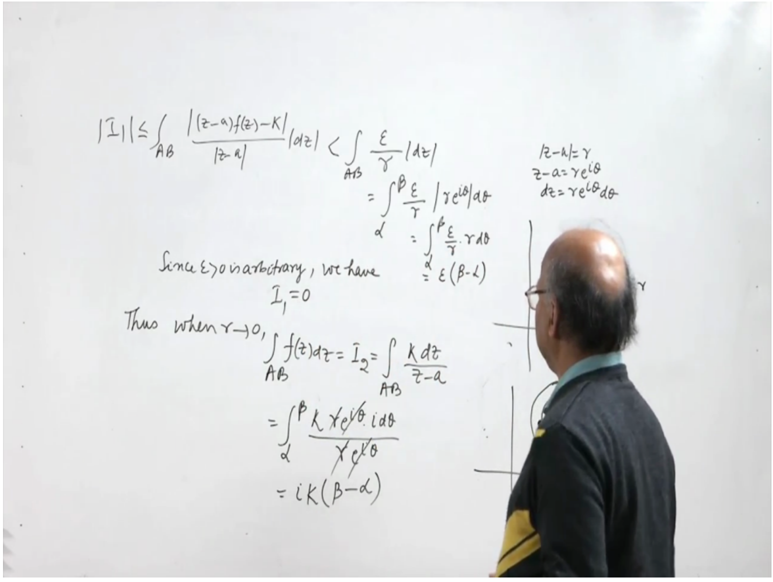

Now since Epsilon greater than 0 is arbitrary, since Epsilon greater than 0 is arbitrary we can make it a small as we please and therefore mod of I1 can be made a small as we please and so we are I1 equal to 0 and thus integral over AB fzdz is equal to I2 when r goes to 0. So thus when r goes to 0 integral over AB fz dz is equal to I2 which is equal to integral over AB K times dz upon z minus a.

Again let us use the parametric representation of the arc AB, so then we have integral alpha to beta K times re to the power i theta, into i d theta divided by re the power i theta they arc AB, let us recall that the arc AB lies on the circle mod of z minus a equal to r and so we can put z minus equal to rei theta and dz equal to rei theta id theta. So this cancels these 2 cancel each other and thus we have i into K beta minus Alpha.

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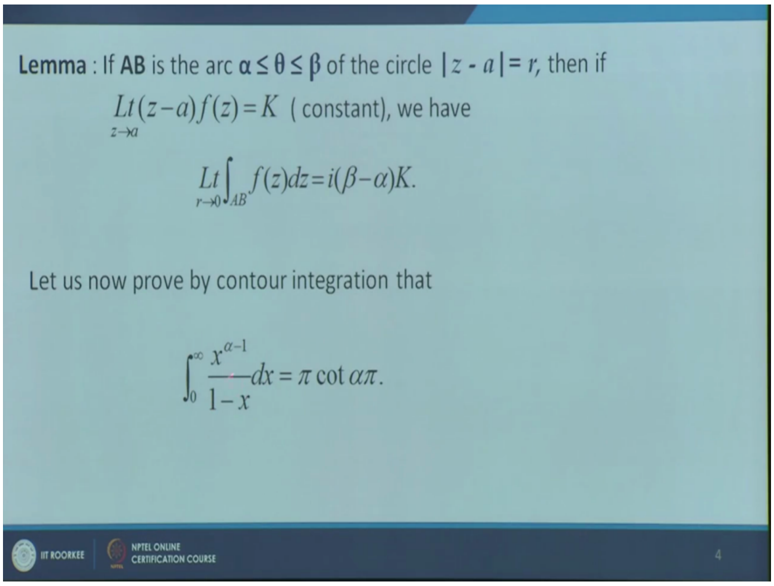

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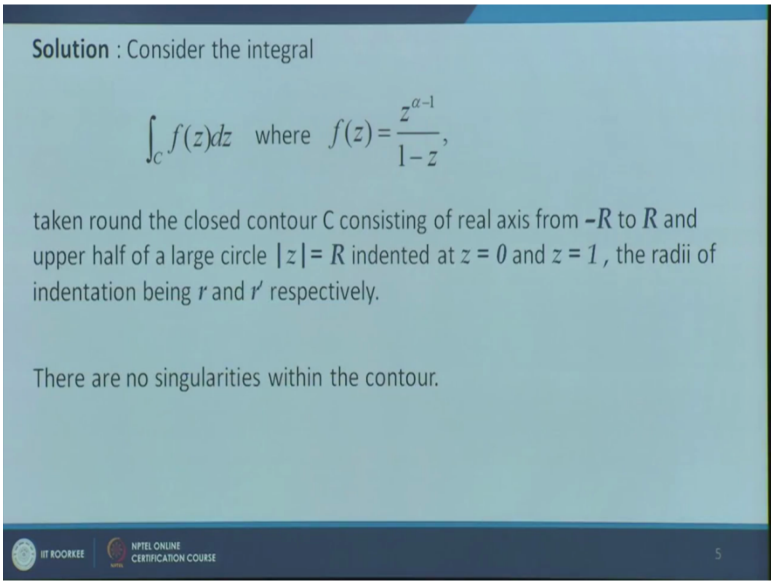

So thus we have when r goes to 0 integral over AB fzdz tends to i into k times beta minus Alpha. So this is the lemma which we will use to evaluate this integral this real integral by using the contour integration. So what we do is, let us consider the contour integral, integral over c fzdz where fz is equal to z to the power Alpha minus 1 over 1 minus z.

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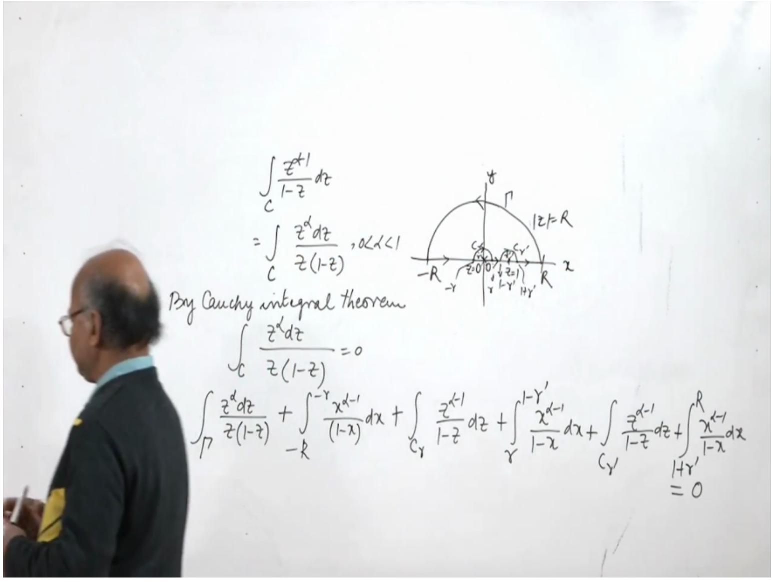

So integral over z to the power Alpha minus 1 over 1 minus z dz, now what is c? c is the close contour consisting of the real axis. So C is taken, so the integral is taken round the close contour consisting of real axis from minus R to R and upper half of the large circle mod of z equal to R. So this is origin this is mod of z equal to R this is the upper half plane, this is the x axis, this is y axis and we move in the counter clockwise direction along the semicircle of radius r with Centre at the origin.

Now let us look at this, this is nothing but since Alpha lies between 0 and 1, we can write there as z to the power Alpha dz divided by z into 1 minus z therefore z to the power Alpha over z into one minus z has similarities at z equal to 0 and z equal to which lies on the real axis. So one similarity lies here, z equal to 0 and another similarity lies here at z equals to 1. we remove these 2 similarities by indentation and what we do is, we take semicircles of radius small r and small r dash.

So let us draw a small circle here of radius r, this is r radius we call it circle cr, this semicircle cr and here we take another semicircle which centre at z equal to one of radius r dash and we call it cr dash. Now when we start moving along the semicircle in the counter clockwise direction when we reach here minus r, we move along the real axis, when we come here this is minus r dash, so when we come here we move along the semicircle in the clockwise direction, when we come here we move along the real axis and then again we move along the semicircle in the counter clockwise direction, when we come here we move along the real axis and reach r.

So now this contour, this contour c which consist of the semicircle from r to minus r then from minus r to minus r dash for from then along cr then from r dash to 1 minus r dash this is 1 minus r dash, this is 1 minus r dash this is minus r dash this is minus r and then this is r and when we reach here this is 1 plus r dash.

So it becomes a simple closed curve when we move on the semicircle let us call it as gamma when we move around the semicircle gamma then the real axis from minus R to r then along semicircle cr then from along real axis from r to 1minus r dash then from 1 minus r dash 2 1 plus r dash along the semicircle cr dash then from 1 plus r dash to r when we move, we are moving along the simple closed curve in which the function fz is z to the power Alpha over z into 1 minus z is analytic.

So by Cauchy integral theorem, integral over c, z the power Alpha dz divided by z into 1 minus z will be equal to 0. Now this is integral over gamma, we can break into parts plus integral over now when we have moved along gamma when we reach here minus r from minus R to minus Rv, we move along the real axis. So z becomes equal to r, so minus r to minus r x to the power alpha over r, you can write x with power Alpha minus 1 upon 1 minus x dx then we are moving along cr in the club y direction cr x to the this is z to the power Alpha minus 1 upon 1 minus z dz and then we are moving along from r to 1 minus r dash from r to 1 minus r dash x the power of minus 1 upon 1 minus xdx then we are moving along cr dash in the club y direction after we reach this point we are moving along the real axis again from 1 plus r dash to r, so this thing is equal to 0.

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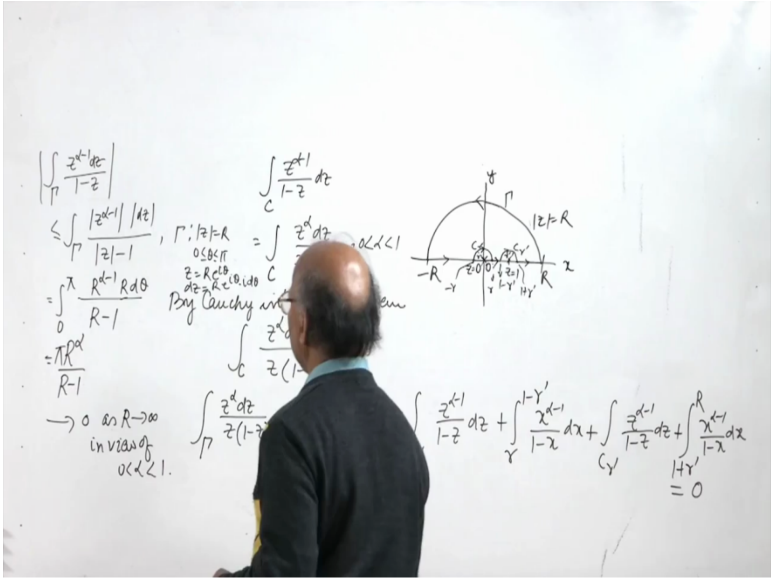

Now what we do is let us evaluate the integral, first we are estimating the integral along gamma, integral over gamma z to the power alpha dz upon z into 1 minus z, so let us see integral over gamma z to the power Alpha minus 1 dz upon 1 minus z. Let’s take mod of this, this is less than or equal to integral over gamma, mod of z to the power Alpha minus 1 then mod of the z and mod of 1 minus z, mod of 1 minus z is greater than or equal to mod of z minus 1, so we have mod of Z minus 1 here along gamma mod z is equal to R and 0 is less than or equal to theta less than or equal to pi because we are taking the semicircle.

So this is equal to integral 0 to pi and we have here R minus 1, this is mod of z to the power Alpha minus 1, so we get R to the power Alpha minus 1 and then mod of dz, z is equal to re to the power i theta, so dz equal to re to the power i theta d theta, so mod of dz will be equal to Rd theta. So this is R to the power Alpha divided by R minus 1 into pi, r to the power Alpha divided by R minus 1.

Now when R goes to infinity, okay when R goes to infinity since Alpha is less than 1, okay. R to the power Alpha over R minus 1 will go to 0 as, so this goes to 0 as R goes to infinity in view of 0 less than Alpha less than 1. So when R goes to infinity integral over gamma z to the power alpha minus 1 over 1 minus z dz goes to 0.

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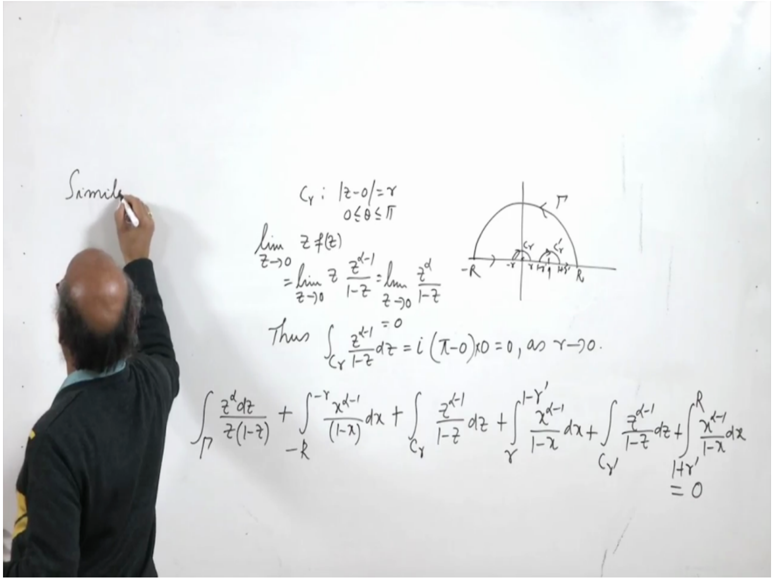

Now let us evaluate the integrals over cr and integrals over cr dash, okay. Now integral over cr, when we evaluate integral over cr, this is 0, this is 1 and this is our gamma this is cr and this is cr dash, this is minus R, this is R, this is one minus R dash, this is one plus R dash and this is R. Now here what we have? This is the indentation along with 0 at centre, okay.

So let us use the lemma, in lemma what we have said, if AB is the arc Alpha less than or equal to theta less than or equal to beta of the circle mod of z minus a is equal to r. So cr is the arc of the circle mod of z minus 0, cr is the arc of the circle mod of z minus a equal to r where 0 is less than or equal to theta less than or equal to pi, let us see the limit here. So limit z tends to a means a tends to 0 because it is the circle with centre at 0. So limit z tends to 0, z minus a means z times fz.

So this is limit z tends to 0, z into z to the power Alpha minus 1 divided by 1 minus z. So this is limit z tends to 0 z to the power alpha divided by 1 minus z. Now Alpha is positive it is greater than 0 less than 1, so this limit is 0 at z tends to 0. So k is equal to 0 this means that integral over cr, so thus integral over cr z to the power Alpha minus 1 upon 1 minus z dz is equal to i times beta minus Alpha that is Pi minus 0 K is equal to 0 here, so into 0 equal to 0 as r goes to 0. So when this radius of this circle cr goes to 0, this integral goes to 0.

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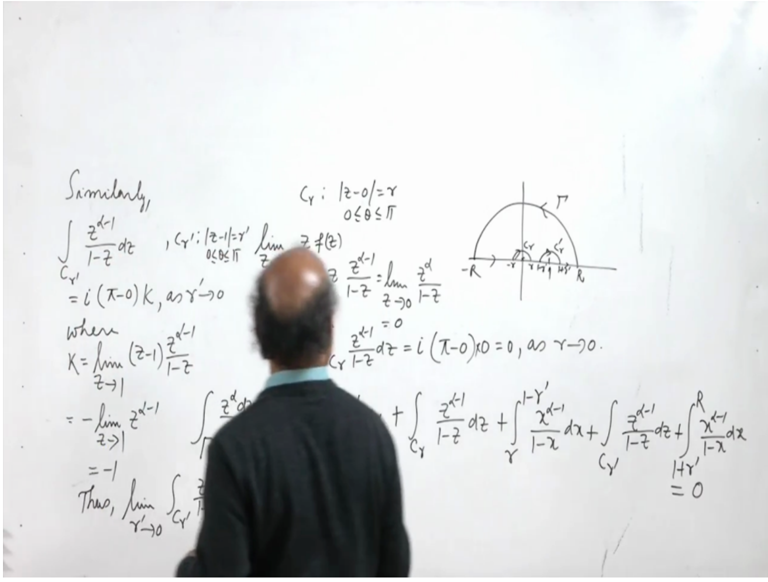

Similarly let us evaluate the other integral. Integral over cr dash z to the power alpha minus 1 upon 1 minus z dz this will be equal to Uh hi times pi minus into k as r dash goes to 0 because the radius here is r dash. Now let us see value at the K where K is equal to limit z tends to A, here the Centre is at at 1 the circle is of radius r dash centred at 1. So cr dash is mod of z minus 1 equal to r dash where 0 is less than or equal to Theta less than or equal to pi, so limit z tends to 1 z minus 1 into fz, fz is equal to z to the power Alpha minus 1 divided by 1 minus z.

So this will be equal to, so this is minus limit z tends to 1, so this will cancel with this we will get minus 1, z to the power Alpha minus 1 so this is equal to minus 1. So we get here minus i pi thus limit r dash tends to 0 integral over cr dash, z to the power Alpha minus 1 divided by 1 minus z dz is equal to minus i pi.

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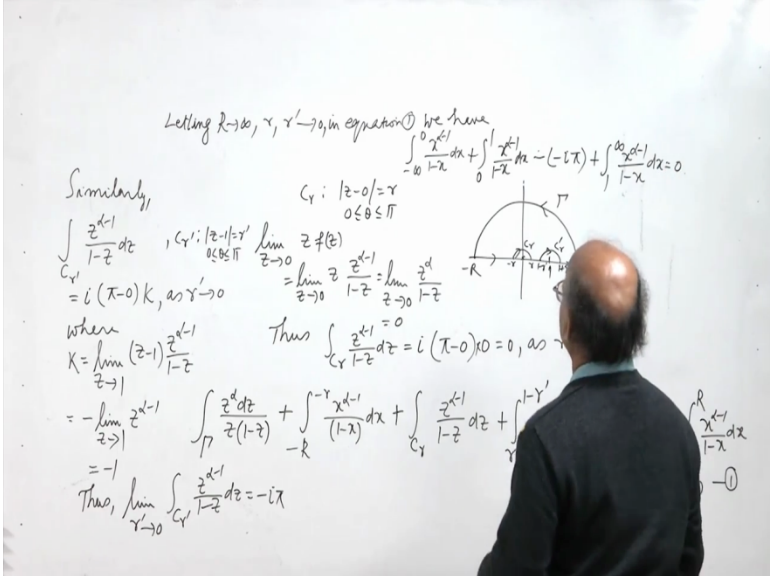

Now let us recall that, so let us now in this equation let me call this as equation 1, in equation one let R go to infinity, this capital R goes to infinity and small r and r dash goes to 0. So letting R go to infinity r, r dash goes to 0 in equation one. So when R goes to infinity this goes to 0, this integral goes to minus infinity to 0, so we have minus infinity to 0 x to the power Alpha minus 1 divided by 1 minus x dx.

Here now this when R goes to 0, integral over cr z to the power Alpha minus 1 into 1 minus z dz goes to 0. Here what we have? Integral over 0 to 1, x to the power Alpha minus 1 upon 1 minus x dx then here when we are moving around cr dash we are moving in the clockwise direction, so with the value of this integral we put minus sign, so minus of minus i pi and then we have integral over 1 to infinity. So this is what we have.

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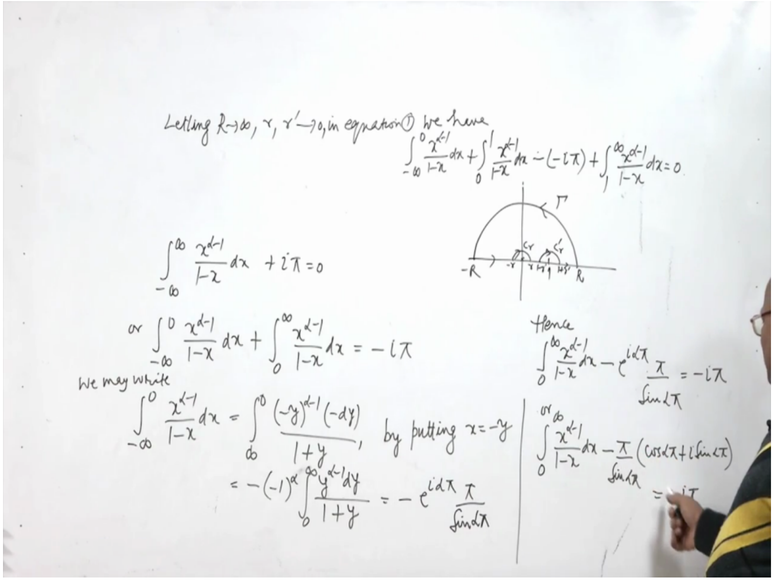

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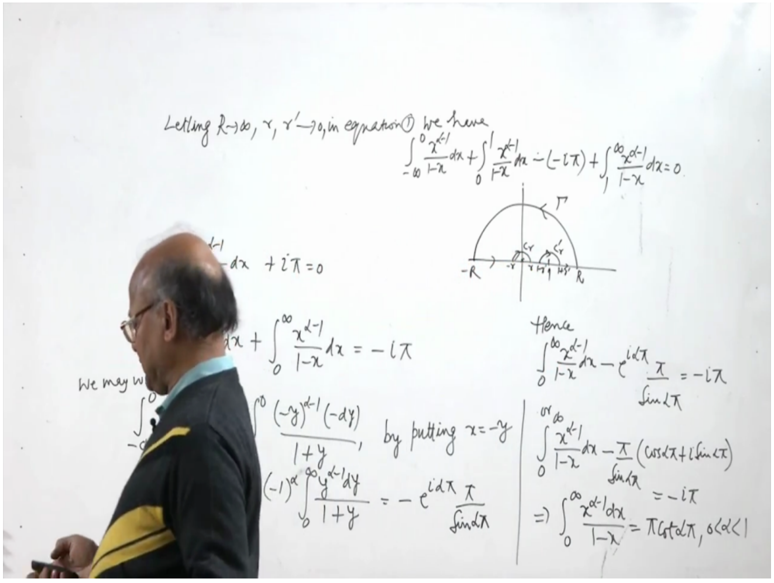

Now let us simplify this, now we are minus infinity to 0 then integral 0 to 1 and then integral 1 to infinity we can combine and write integral over minus infinity to infinity x to the power Alpha minus 1 over 1 minus x dx plus i pi equal to 0. we can say integral over minus infinity to 0, x to the power Alpha minus 1 upon 1 minus x dx plus integral over 0 to infinity, x to the power Alpha minus 1 upon 1 minus x dx equal to minus i pi.

Let us now change the integral minus infinity to 0 to 0 to infinity. So integral over minus infinity to 0 we may write integral minus infinity to 0 x to the power Alpha minus 1 upon 1 minus x dx equal to infinity to 0 minus y to the power alpha minus 1 minus dy divided by 1 plus y by putting x equal to minus y. Now this is equal to minus times minus 1 to the power, alpha minus 1 minus 1 to the power alpha we may write this is minus 1 to the power Alpha y dy, y to the power alpha minus 1 dy upon 1 plus y integral 0 to infinity and we would negative sign when we change the limit of limits of integration.

So this is equal to minus e to the power i Alpha pi, we know that e to power i pi is minus one, so minus 1 to the power Alpha we can write as e to power Alpha pi and integral 0 to infinity y to power Alpha minus 1 dy over 1 plus y we know the value of this Pi over sine Alpha y. So let us put this value here, so we will then have integral over 0 to infinity x to the power Alpha minus 1 upon 1 minus x dx and then let us put the value of the integral.

So minus e to the power i Alpha pi into pi upon sine Alpha pi equal to minus i pi. So this is equal to integral 0 to infinity x to the power Alpha minus 1 upon 1 minus x dx and here I can use the Euler's formula, so pi upon sine Alpha pi and then e to the power Alpha I can write as cos Alpha pi plus i sine Alpha pi, so this is equal to minus i pi. Now when you multiply minus pi sine alpha pi here then you see that sine Alpha pi will cancel minus i pi will cancel with minus i pi and we will get integral 0 to infinity x the power Alpha minus 1 dx divided by 1 minus x equal to pi cot Alpha pi. So this is how we show that integral 0 to infinity x to the power Alpha minus 1 upon 1 minus x dx is equal to pi cot Alpha pi.

So this is the proof will make use of this in getting a solution of the Cauchy integral equation of the second kind, so so that will win the next lecture, so with this I would like to conclude my lecture, thank you very much for your attention.