Integral equations, calculus of variations and their applications
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Lecture 31
Cauchy type integral equations-1

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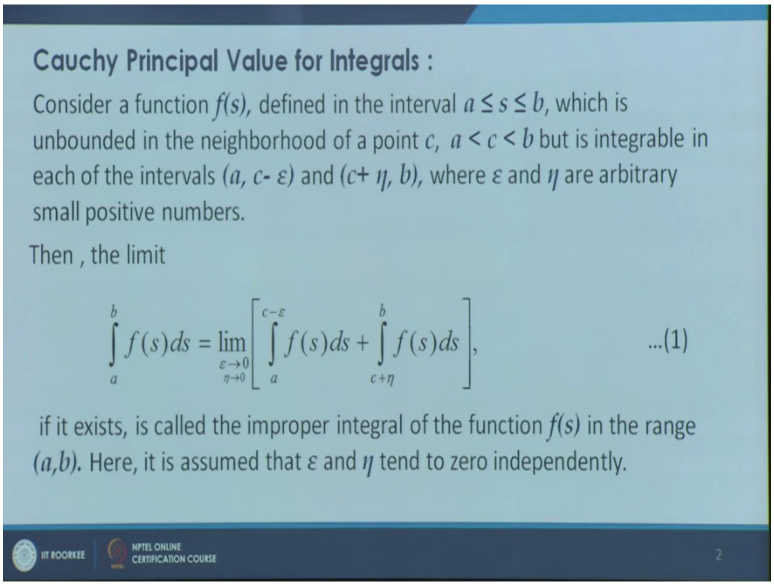

Hello friends welcome to my lecture on Cauchy type integral equations, we will have series of lectures on this topic. So this is first of the first of those lectures. Let us first define Cauchy principle value for integrals. Suppose we have a function Fs which is defined in the interval a less than or equal to s less than or equal to b and it is unbounded in the neighborhood of a point c which lies in between a and b.

And if we assume that this function f is integrable in each of the intervals a to c minus Epsilon and c plus eta to b where eta, Epsilon and eta are arbitrary small positive real numbers then the limit integral a to b fsds which is equal to limit Epsilon tends to 0, eta tends to 0 integral over a to c minus Epsilon fsds plus integral over c plus eta to b fs ds if it exists it is called the improper integral of the function fs in the range a,b.

Now here Epsilon and eta tends to 0 independent of each other. Now it may happen that the limit here the limit here may not exist. So if the limit does not exist then Epsilon and eta tends to 0 independently of each other and exists if Epsilon and eta related for example integral z to c minus Epsilon, suppose we take the function fs equals to 1 over s minus c where the real numbers c lies between a and b than the function fs equal to 1 over s minus c will become unbounded in neighborhood of c.

So if you want to integrate the function 1 over s minus c over the interval a to b then we will write it as integral a to c minus Epsilon ds over s minus c plus integral over c plus eta to b ds over s minus c and when you integrate this, what you will get is? ln b minus c over c minus a plus ln Epsilon over eta. If Epsilon and eta tends to 0 independently of each other than this value ln Epsilon over eta will also very.

So if you consider Epsilon and eta to be related than the preceding limit exist for example reconsidering the special case Epsilon equals to eta then when you take Epsilon equals to eta here then ln 1 is equal to 0. So integral a to c minus Epsilon ds over s minus c plus integral c plus eta to b ds over s minus c will be ln b minus c over c minus a as Epsilon goes to 0.

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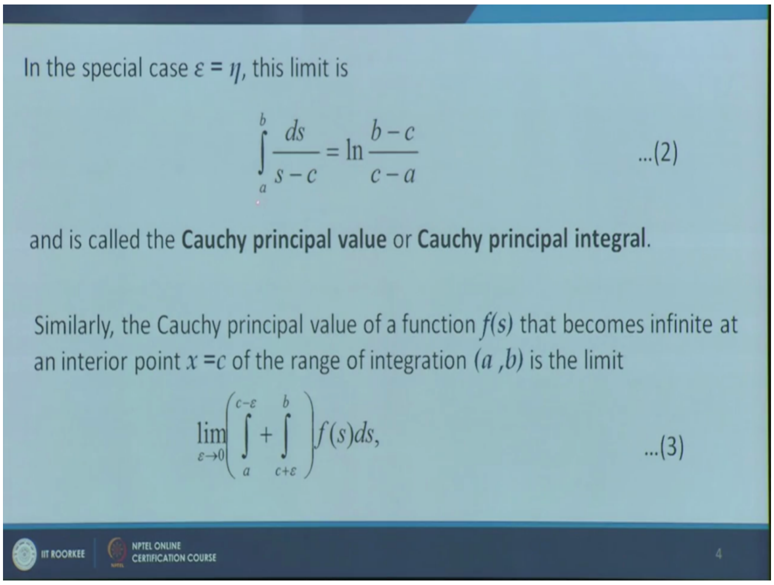

So integral a to b ds over s minus c will be ln b minus c over c minus a this value of the integral is called as the Cauchy principal value will value or we call it as a Cauchy principal integral. Similarly the Cauchy principal value of function fs which becomes in finite at an interior point x equals to c of the range of integration a,b is the limit. So in general we can define the Cauchy integral principle value of a function Fs s limit Epsilon tends to 0, integral a to c minus Epsilon plus integral c plus Epsilon to b fsds the clinch we take eta to be equal to Epsilon.

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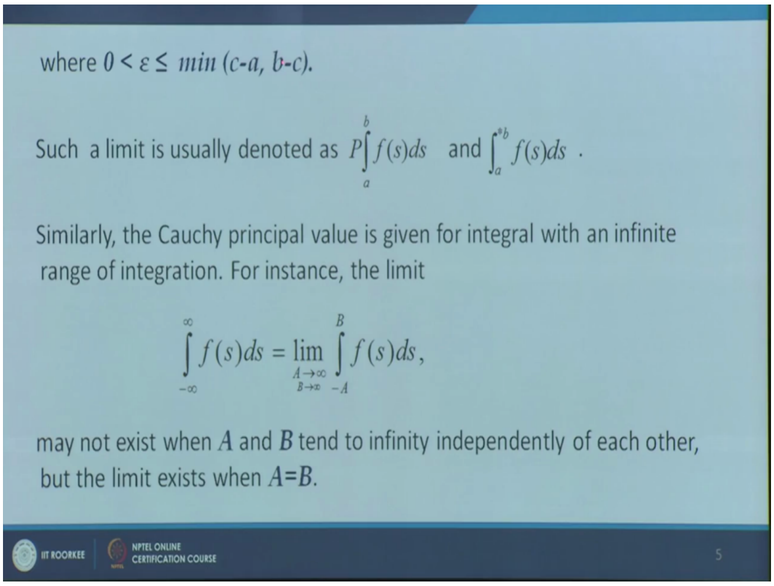

Then when this limit exists it is called as the Cauchy principal value of the function of the integral a to b fsds. Now here Epsilon is greater than 0 but less than or equal to minimum of c minus a, b minus c because we are considering a neighborhood of the point c, so Epsilon has to be like that. Now such a limit the notation for such a limit is the integral a to b fsds this P denotes the Cauchy principal value of the integral.

We also write it as integral a to star b fsds, so this star denotes the function fs is unbounded at some point and the open interval ab and we are considering the Cauchy principal value of the integral. Similarly if you have to consider, the improper integral where the limits of integration are in finite that because principal value is given for the integral within infinite range of integration like this.

Integral minus infinity to infinity fsds equals to limit A tension infinity B tension infinity minus A to B fsds. So this integral is defined like this, now these limits may not exist when A and B tend to infinity independently of each other but the limit exists when A is equal to B then we call it as the Cauchy principle value.

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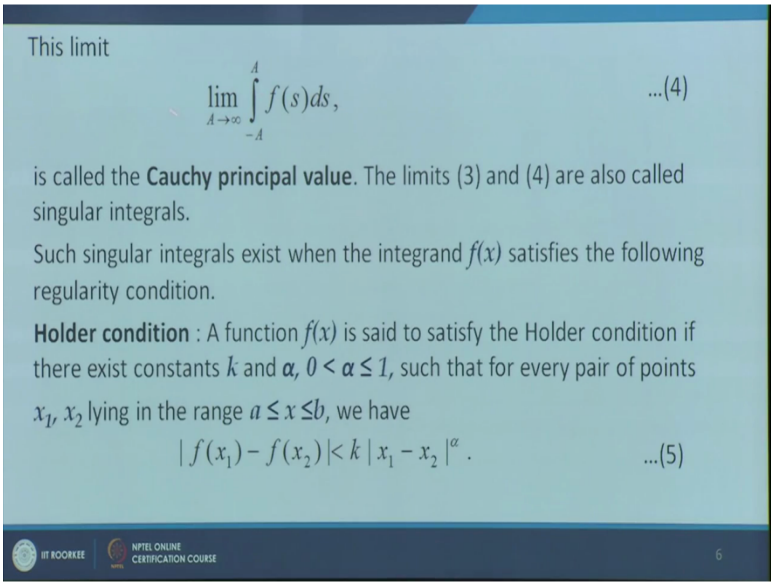

So if limit A tends to infinity minus A to A fsds is called the Cauchy principal value of the integral over minus infinity to infinity fsds. Now the limits 3 and 4 are also called singular integrals. So this is also called a singular integral, similarly the 3 this is also called this one as the singular integral.

Now such singular integrals exist when the integrand fx satisfies the, so there is a condition which the function fx has to satisfy that the such singular integrals exist. So the condition is the called the older condition. The function fx is set to satisfy the older condition if we can find constants k and Alpha where 0 is less than Alpha are less than or equal to 1 such that whenever you take any 2 points x1, x2 in the range a less than or equal to x less than or equal to b then fx1 minus fx2 is less than k times mode of x1 minus x2 to the power Alpha.

Now in the particular case when alpha equals to 1 this condition is also known as the Lipschitz condition. A Function which satisfies the older condition is called as holder continuous and when alpha is equal to 1 condition is called Lipschitz Condition and the furniture is called as the Lipschitz continuous function. The older condition can also be extended to the case of functions of more than 1 variable for example if you take the kernel Kxt in the integral equation then the kernel Kxt is older continuous with respect to both the variables x and t.

Is the can find constants k and Alpha the again 0 less than Alpha less than or equal to 1 such that mod of K x1t1 minus Kx2t2 is less than k times mod of x1 minus x2 to the power Alpha plus mod of t1 minus t2 to the power Alpha. Now where x1t1 and x2t2 are there in the range of integration of Kxt.

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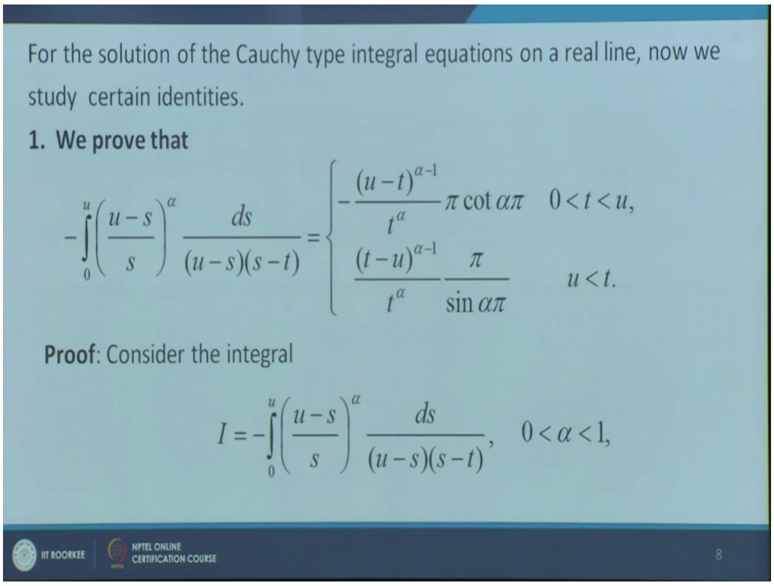

Now for the solution of the Cauchy integral equations which we are going to discuss now or little later. For the solution of Cauchy type integral equations on a real line, let us study some identities which we shall use in order to find a solution of Cauchy type integral equations. So the first identity is minus integral 0 to u, u minus s over s to the power Alpha ds upon u minus s into s minus t the value of this integral is minus u minus t to power alpha minus 1 over t to the power Alpha pi cot Alpha pi when 0 is less than t less than you and her remember Alpha is lying between 0 and 1, so 0 is less than alpha less than one and t minus u to the power Alpha minus 1, t to the power Alpha pi over sin alpha pi u less than t. So here the alpha is the condition on alpha is that 0 less than Alpha less than one.

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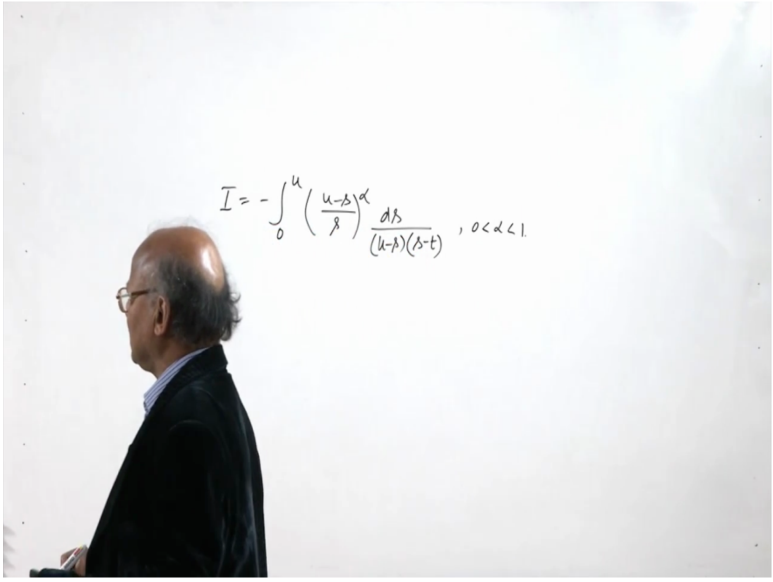

Now in order to proof this identity consider the integral I equal to minus 0 to u, minus 0 to u, u minus s divided by s raise to the power alpha ds upon u minus s into s minus t, okay. Now let us see how we transform this integral.

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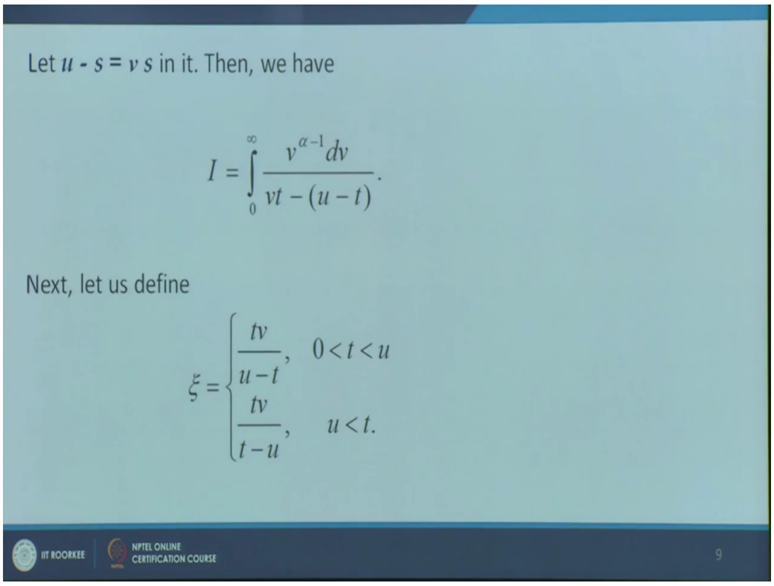

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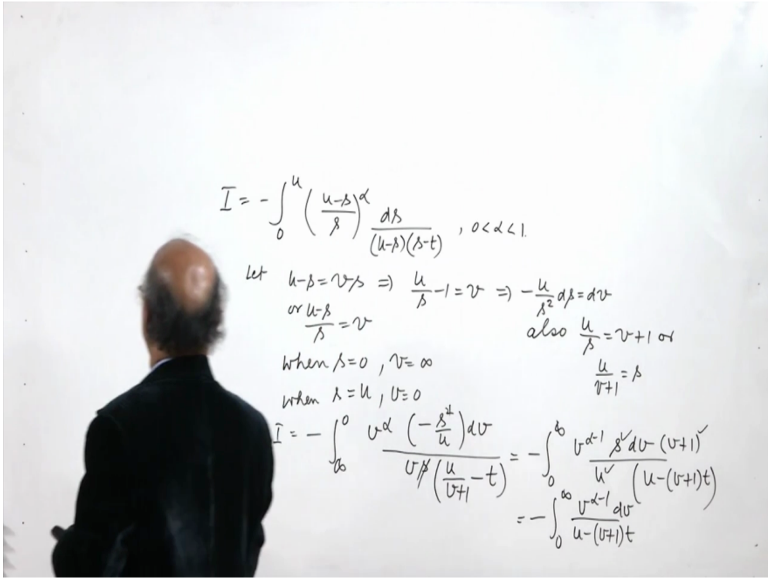

So let us put u minus equal to vs to identify this equality. So let us put let u minus s equal to vs when u minus s equals to vs, what we will get from here? Or u minus s over s is equal to v. So this will imply that u upon s minus 1 equal to b. So this will imply that minus u by s square ds is equals to dv and also the value of as we can find. So also u by s equals to b plus 1, u by s is equal to b plus 1, r we can say u upon b plus 1 is equal to s.

So let us make the substitution in the above integral, we can see here that when s equals to 0 when s equals to 0, v equals to infinity and when s equals to u v equals to 0. So when s equals to 0, v becomes infinity and when s equals to 0, when s equals to u, b equals to 0.

So I will be equal to minus integral infinity to 0, u minus s by s is equal to b. So b to the power Alpha ds is equal to minus s square by udv and then u minus s is equal to vs and s minus te u minus s equals to vs we can also write s minus t equals to u upon v plus 1 s is equal to u upon b plus 1 minus t, okay.

So what do we get? 0 to infinity, now v to the power Alpha upon v gives you v to the power Alpha minus 1, okay. And one s will cancel with one s here, so we will get sdv by u into v and we will put a negative sign here. So what we get is, v to the power Alpha minus 1 then we get sdv divided by u, v we have already cancelled, so we have only u here v we have already cancelled.

So we have only u here and then we get u minus b plus 1 into t by u, so this is equal to v plus 1 divided by u and what we have here? We should get v to the power Alpha minus 1 dv s upon u, okay. Yeah v plus 1 is equal to u upon s, okay. So yeah, so s upon u into v plus 1, okay. s upon u into v plus 1 is equal to 1, so this is equal to minus integral 0 to infinity e, v to the power Alpha minus 1 divided by u minus v plus 1 into t, dv.

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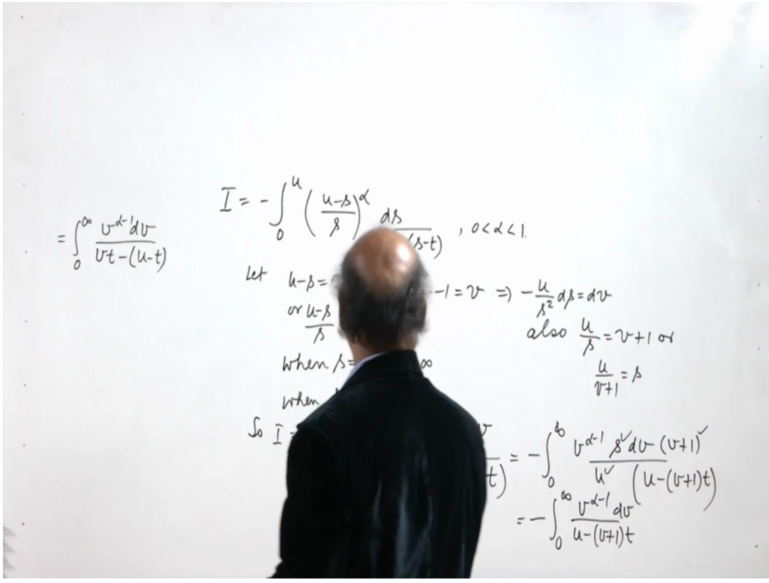

Or we can say that it is equal to integral 0 to infinity v to the power Alpha minus 1dv divided by vt minus u minus t. So this is what we get by making the substitution we arrive at I equals to, this is integral I, so I equals to integral 0 to infinity v to power Alpha minus dv upon vt minus u minus t.

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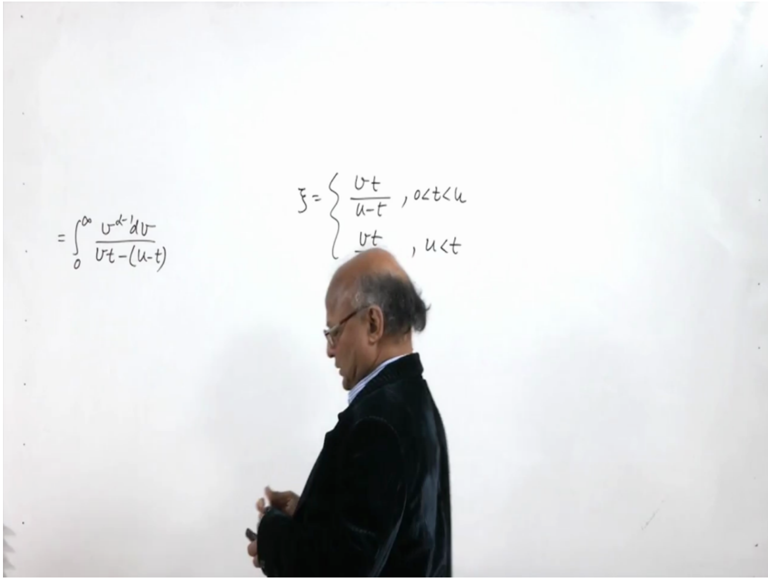

Now let us define Xi like this then, so define Xi equal to vt, vt upon u minus t incase 0 less than t less than u and vt upon t minus u in the case u less than t, okay. Let us see, so I this is I now, so let us see when we get the substitution, what do we get?

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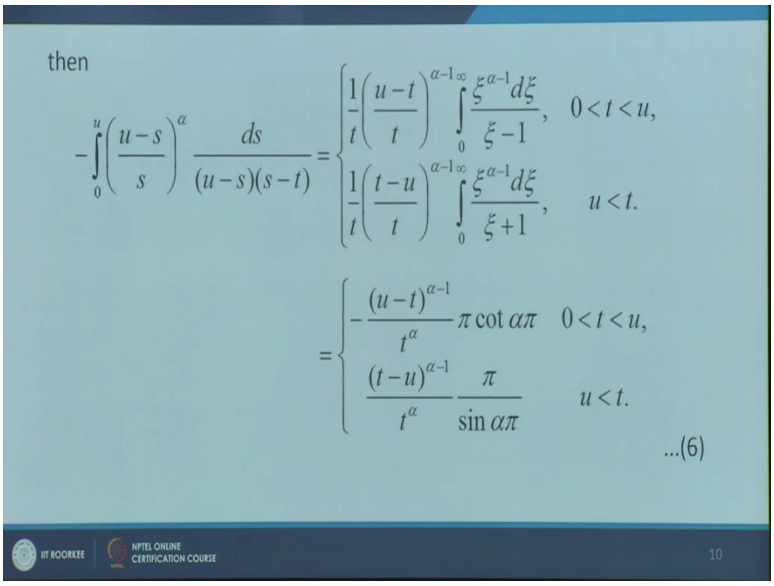

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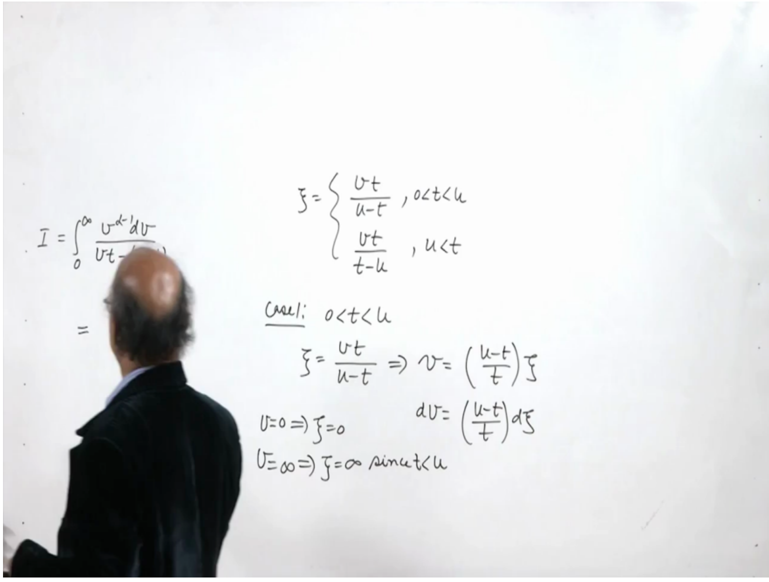

So then this I becomes, okay. So let us say when we take 0 less than t, okay. Let's take the case 1, 0 less than t less than u, let us take this case. So in this case we are making the substitution Xi equal to vt upon u minus t. So this gives you what? v is equal to u minus t upon t into Xi and dv equals to u minus t upon t dXi. Now here what happens? When v is equals to 0, Xi equals to 0 when v equals to infinity Xi will be infinity because t is less than u. So v equals to infinity gives us, sin ut is less than u.

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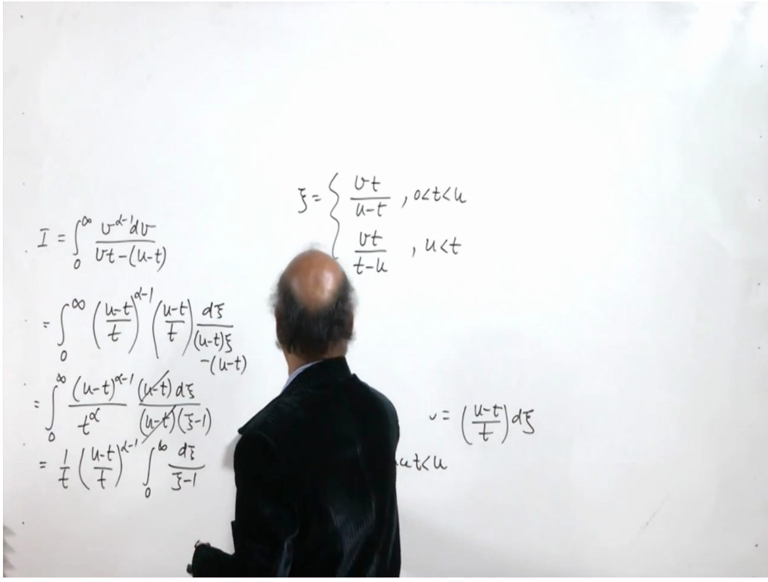

So the limits of integration will remain 0 infinity and what we get? v equals to u minus t by t. So u minus t divided by t raise to the power Alpha minus 1, dv will be u minus t over t, dXi divided by vt, vt is u minus t into Xi then minus u minus t. So this is equal to integral u minus t raise to the power alpha minus 1 divided by, and here u minus t dXi divided by u minus t into Xi minus 1 we get, so 0 to infinity.

So this will cancel and we will write it as 1 by t, u minus t upon t raise to the power Alpha minus 1, 0 to infinity dXi upon Xi minus 1. Similarly and this I, I is nothing but this integral minus integral 0 to u, u minus s by s to the power Alpha ds over u minus s into s minus t. Similarly we can show when t is greater than u?

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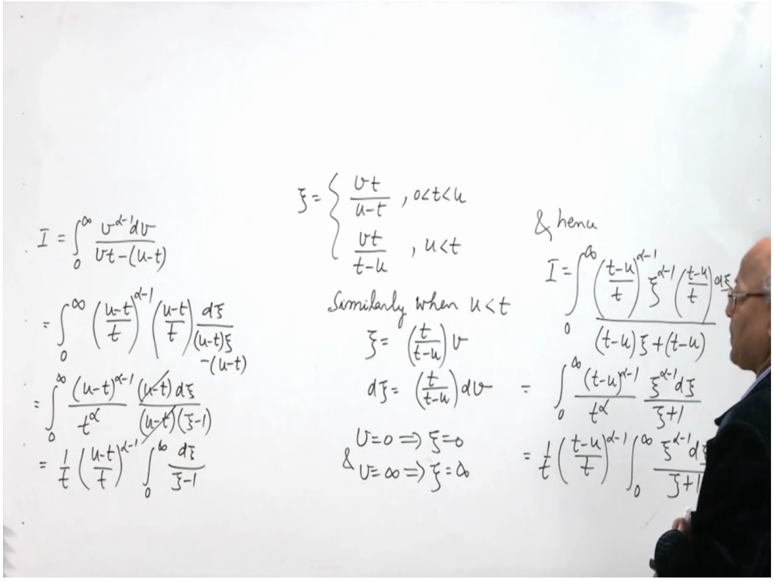

Similarly when u is less than t, now when u is less than t, Xi we are writing as t upon t minus u into b, so dXi equals to t upon t minus u dv. Further Xi equals to 0, so we are writing the value of I now for the case when u is less than t. So Xi is equals to 0 gives you v equal to 0 and Xi equals to infinity, now when u is less than t, t upon t minus u is a positive quantity, so Xi equals infinity gives you v equal to infinity, sorry v we equals to nor we have to write it like this.

v equal to 0 gives you, v equal to 0 gives you Xi equal to 0 and v equal to infinity gives you Xi equal to infinity and hence I becomes integral 0 to infinity, v to the power Alpha minus 1, so v the power Alpha minus 1 makes it t minus u by t raise to the power Alpha minus 1, Xi to the power Alpha minus 1, v is equal to t minus u by t into Xi then dv, dv is equal to t minus u by t, dxi.

dxi divided by vt, vt is equal to t minus u into Xi minus u minus t that can be written as plus t minus u. So this is equal to we can write it as, now okay. So this is integral 0 to infinity when t minus u will cancel when t minus u here, so t minus u to the power Alpha minus 1 divided by t to the power alpha and dxi, Xi to the power alpha minus 1 into dXi divided by Xi plus we get. So this can also be written as 1 by t, t minus u upon t raise to the power Alpha minus 1 integral 0 to infinity xi to the power alpha minus 1 dXi upon Xi plus 1.

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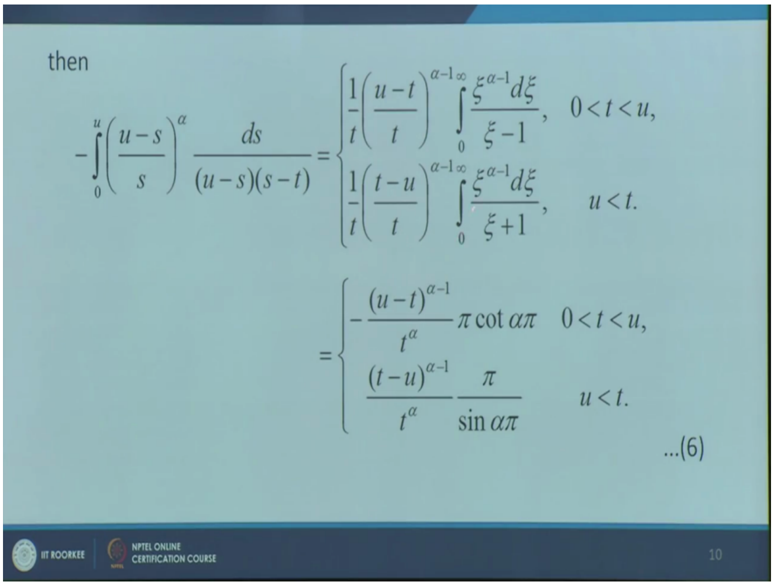

So we get this I, the expression for I which is minus integral 0 to u, u minus s by s to the power alpha, ds over u minus into s minus t than 0 it as this integral 0 less than u less than u and can be written like this when u is less than t.

Now these integrals can be further simplified and we can get the value of the integrals and we can get that for the case 0 less than t less than u the value of the this I, this value of this integral I is minus u minus t raise to the power Alpha minus 1 t to the power alpha pi cot alpha pi and t minus u the power Alpha minus 1 over t to the power alpha, pi sine alpha pi when u is less than t.

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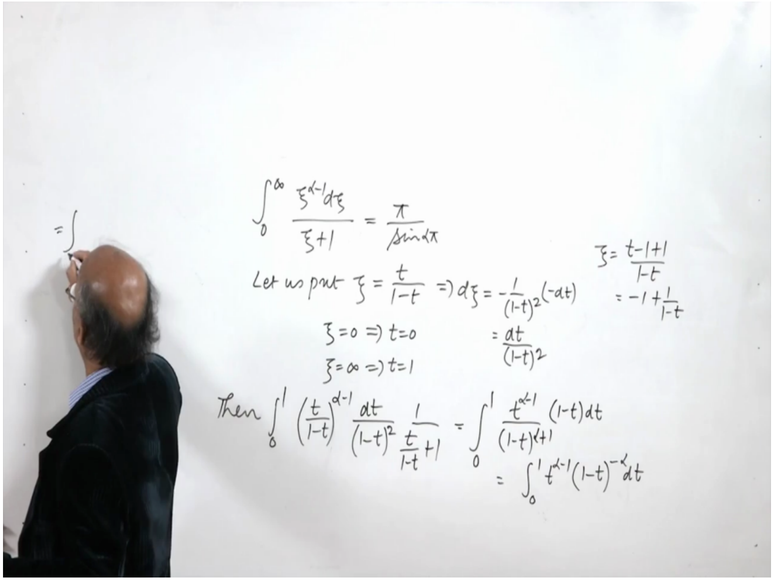

Now let us first find the value of the integral 0 to Xi, Xi to the power Alpha minus 1 dXi over Xi plus 1 and show that this is equal to pi over sine alpha pi. So integral 0 to Xi, this is equal to pi over sine alpha pi.

“Professor-student conversation starts”

Sir 0 to infinity.

“Professor-student conversation ends”

0 to infinity, it is 0 to infinity, okay. So what we do here is that we will convert it into beta function, so let us put Xi equal to t upon 1 minus t when we put Xi equal to t upon 1 minus t, Xi equal to 0 this t equal to 0 and Xi equal to infinity gives t equal to 1, okay. So let us call , okay so let us write integral 0 to infinity then integral 0 to infinity Xi to the power Alpha minus 1, so t upon 1 minus t raise to the power Alpha minus 1, we have.

And this gives you dXi equals to, we can write it as Xi equal to t minus 1 plus 1 divided by 1 minus t which is minus 1 plus 1 upon 1 minus t. So dXi, dXi will be equal to minus 1 upon 1 minus t whole square into minus dt, so this is dt upon 1 minus t whole square. So we can put the value here, so t upon 1 minus t raise to power Alpha minus 1 for Xid power Alpha minus 1 for dXi we write dt upon t minus 1 minus t whole square and 1 over Xi plus 1, so 1 over Xi plus 1 will be 1 over t upon 1 minus t plus 1.

And what we get is, this is integral 0 to infinity t to the power alpha minus 1 upon 1 minus t raise to the power Alpha minus 1 plus 2, so alpha plus 1 and here we get t plus 1 minus t which is equal to 1, so this is one minus t here. So this is 0 to 1 and here also it is 0 to 1 because the limits of integration for t are 0 to 1. So this is equal to integral 0 to 1 t raise to the power Alpha minus 1, 1 minus t raise to the power minus alpha dt.

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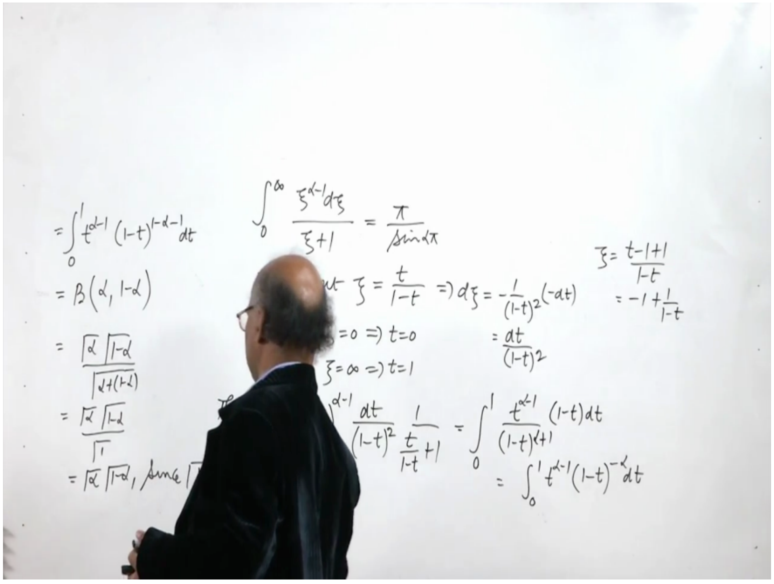

We can also write that as in the form of beta function P to the power Alpha minus 1, 1 minus t raise to the power 1 minus Alpha minus1dt. So this is equals to beta function, alpha beta (()) (26:43) that is beta, alpha 1 minus Alpha. Now let us express it in terms of gamma functions. So gamma alpha, gamma 1 minus Alpha divided by gamma alpha plus 1 minus Alpha which is equal to gamma alpha, gamma 1 minus Alpha divided by gamma 1. Now by the properties of gamma function gamma 1 is equal to one. So we get gamma alpha gamma 1 minus Alpha. Since gamma 1 is equal to one, so we get gamma alpha gamma 1 minus Alpha, since gamma 1 is equal to one

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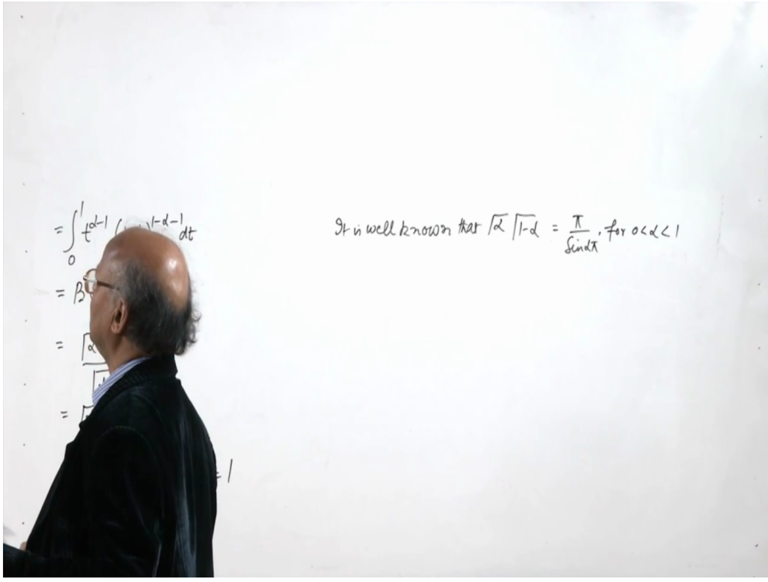

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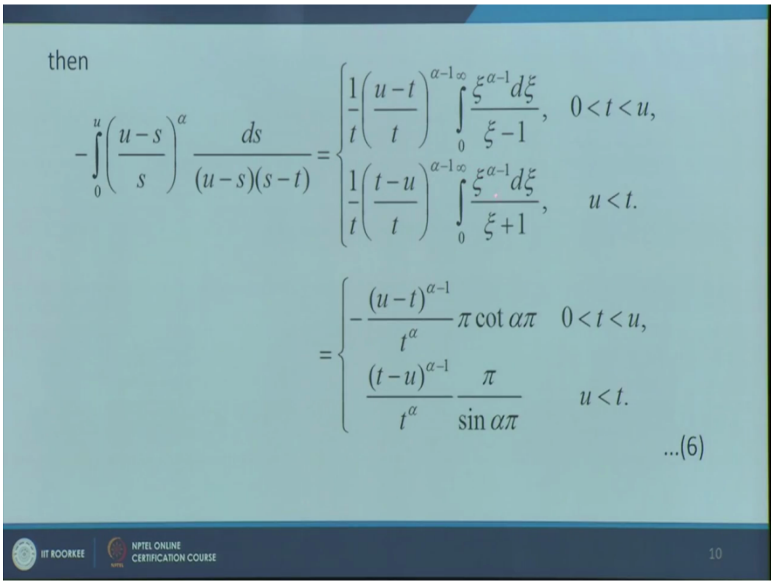

Now it is well-known that, it is well-known that gamma alpha, gamma 1 minus Alpha is equal to pi over sine alpha pi and for 0 less than alpha less than one. So integral 0 to infinity Xi to power Alpha minus 1 dXi upon Xi plus 1 is pi over sine alpha pi. Similarly we can find the integral 0 to infinity Xi to power Alpha minus 1 dXi upon Xi minus1 and can get pi cot alpha pi.

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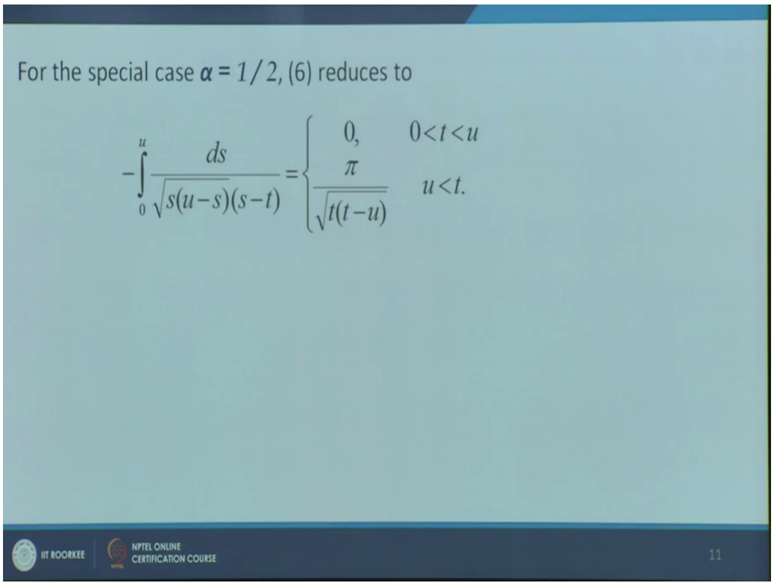

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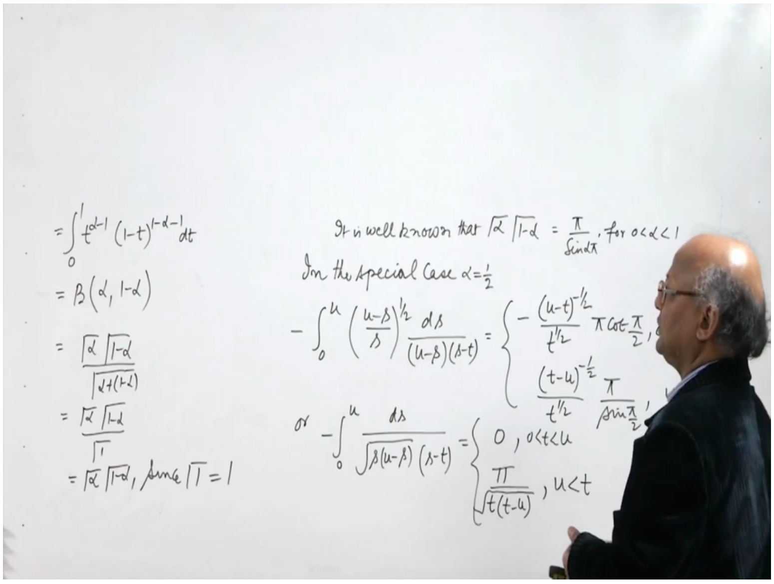

Now when we take the special case alpha equal to half here, so when we take alpha equal to half, what we will get? So in the special case alpha equal to half minus integral 0 to u, u minus s by s to the power half ds upon u minus s into s minus t this is equal to minus u minus t raise to the power minus half upon t to the power half or pi cot pi by 2 in case 0 less then t less than u and this is t minus u to the power minus half divided by t to the power half pi over sine pi by 2 when t is greater than u, or u is less than t.

Now this can be expressed in an alternative way, like this. We can also write it as minus integral 0 to u, u minus s to the power half will cancel with this one and will get ds upon square root s into u minus s into s minus t equal to, now cot pi by 2 is cos pi by 2 over sine pi by 2. So cos pi by 2 is 0, so this is 0. So 0 when 0 is less then t less than u and pi over sine pi by 2, this is pi because sine pi by 2 is one.

So t minus u, so this can be written as 1 over t times t minus u into pi, so Pi over square root t into t minus u when u is less than t. So this is the case when alpha is equal to half. The identity over identity reduces to this value. So we will make use of this identity when we discuss the solution of caution integral equation on a real land.

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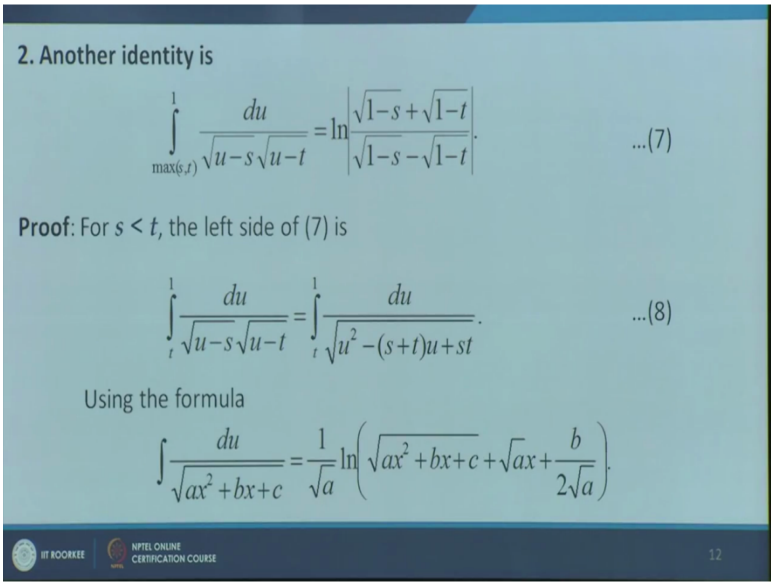

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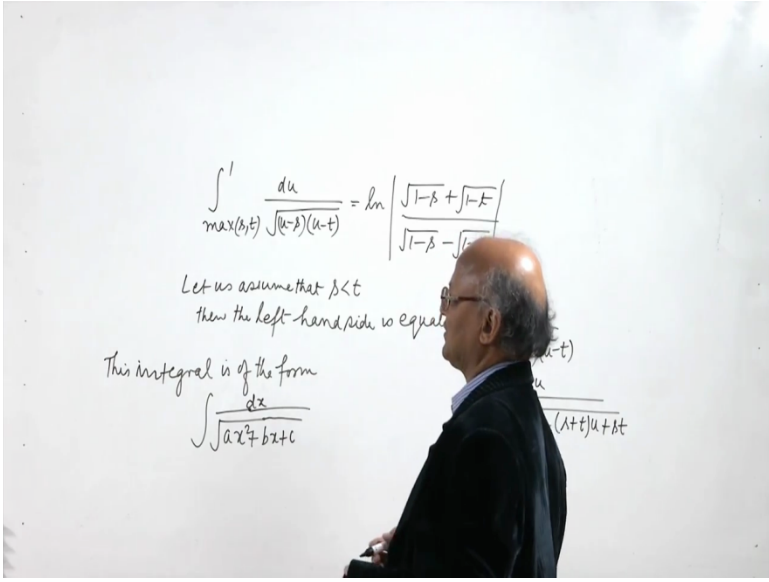

Now another identity let us discuss, so we are going to integrate du upon square root u minus s, u minus t over the interval maximum of st to 1 and we will show that it is equal to ln mod of square root 1 minus s plus square root 1 minus t over square root 1 minus s minus square root 1 minus t. So integral du upon square root u minus s into u minus t we want to prove that it is ln mod of square root 1 minus s plus square root 1 minus t divided by square root 1 minus s minus square root 1 minus t.

So there are 2 cases, one case is when s is less then t and the other case is when s is greater than t. So let us first prove of s less than t. So let us assume that s is less then t, so then the left inside is equal to integral over t to 1 du over square root u minus s into u minus t which we can also write as integral t to1 du divided by square root u square minus s plus t into u plus st.

Now this integral is of the form, this integral is of the form integral one over square root ax square plus bx plus c dx upon square root ax square plus bx plus c and we have the formula for the integral of 1 over square root ax square plus bx plus c which is given by 1 by root a ln of square root ax square plus bx plus c plus square root a into x plus b upon 2 square root to a. Here we have left arbitrary constant, a constant term here because that will not be used when you put it the limits of integration because it is a finite integral. So in the case of a finite integral the constant term that comes here we can ignore.

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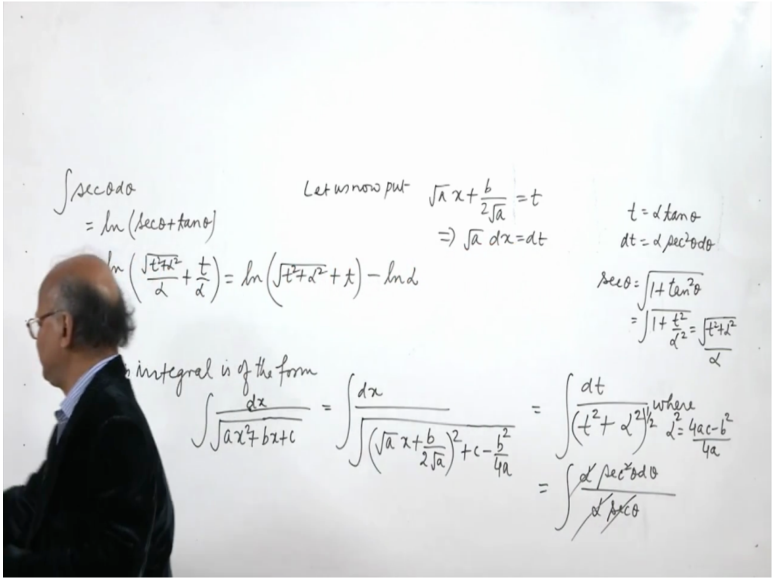

So let us see how we get this integral? So this is equal to the can write it as dx upon square root, root a into x plus b by 2root a whole square and then we have c minus b square by 4a, when you square this you get ax square then you get twice root a into x plus b upon 2root a, so we will get bx and then we have square of this which is b square by 4a, b square by 4a we have subtracted here. So we can write ax square plus bx plus c like this and let us now put square root a into x plus b upon 2 square root a equal to t which will imply that root a dx equals to dt. So we will have here integral dt upon t square plus Alpha square where alpha is equal to 4ac minus b square divided by 4a, alpha square is this.

So now we have to integrate raised to the power half, so we have to integrate 1 over square root t square plus Alpha Square as what we can do is, let us put t equals 2 alpha ten theta. So this will become alpha sec Theta d Theta divided by dt equal to alpha sec Theta d Theta and the denominator it will become alpha square sec Theta square root, so alpha sec theta.

So this will cancel with this when sec Theta will cancel and integral of sec theta d theta is ln sec theta plus tan theta, we are not writing constant of integration because we are going to evaluate this definite integral. In the definite integral the constant is not required. So we have this ln, now sec theta is what? Sec Theta is equal to square root 1 plus tan square Theta and tan theta is t by alpha, so this is one plus t square by alpha square. So this is under root p square plus Alpha Square divided by alpha. So this is under root t square plus Alpha Square divided by alpha and tan theta is also equal to t by alpha. So this is equal to ln under root t square plus Alpha square plus t minus ln alpha.

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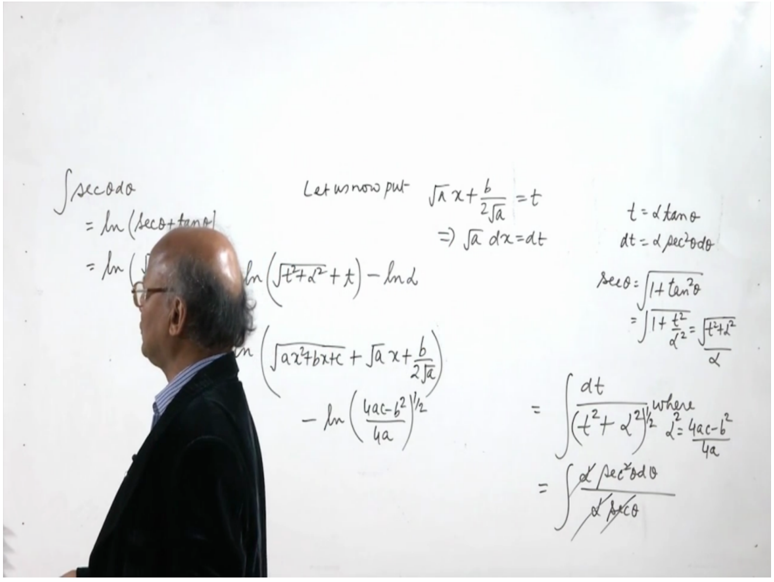

Let us substitute the value of t here, if we substitute the value of t, what we will get? So t square plus Alpha Square, t square plus Alpha Square is nothing but ax square plus bx plus x. So this is ax square plus bx plus c plus t, t is t is equal to root a into x plus b over 2root a, this is expression for ln under root t square plus a square plus t minus ln alpha. Alpha is alpha is square root of 4ac minus, 4ac minus b square upon 4a raised to the power half.

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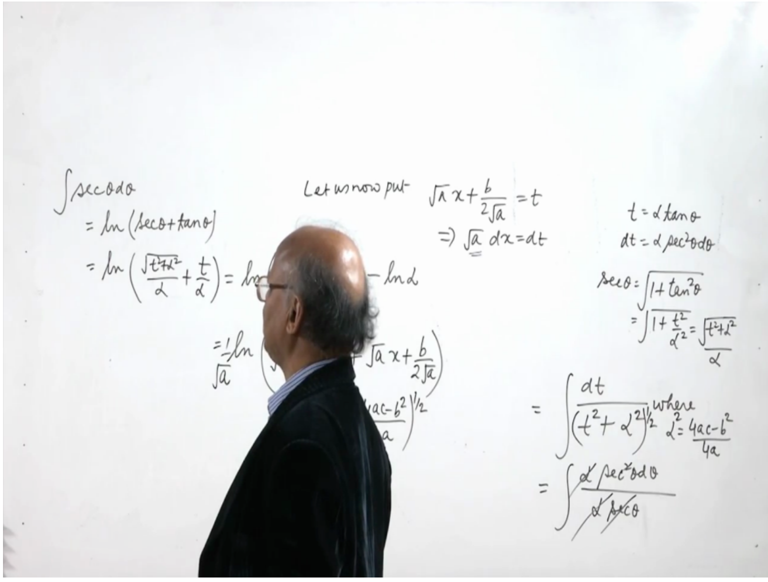

Now we need to use this integral here to compute this definite integral. So we are not considering this constant term, we are simply taking, okay 1 over root a is also there, this root a, we have missed here. So 1 over root a here, we have missed all along. So one over root a ln under root ax square plus bx plus c plus root ax 1 over b over 2 root a.

We are simply considering this part 1 over root a will be multiplied here also. So we are not considering this particular which is of constant because it will get cancelled when we calculate this definite integral.

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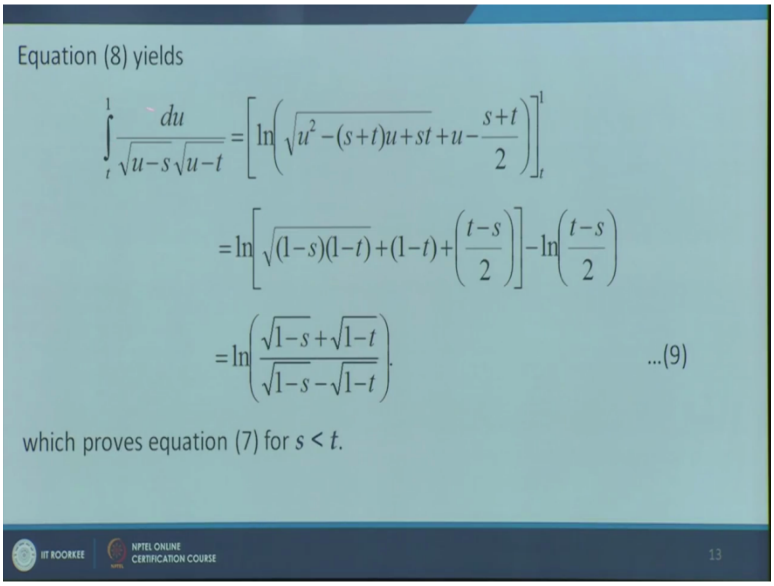

Now we use this standard formula to evaluate this definite integral and then the integral t to 1 du upon square root u minus s square root u minus t will be equal to ln under root u square minus s plus t into u plus st plus u minus s plus t by 2. Now let us put the limits, upper limit is 1, lower limit as t then we shall be getting this ln under root 1 minus s into 1 minus t plus 1 minus t plus t minus s by 2 minus ln t minus s by 2 and which we can easily show because ln under root, this is nothing but let us see.

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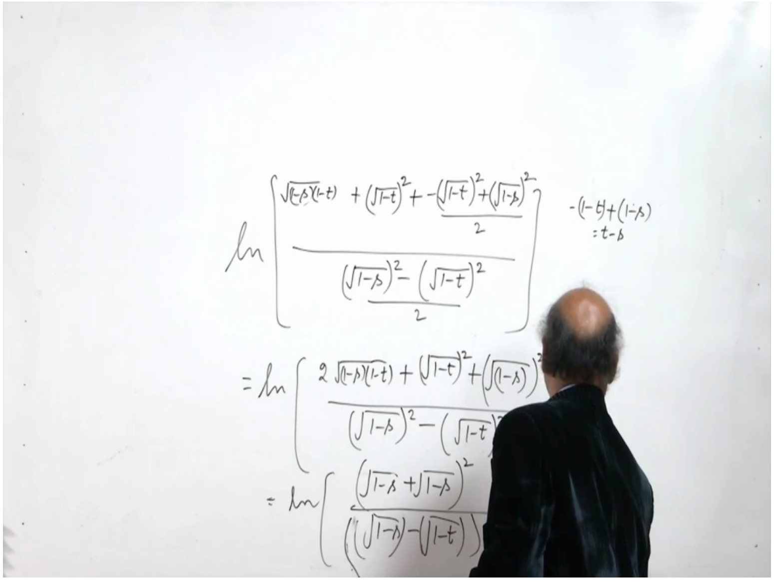

How we get this expression? This one is nothing but ln under root 1 minus s into 1 minus t plus 1 minus t which we can be written as 1 minus t whole square under root 1 minus t whole square plus t minus s by 2. So t minus s by 2 we can write as under root 1 minus t whole square minus under root 1 minus s whole square divided by 2, this will be giving 1 minus t minus, 1 minus t minus 1 minus s, so this is s minus t so I can write like this, so that it is t minus s.

So t minus s can be written like this, this divided by t minus s by 2. So t minus s by 2 again can be written as like this under root 1 minus s whole square, we can write like this. So ln of this quantity and this is equal to ln of , here we can write it as 2 times square root 1 minus s into 1 minus t and this is under root 1 minus t whole square from this, we are subtracting under root 1 minus t whole square by 2. So we will get plus sign, so we will have under root 1 minus t whole square plus under root 1 minus s whole square.

And then in the denominator we will have under root 1 minus s whole square minus under root 1 minus t whole square and this, now this is what? ln under root 1 minus t plus under root 1 minus s minus under root 1 minus t plus whole square denominator can be written like this and we have under root 1 minus s minus greatest factor in the denominator.

So this cancels with numerator this one and we get ln under root 1 minus s plus 1 minus t here ln under root 1 minus s plus under root 1 minus t upon under root 1 minus s minus under root 1 minus t. Now which proves, this the result the identity for s less than t. Similarly when we take s greater than t we can show that it is integral maximum of s and t will be there s, so integral s du upon under root u minus s under root u minus t is equal to ln under root 1 minus t plus under root 1 minus over under root 1 minus t minus under root 1 minus s.

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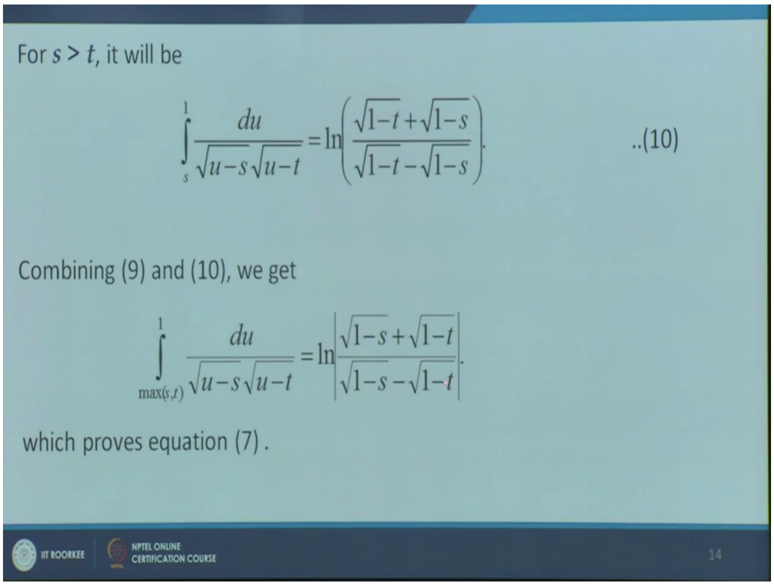

Now so we can combine equality and this equality and write integral over maximum st to 1 du over under root u minus s u minus t to ln pf under 1 minus s plus under root 1 minus t upon under root 1 minus s minus under root 1 minus t because when s is less then t then under root 1 minus s will be more than under root 1 minus t. So this model will disappear and when s is greater than t then under root 1 minus s will be less than under root 1 minus t. So this will be changed to under 1 minus t minus under root 1 minus s and we will get this.

So this proves the second identity, so with this I would like to conclude my lecture, thank you very much for your attention.