Integral Equations, Calculus of Variations and their Applications
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Lecture 30
Singular Integral Equation II

Hello friends, I welcome you to my second lecture on singular integral equations. So here we shall consider the general form of the form of Abel singular integral equation. The general form of the Abel singular integral equation is given by f x equal to integral a to x, y t d t over h x minus h t raise to the power alpha and where alpha lies between 0 and 1. In the previous lecture we had considered the Abel integral equation to be f x equal to integral 0 to x, y t d t over x minus t raise to the power alpha.

So here this is the general form where h t is a strictly monotonically increasing and differentiable function in the open interval a b and that h prime t is not equal to 0 in the open interval a b. So when you take h t equal to t here then h t is more strictly monotonically increasing and also differentiable and h prime t is equals to 1, so all the conditions here are satisfied. So when you take h t equal to t, this form reduces to the Abel singular integral equation and therefore we call it as the general form of the Abel singular integral equation.

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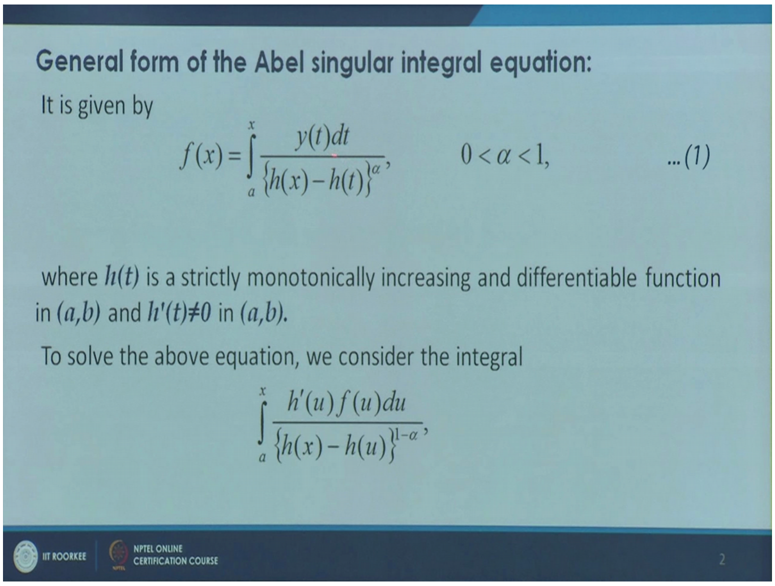

So we shall see how we can solve this general form of the Abel singular integral equation? What we do is to solve the above equation let us consider this integral. Integral a to x, h prime u, f u d u over h x minus h u to the power 1 minus alpha and substitute for f u from the equation 2. So in this integral the value of f u is substituted.

So integral a to x, h prime u into f u d u divided by h x minus h u in power 1 minus alpha will become, the value of f u will be integral a to x, y t d t divided by h x minus h u minus h t raise to the power alpha d u. So let us substitute the value of f u from the given integral equation. In this integral a to x, h prime u, f u d u by h x minus h u to the power 1 minus alpha. Then we get this integral.

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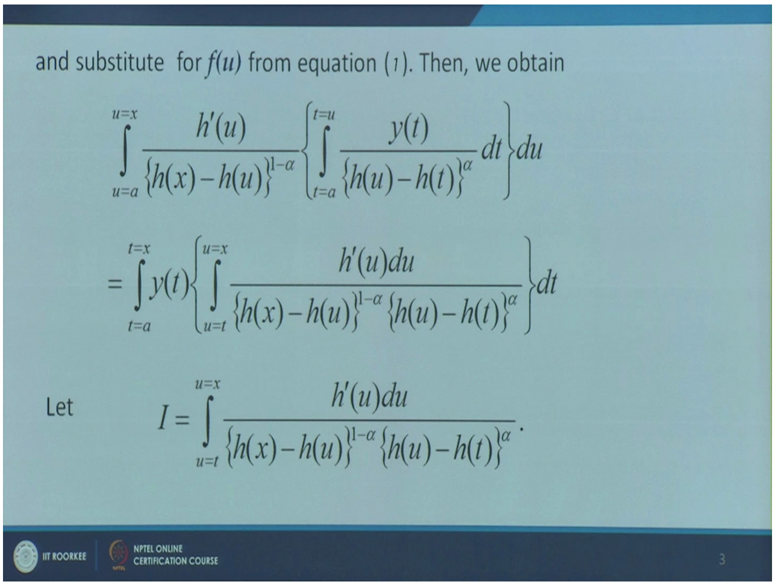

Integral the limits of integration for u are from u equal to a to u equal to x, h prime u over h x minus h u raise to the power 1 minus alpha. And the integration limits for t are t equal to here it should be u because we have replaced by x by u. So t varies from a to u and then y t over h u minus h t raise to the power alpha d t d u. Now let us check the order of integration here. So what we will do? A is less than or equal to t less than or equal to u, a t is equal to a here and then u varies from a and u goes up to x.

So u equal to a here and u equal to x here. So what we have? T varies from a to t equal to u. On this line t is equal a, so on this line t varies from a to u, okay. And u varies from a to x. So I think this is the figure. So this is we have this region. Now let us take a vertical (la) strip here. So when I take a vertical strip then for the vertical strip u will vary from t to u equal to x.

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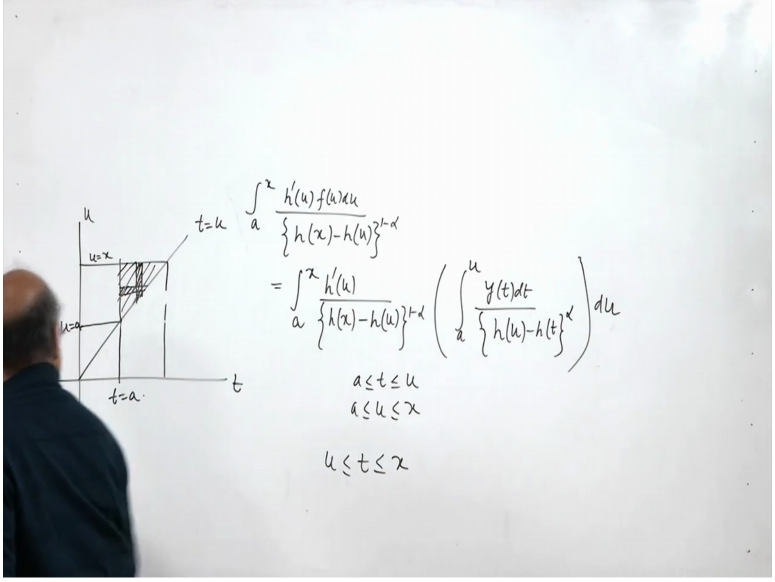

So now u varies from u varies from t to x and (vari) t varies from, because t is a here. So t varies from a to and here it is x, so a to x.

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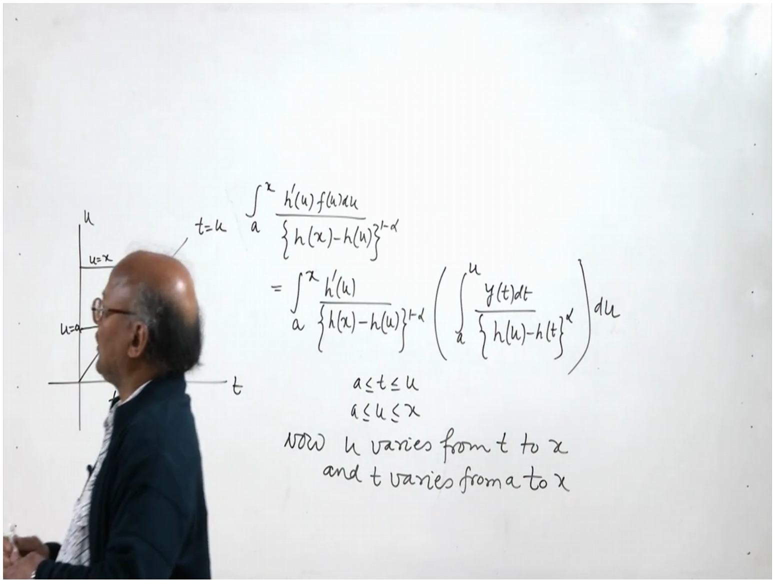

And then what we have is this integral where t varies from a to x y t, u varies from t to x. So h prime u d u divided by h x minus h u to the power 1 minus alpha, h u minus h t to the power alpha. So this is the correct figure where t we are taking on the horizontal axis, u on the vertical axis and the region shaded is the region over which we are integrating. So let us not take i to be equal to integral u equal to t to u equal to x, h prime u d u over h x minus h u to the power 1 minus alpha, h u minus h t to the power alpha.

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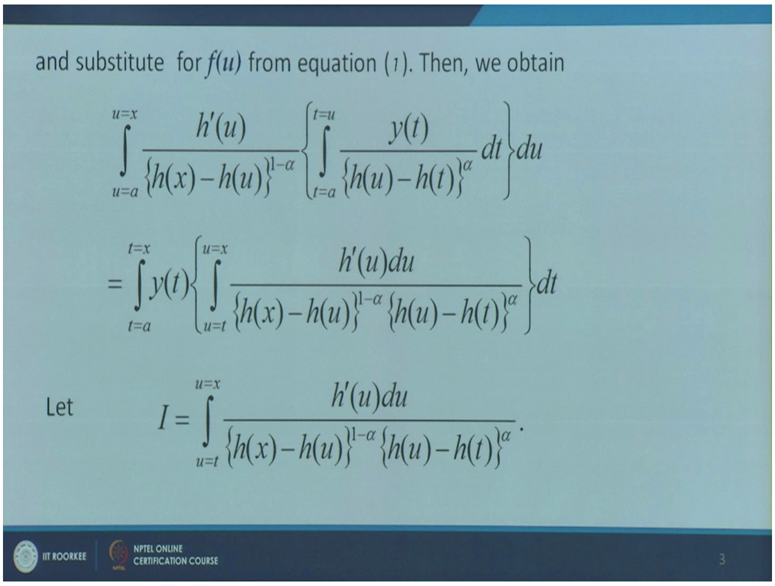

And then let us put h x minus h u over h x minus h t equal to s. Then from this substitution what we will get? H u will be equal to minus h x minus h t into s, so that h prime u d u is equal to minus h x minus h t into d s. Now by making the substitution what we will get is, so i will become i is equal to u equal to t to u equal to x and then we have h prime u d u divided by h x minus h u to the power 1 minus alpha and h u minus h t to the power alpha.

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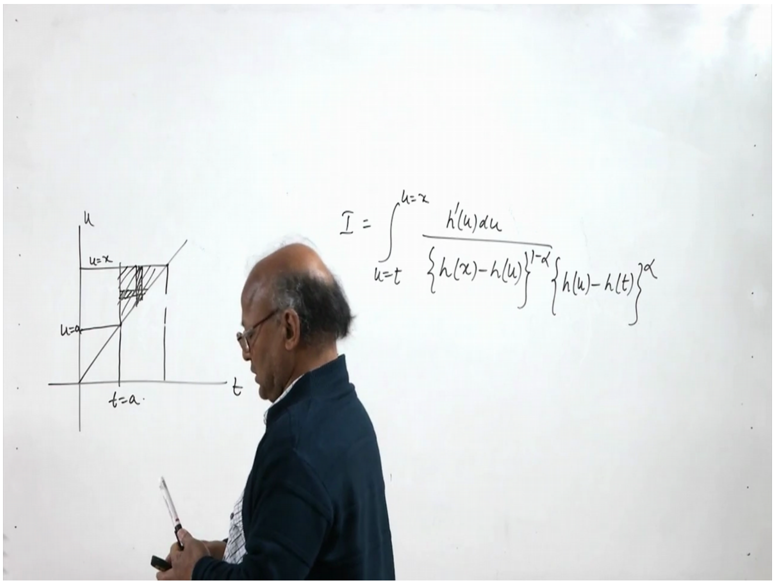

Now let us make the substitution. H x minus h u divided by h x minus h t equal to s, so that h prime u d u is equal to minus h x minus h t into d s. Now with this substitution i becomes i equal to when u is equal to t. So we will have h x minus h t divided by h x minus h t. So lower limit will be 1 and when u becomes equal to x, h x minus h x will give you 0. Then h prime u d u is minus h x minus h t d s divided by h x minus h u is s to the power 1 minus alpha, h x minus h t to the power 1 minus alpha.

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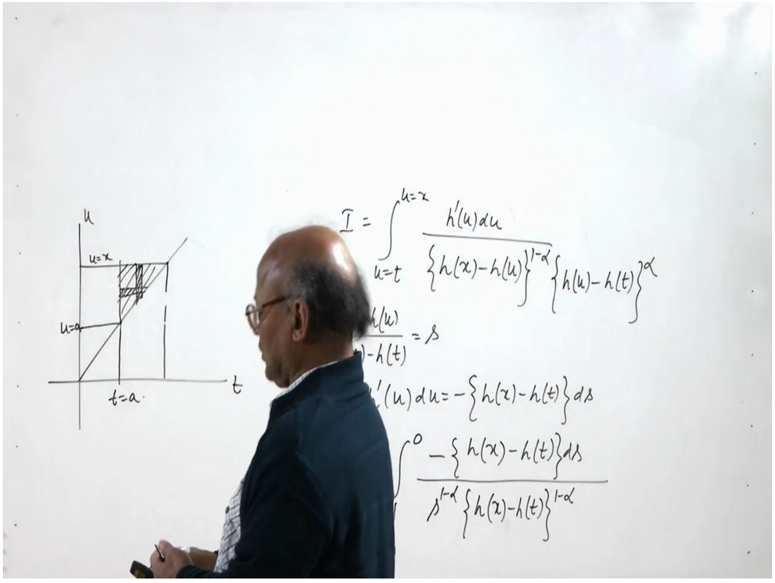

And h u is equal to h x minus h x minus h t into s. So h u minus h t will become h x minus h t into 1 minus s, so we will have h x minus h t into 1 minus s raise to the power alpha. What will happen is then this will be equal to so i will be equal to integral 0 to 1.

Now h x minus h t to the power 1 minus alpha and h x minus h t to the power alpha will cancel with h x minus h t and we shall have d s divided by s to the power 1 minus alpha and 1 minus s to the power alpha which can be written as 0 to 1, s to the power alpha minus 1, 1 minus s to the power minus alpha which is nothing but integral 0 to 1 s to the power alpha minus 1, 1 minus s to the power 1 minus alpha minus 1 d s. So we get it as beta function alpha 1 minus alpha.

And this is gamma alpha gamma 1 minus alpha divided by gamma 1. Gamma 1 is equal to 1, so it is gamma alpha gamma 1 minus alpha. But 0 is less than alpha less than 1 so this is sin alpha pie divided by pie.

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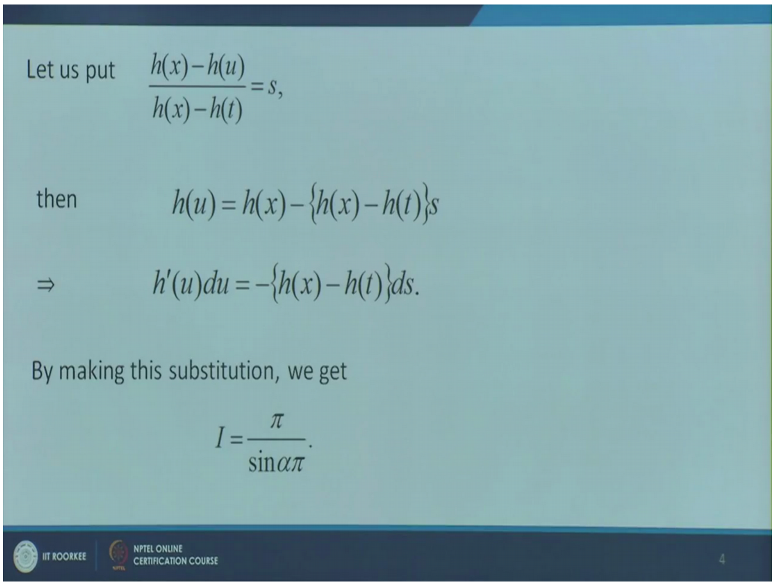

So we get the value of i as pie over sin alpha pie. This is pie over sin alpha pie. And so thus we have this integral a to x, h prime u f u d u over h x minus h u to the power 1 minus alpha as integral over t equal to a to t equal to x, y t pie over sin alpha pie d t. So integral over a to x, y t d t is equal to sin alpha pie over pie integral a to x, h prime u f u d u over h x minus h u to the power 1 minus alpha.

Now we will differentiate both sides with respect to x by the Leibniz rule and when we differentiate the integral over a to x, y t d t with respect to x by the Leibniz rule then we get y x here. So y x equal to sin alpha pie over pie, d over d x of the integral a to x, h prime u f u d u over h x minus h u to the power 1 minus alpha. We do not apply the Leibniz rule inside the integral here because it will lead us to the divergent integral.

So y t is equal to replacing x by t we get y t equal to sin alpha pie over pie, d over d t of integral a to t, h prime u f u d u over h t minus h u to the power 1 minus alpha.

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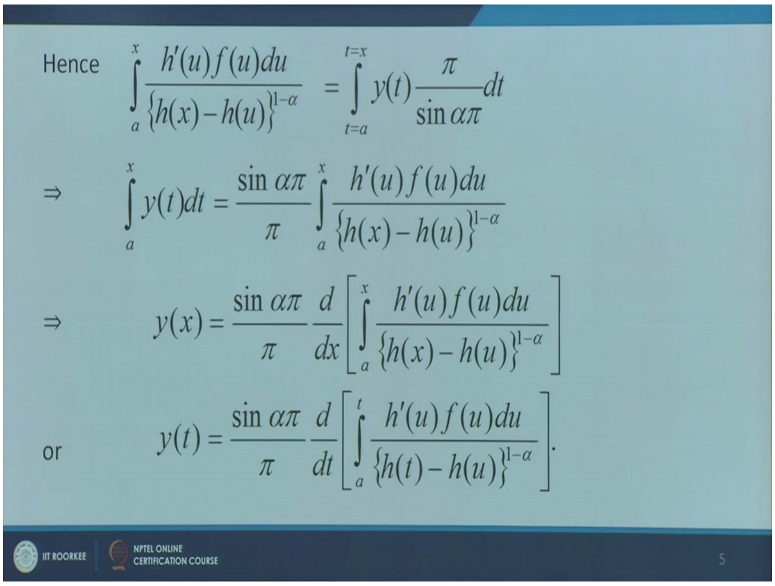

Now let us take an example on this. Similarly the integral equation, there is another general form of the Cauchy integral equation where we have f x equal to the lower limit is variable. In the previous case upper limit was variable, lower limit was a constant. Here upper limit is a constant, lower limit is a variable. So f x is equal to x to b, y t d t over h t minus h x to the power alpha, 0 less than alpha less than 1.

So here again a less than x less than b and h t is a monotonically increasing function and h prime t is positive. So this has the solution y t equal to minus sin alpha pie over pie, d over d t integral t to b, h prime u f u d u over h u minus h t to the power 1 minus alpha. This solution we can arrive at by (sim) simply following the steps of the solution of the previous general form.

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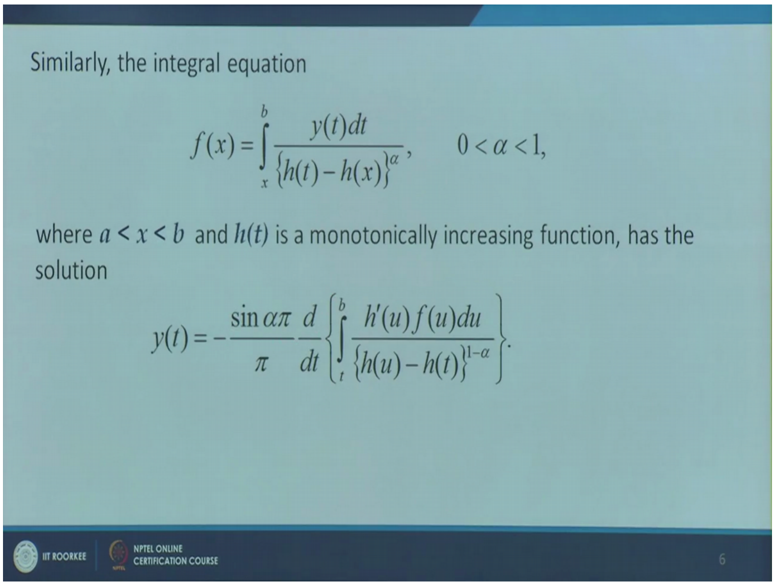

So let us take a example on this. So let us consider f x equal to integral a to x, y t d t over cos t minus cos x to the power half. So here f x is the given function, y t is the unknown function. So when you compare it with the general form of the Abel integral equation then what we get is alpha is equal to half and h x equal to cos x. So this will be h t equal to cos t and alpha is equal to half.

Now sorry not cos t, we will write it as, because we have to consider the function h t in such a way that h t is monotonically increasing and its derivative is positive. So let us write it like this. So y t d t equal to y t d t upon 1 minus cos x minus 1 minus cos t raise to the power half. Then when we compare it with the given general form of the Abel integral equation, alpha is equal to half and h t is equal to 1 minus cos t. So we get here h prime t equal to sin t.

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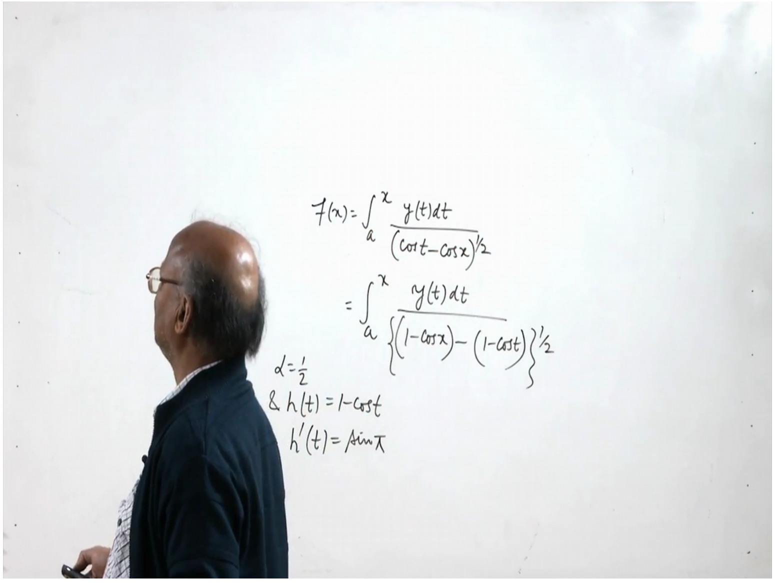

Now the interval given is 0 pie where x lies between 0, a and b. Okay, so what we have here? this is sin t and we know that sin t is positive in 0 pie interval. So h t is monotonically increasing because h prime t is positive and h prime t is (eq) equal to sin t which is positive in the 0 point interval. So the solution of this (gen) Abel integral equation will be given by y t equal to sin alpha pie over pie, d over d t of integral a to t, h prime u.

H prime u will be equal to sin u, into f u d u divided by h t minus h u. So 1 minus cos t minus 1 minus cos u, so that is equal to sin alpha pie over pie, d over d t, integral a to t, sin u f u d u divided by cos u. And this is power 1 minus alpha here so 1 minus half. So we have cos u minus cos t raise to the power half. So this is the solution of the given integral equation.

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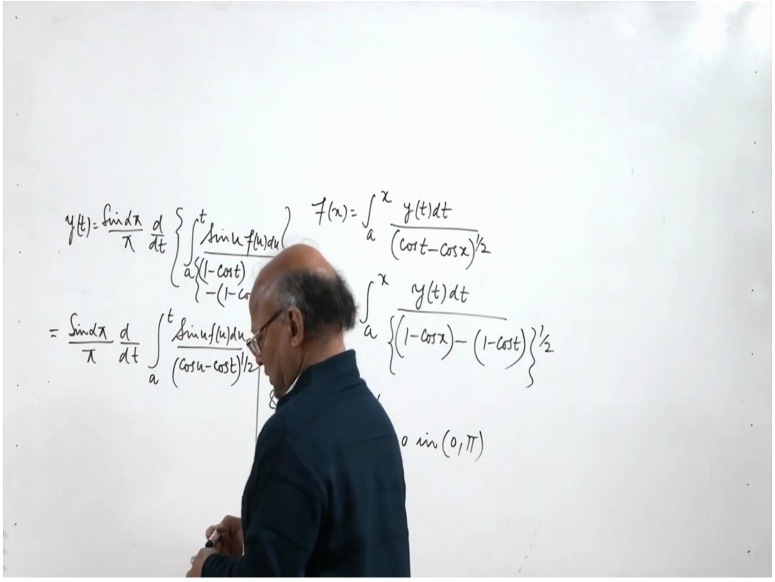

F x is equal to integral a to x, y t d t over cos t minus cos x raise to the power half, y t equal to 1 over pie. This is alpha is 1 by 2, so sin pie by 2 over pie we have. So sin pie by 2 is 1, so we have y t equal to 1 over pie, d over d t, integral a to t, f u sin u d u over under root cos u minus cos t.

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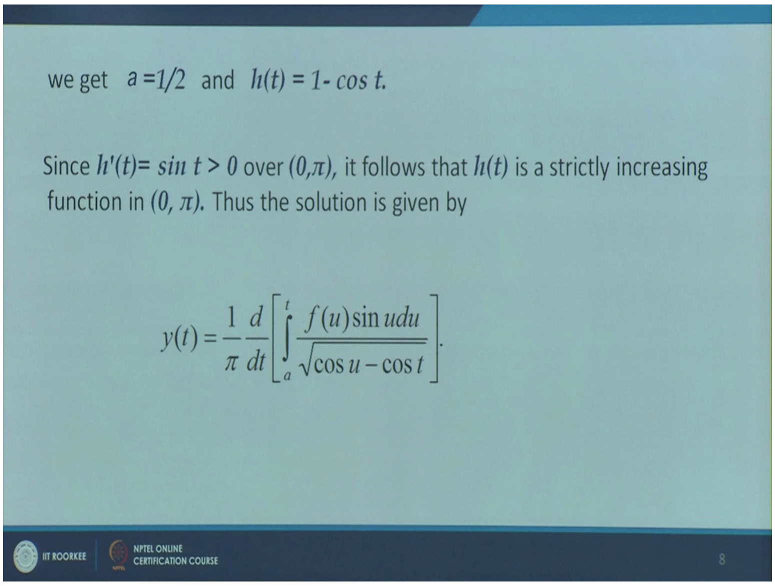

So let us consider another example on the general form of the Abel integral equation. Here we have the lower limit as a variable x and the upper limit is a constant. So f x is integral over x to b, y t d t over t square minus x square to the power alpha. Alpha is lying between 0 and 1 and 0 is less than or equals to a less than x less than b. Now when we compare it with the general form of the Abel integral equation, we have f x equal to x to b, y t d t over h t minus h x to the power alpha.

So h t is equal to t square here. So h t equal to t square is strictly monotonically increasing function in a b because h prime t is equal to 2 t and so over the open interval a b because (cou) a could be 0 also. So in the open interval a b it is strictly (19:45).

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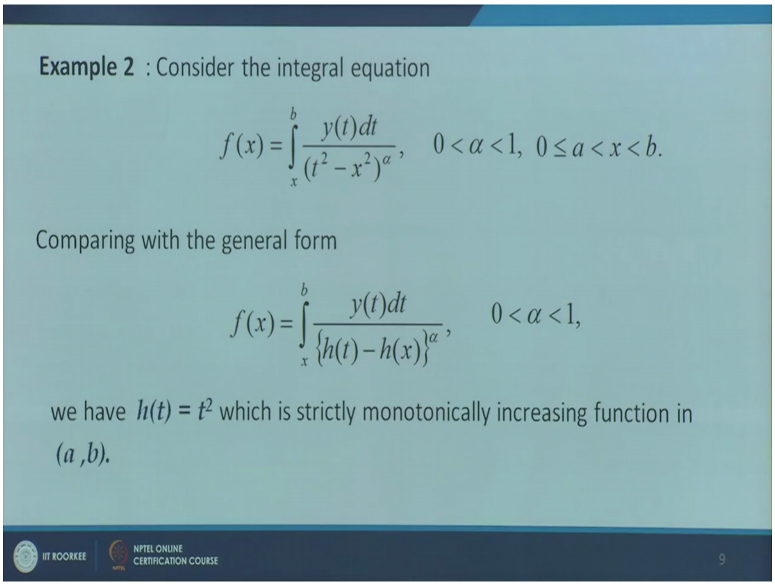

And therefore we can find its general solution. We can find the solution of this integral equation. The solution of this integral equation let us find by the formula this one. So y t is equal to minus sin alpha pie over pie, d over d t, integral t to b, h prime u f u d u divided by h u minus h t raise to the power 1 minus alpha. So this is minus sin alpha pie divided by pie, d over d t, integral over t to b, h prime u is 2 u, so 2 u f u d u divided by u square minus t square raise to the power 1 minus alpha.

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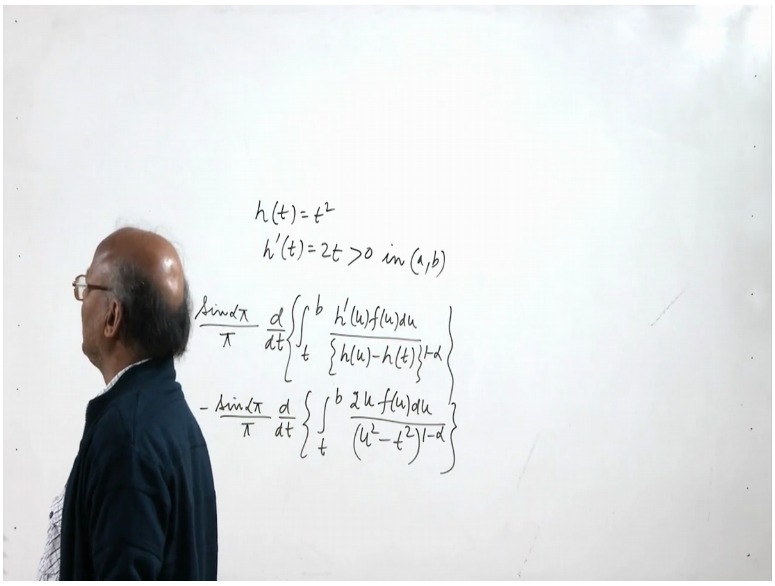

So we get the solution of the integral equation as y t equal to minus 2 sin alpha pie over pie, d over d t, integral t to b, u f u d u over u square minus t square raise to the power 1 minus alpha.

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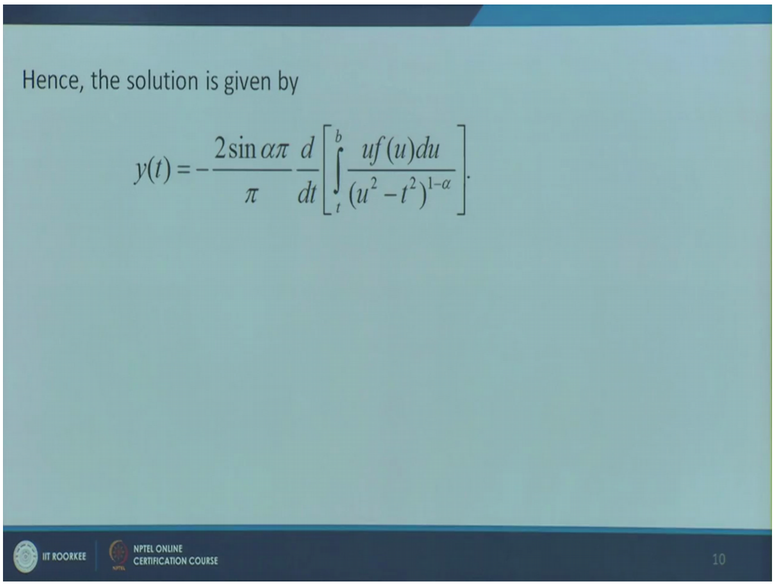

With that I would like to conclude my lecture. In the next lecture we shall discuss the Cauchy integral equations. Thank you very much for your attention.