Integral Equations, Calculus of Variations and their Applications
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Lecture 29
Singular Integral Equations I

Hello friends, I welcome you to my lecture on singular integral equations. There will be two lectures on this topic. This is the first lecture on the singular integral equations. Let us see what do we mean by a singular integral equation. Integral equation is called singular if either the range of integration is infinite or the kernel has a singularity within the range of integration. For example let us consider the equation f x equal to integral 0 to infinity, sin x t, y t d t.

So this is an integral equation where the kernel K x t is sin x t and the limits of integration are 0 and infinity. So, either one of the two limits is infinite or both the limits could be infinite. So here the range of integration is infinite therefore this is a singular integral equation. And the second example is f x is equal to integral 0 to infinity, e to the power minus x t, y t d t.

So here again the range of integration is infinite so it is a singular integral equation. And here you can say that although the limits of integration are not infinite but the kernel K x t is 1 over square root x minus t, so it becomes infinite at t equal x, and therefore this is again a singular integral equation.

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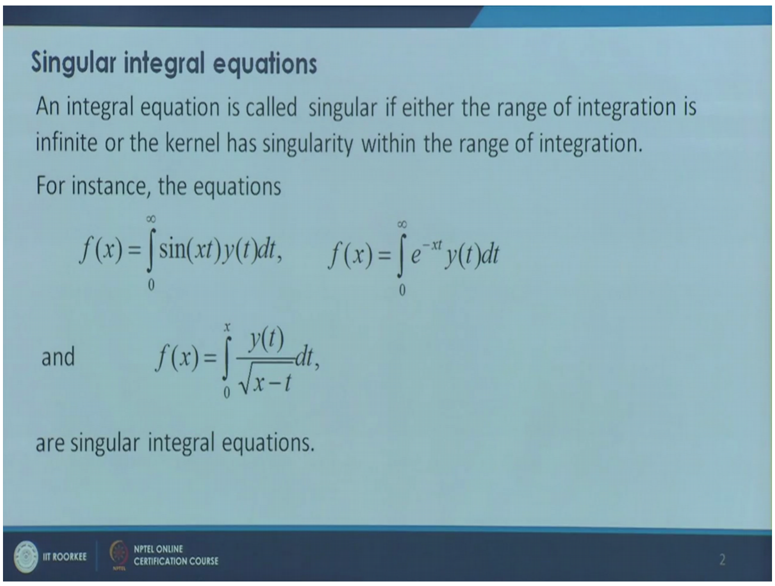

We shall be seeing how we can solve such integral equations. Let us begin with the Abel integral equation, f x equal to integral 0 to x, y t d t divided by x minus t to the power alpha, where alpha is lying between 0 and 1. Here f x is a known function and y t is the unknown function. So we are going to see how we can determine this unknown function y t.

Now this equation 1 is the integral equation formulation of the problem. You see when you take alpha equal to half then this equation is nothing but the equation which we discussed while we took up the case of a material point moving under the influence of gravity along a smooth curve from a vertical height x to a fixed point O on the curve.

In the first lecture we had discussed this example which is a problem in mechanics and we had got an integral equation of this type where alpha was equal to half. So here we are considering a general case where alpha could be any real numbers between 0 and 1.

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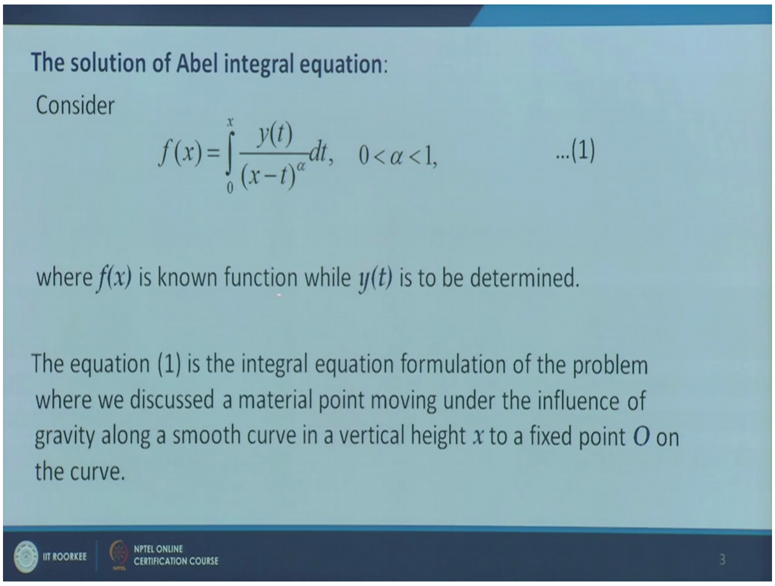

Now to solve this integral equation 1 what we do is we multiply both sides of this equation by 1 over u minus x to the power 1 minus alpha and then integrate with respect to x from 0 to u.

So when you multiply both sides by 1 over u minus x to the power 1 minus alpha and integrate over the interval 0 to u then what you get is integral 0 to u, f x d x over u minus x to the power 1 minus alpha equal to integral x equal to 0 to x equal to u, 1 over u minus x to the power 1 minus alpha and t varies from 0 to x, y t over x minus t raise to the power alpha d t, d x. Now what we are going to do is we are going to change the order of integration on the right side of this equation.

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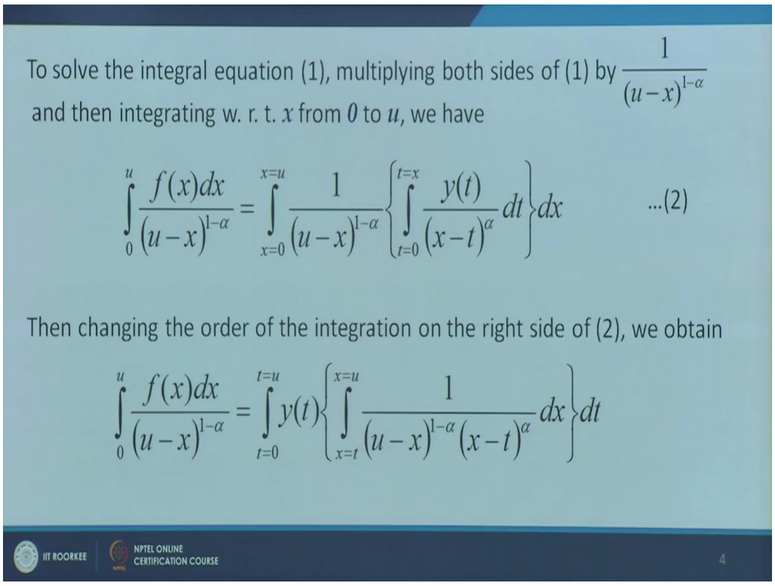

So here let us say this is t axis, this is u axis and t varies from 0 to x. So we have this line. This is t equal to u. Not like this. We have to take x and t. This is let us say x, this is t, this is t equal to x. So then t varies from 0 to t equal to x, t is 0 here and t is x here. So t varies from 0 to x and x varies from 0 to x equal to u. So this is x equal to u.

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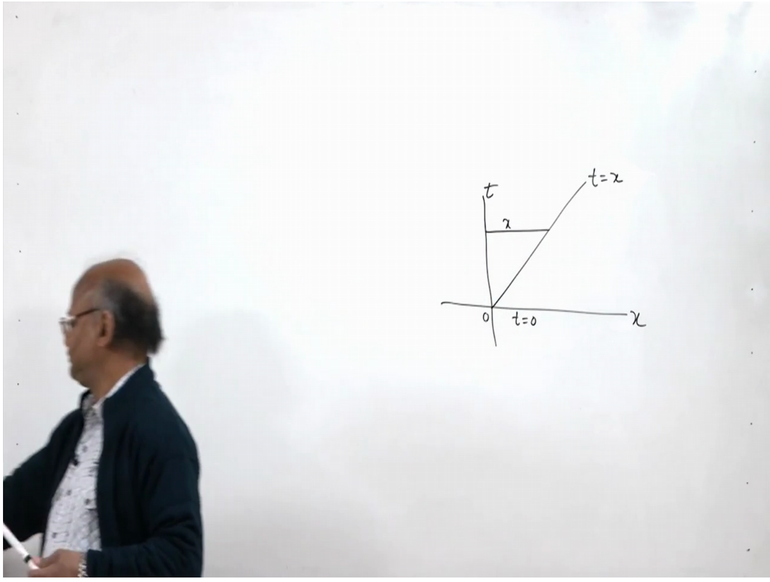

So t varies from 0 to t equal to x and x varies from 0 and goes up to u. So this is the region over which we are integrating, okay. Now let us take a horizontal strip in this to change the order of integration. When you take a horizontal strip in this region then for the horizontal strip x varies from t to x equal to u. So x varies from t to u and t varies from 0 to u because at this point when x is u, t is also u.

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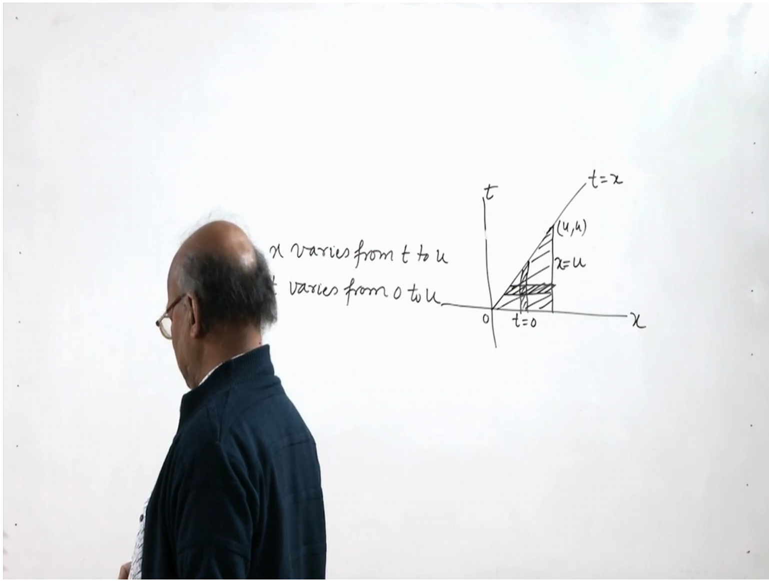

So when you change the order of integration you see that t varies from 0 to u and x varies from t to u. So after changing the order of integration we get this.

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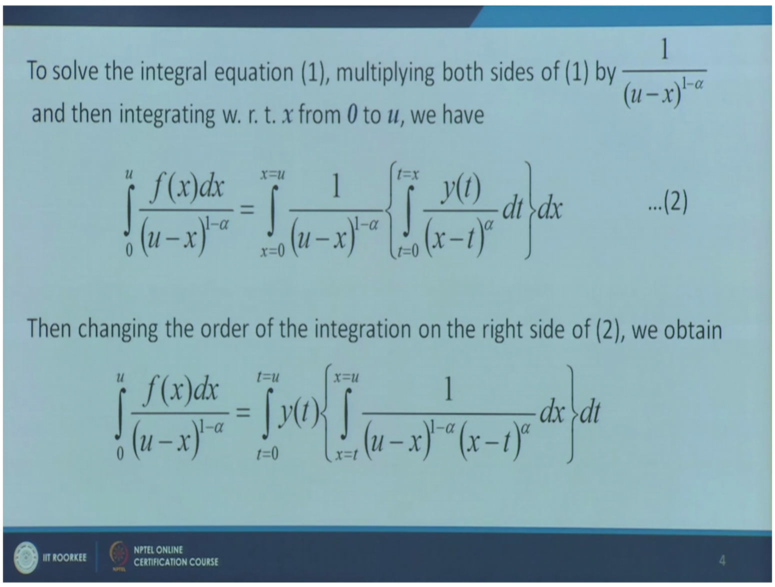

Now what we do is let us define I equal to integral over x to t to x equal to u, 1 over u minus x to the power 1 minus alpha, x minus t raise to the power alpha, d x. That is this integral. This integral let us define as I.

So we will be evaluating the integral inside this curly bracket. We are denoting it by I. So when I is equal to this let us define this transformation u minus x over u minus t equal to x. So if you define u minus x over u minus t equal to s then x will be equal to u minus u minus t into s, so that d x is equal to minus u minus t into d s. Let us substitute this transformation here.

So what we have? When your x is equal to t so that u minus t over u minus t will become 1, so the integral limits for s is 1 here, lower limit. And when x is equal to u, you get s is equal to 0. So the upper limit is 0 and d x becomes minus u minus t into d s divided by u minus x. So u minus x when you calculate to the power 1 minus alpha, you get u minus t to the power 1 minus alpha into s to the power 1 minus alpha.

And then when x minus t raise to the power alpha when we calculate, x minus t raise to the power alpha gives you u minus t raise to the power alpha into 1 minus s raise to the power alpha. So this u minus t divided by u minus t raise to the power 1 minus alpha into u minus t raise to the power alpha gets cancelled. And what we have is with this negative sign you can change the limits of integration.

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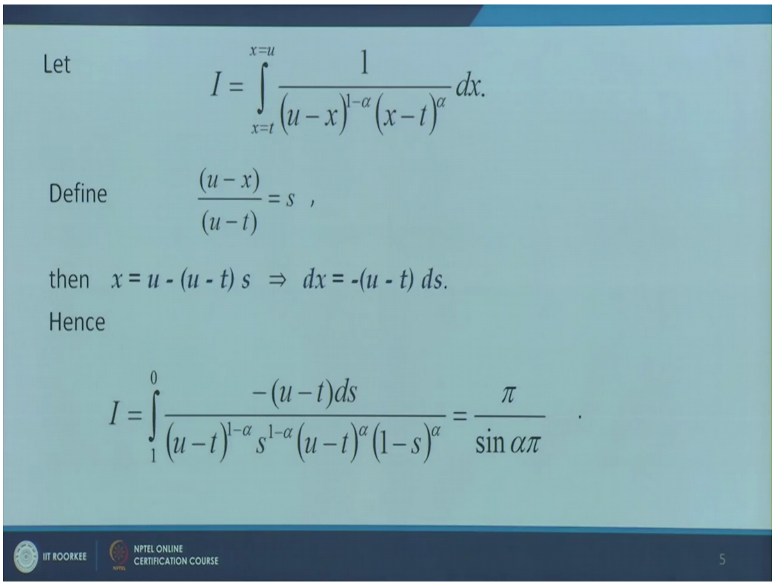

So I becomes 0 to 1. I is equal to 0 to 1, d s over s to the power 1 minus alpha and 1 minus s to the power alpha which we can write as integral 0 to 1, s to the power alpha minus 1 into 1 minus s to the power minus alpha, d s. This also equal to 0 to 1, s to the power alpha minus 1, 1 minus s to the power 1 minus alpha minus 1, d s. Now let us use the definition of beta function. So by the definition of beta function this becomes beta alpha 1 minus alpha.

And when we express beta function into gamma function we write it as gamma alpha into gamma 1 minus alpha divided by gamma 1. Gamma 1 is known to be equal to 1, so this is gamma alpha into gamma 1 minus alpha.

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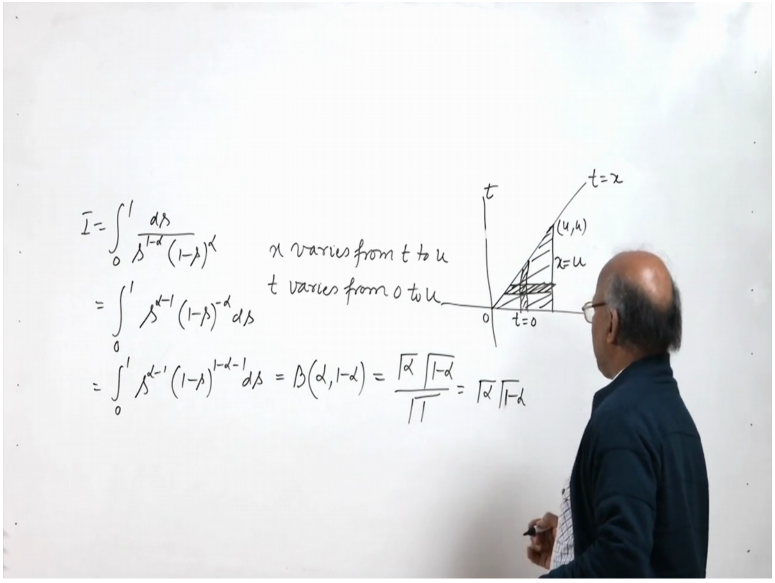

Now here we are given that 0 is less than alpha less than 1, so when 0 is less than alpha less than 1, gamma alpha into gamma 1 minus alpha is pie over sin alpha pie. This is a well known result. So when 0 is less than alpha less than 1, let us use the result that gamma alpha gamma 1 minus alpha is pie over sin alpha pie. So that is how we get the value of I to be pie over sin alpha pie.

Let us put this value here. Then the right hand side becomes integral 0 to u, y t pie over sin alpha pie, d t. So we can take pie over sin alpha pie to the other side.

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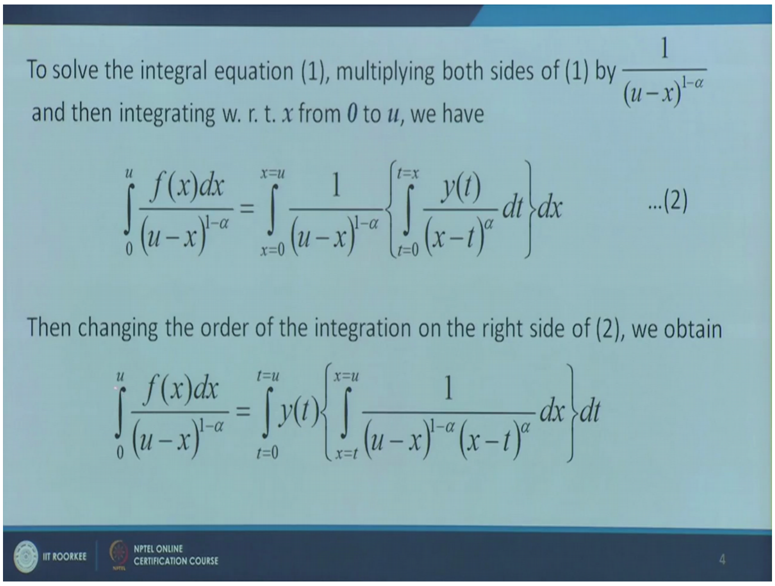

What we get is the integral of y t. So the integral 0 to u, y t d t becomes sin alpha pie over pie integral 0 to u, f x d x over u minus x to the power 1 minus alpha. Now let us differentiate both sides with respect to u. Differentiating both sides with respect to u by the Leibniz rule, see we have integral 0 to u, y t d t equal to sin alpha pie over pie, integral 0 to u, f x d x over u minus x raise to the power 1 minus alpha.

So let us differentiate both sides with respect to u. Then d over d u integral 0 to u, y t d t will be equal to sin alpha pie over pie, d over d u integral 0 to u, f x d x upon u minus x raise to the power 1 minus alpha.

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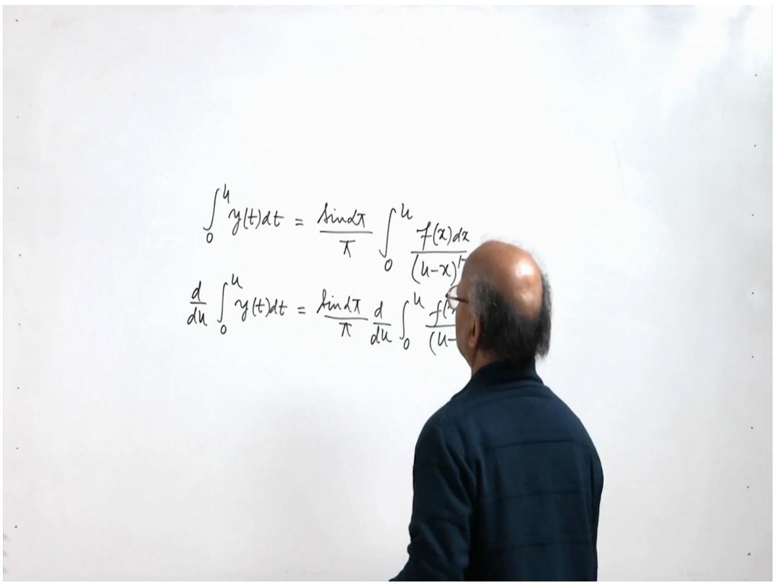

So by using the Leibniz rule what we get is the left hand side becomes integral 0 to u plus y u. We cannot just like that differentiate here on the right side with respect to u by Leibniz rule because then the integrant becomes unbounded. So we have d over d u. We cannot differentiate here on the right side with respect to u inside the (inte) integrant because then the integral becomes divergent.

So we put it like this. Now here what we have is, this is 0 so we have y u. This one is 0, we have y u here, then this is 0, y u is equal to sin alpha pie over pie, d over d u, 0 to u, f x d x over u minus x to the power 1 minus alpha.

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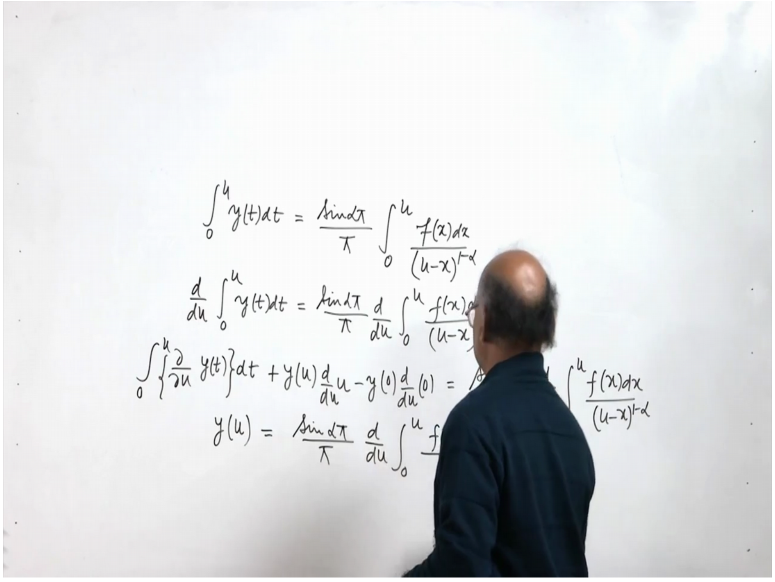

Or we can replace u by t. So y t is equal to sin alpha pie over pie, d over d t, 0 to t, f x d x over t minus x to the power 1 minus alpha. So this is how we get the solution of the Abel integral equation. The unknown function y t is given by sin alpha pie over pie, d over d t, integral 0 to t, f x d x over t minus x to the power 1 minus alpha.

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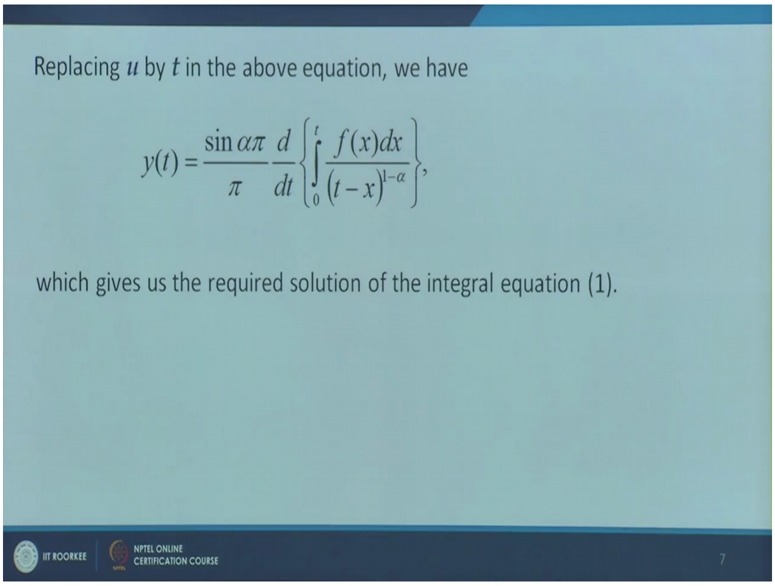

Now let us look at example on this Abel's integral equation. Consider x equal to 0 to x, y t d t over x minus t raise to the power half. So when we compare this integral equation with the standard integral equation, the standard integral equation is this one. F x equal to 0 to x, y t d t over x minus t raise to the power alpha.

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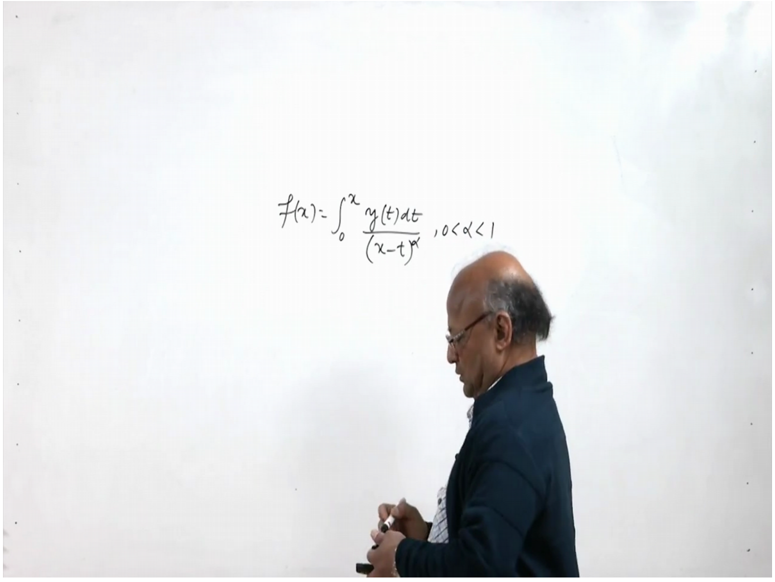

So when we compare with this integral equation what we have here is, here in this example we have f x equal to x and alpha is equal to half. Now let us recall the solution of the integral equation, y t is equal to sin alpha pie over pie, d over d t, integral 0 to t, f x d x over t minus x to the power 1 minus alpha. So let us substitute f x equal to x here and alpha equal to half. So y t is equal to sin pie by 2 divided by pie, d over d t, integral 0 to t, x d x divided by t minus x to the power half, okay.

Now what we do is let us define t minus x equal to s square. So when we define t minus x equal to s square, d x becomes minus 2 s d s and the integral 0 to t, x d x over t minus x to the power half becomes integral over root t to 0 because when x is 0 here, s square is equal to t, so s will be root t. And when x is equal to t, s square will be 0, so s will be 0. So integral over root to 0 and then here in place of x we shall put t minus s square and d x we shall be replacing by minus 2 s d s. And in the denominator t minus x to the power half will become s.

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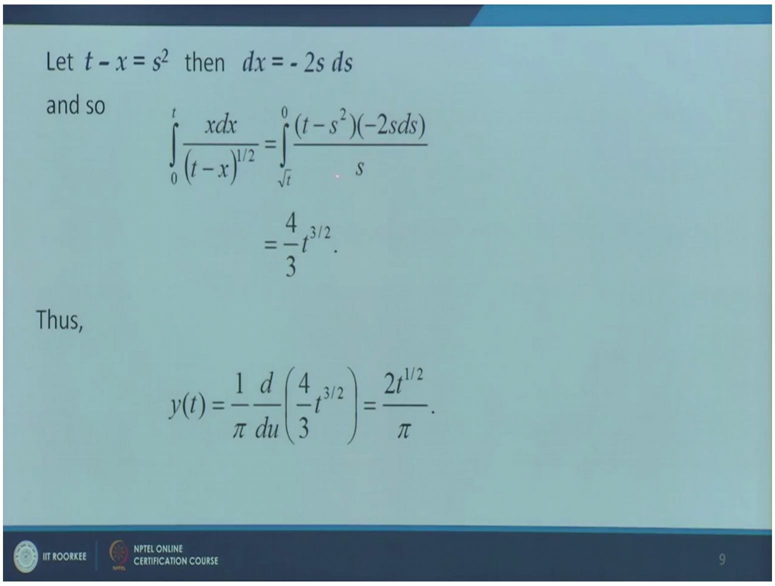

So this s gets cancelled and what we have is with the negative sign we can change the limits of integration and we have the limits of integration as 0 to root t, t minus s square, d s, this is what we have. Now let us integrate. So 2 times t s minus s cube by 3, 0 to root t. So this will be 2 times t to the power 3 by 2 minus t to the power 3 by 2 divided by 3. So this is 2 into 2 by 3, t to the power 3 by 2.

What we get is 4 by 3, t to the power 3 by 2. And thus y t becomes sin pie by 2 is 1. So 1 over pie, d over d t of 4 by 3, t to the power 3 by 2. So this is 1 over pie, 4 by 3 into 3 by 2 into t to the power half. And this gives you 2 over pie root t.

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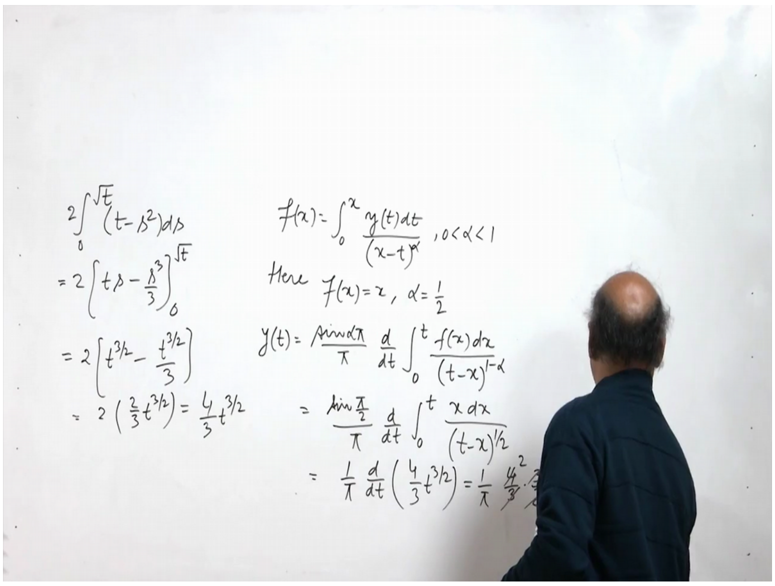

So y t equal to 2 times t to the power half divided by pie is this solution of this Abel type integral equation. Let us discuss one more example on Cauchy integral equation. Let us consider x square equal to integral 0 to x, y t d t over x minus t raise to the power 2 by 3.

Then comparing it with the general form the Abel integral equation we have f x equal to x square and alpha equal to 1 by 3 because the Abel integral equation is given by f x equal to integral 0 to x, y t d t divided by x minus t raise to the power 1 minus alpha where 0 is less than alpha less than 1.

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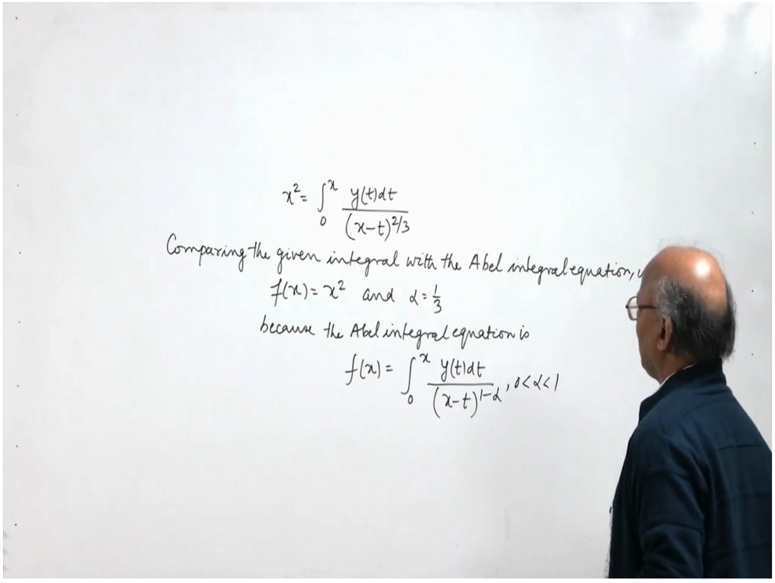

So when we compare the given integral equation with the Abel integral equation we find that f x is equal to x square and alpha is equal to 1 by 3. Now let us recall the solution of the integral equation. The solution of the integral equation is then given by y t equal to sin alpha pie over pie, d over d t, integral 0 to t, f x d x divided by t minus x raise to the power 1 minus alpha. So let us substitute the values of alpha and f x here.

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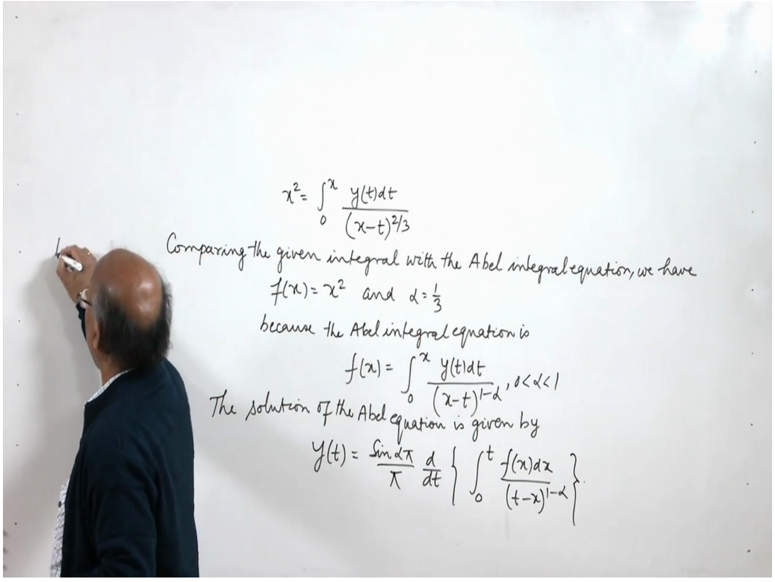

Then we have y t is equal to, so alpha is 1 by 3 so we have sin pie by 3 divided by pie, over d t of integral 0 to t, x square d x divided by t minus x raise to the power 2 by 3.

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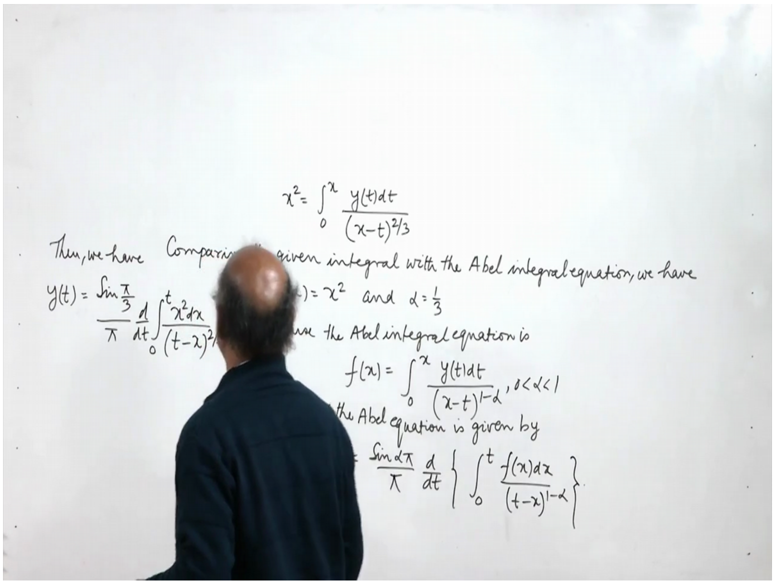

Now let us find this integral, let us evaluate this integral. So let us put t minus x equal to u cube. Then minus d x is equal to 3 u square d u. And so integral 0 to t, x square d x divided by t minus x raise to the power 2 by 3 becomes, when x is equal to 0, u is equal to t to the power 1 by 3. And when x is equal to t, u cube is 0, so u is 0. And x is equal to t minus u cube. So we have t minus u cube raise to the power 2.

Then d x is minus 3 u square d u divided by t minus x raise to the power 1 by 3 is u, so the denominator becomes u square. We can cancel u square and then what we get is integral 0 to t raise to the power 1 by 3, 3 times t minus u cube whole square d u which is equal to 3 times integral 0 to t raise to the power 1 by 3, t square minus 2 t times u cube plus u to the power 6 d u.

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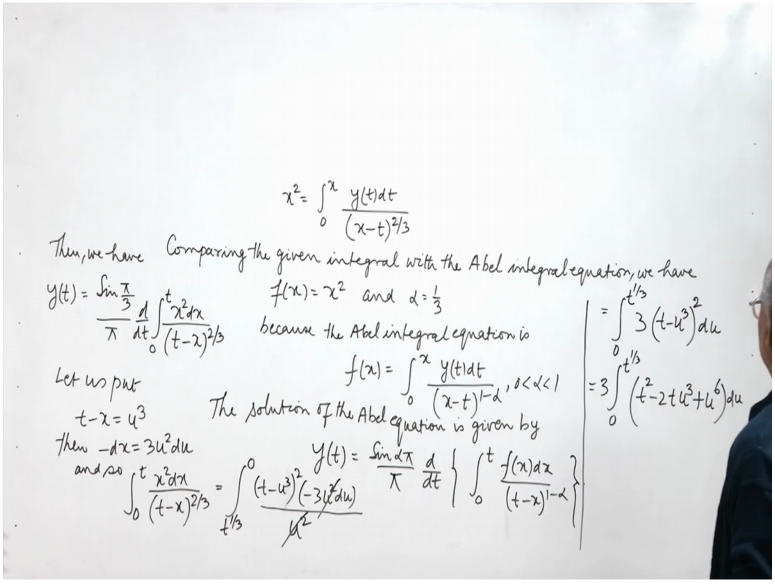

Or we can write it as 3 times t square into u minus 2 t, u 4 by 4 plus u to the power 7 by 7. And which is equal to 3 times t square into t to the power 1 by 3. So that becomes t to the power 7 by 3 minus I get half, t into t to the power 4 by 3. That also becomes t to the power 7 by 3. And then we have 1 by 7, u to the power 7. So we get t to the power 7 by 3. At the lower limit this expression is 0.

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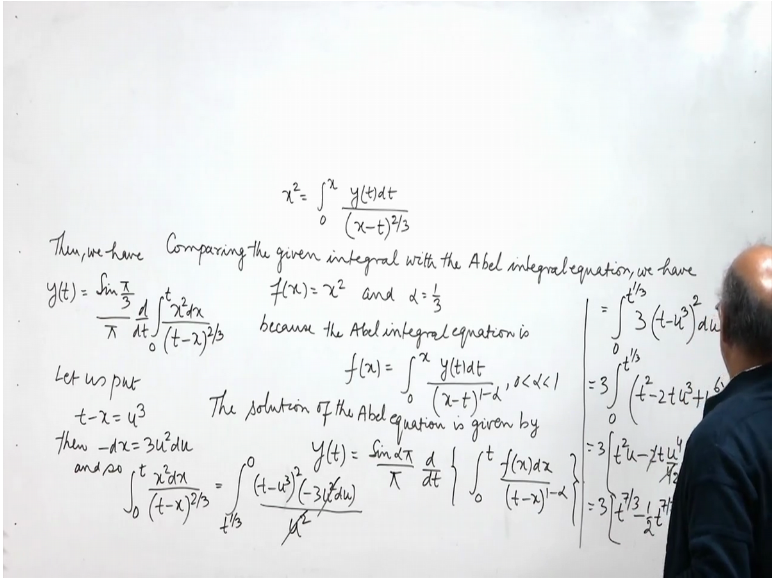

So what we have is, let us simplify it further. So 3 times, let us take t to the power 7 by 3 common, so 1 minus 1 by 2 plus 1 by 7, t to the power 7 by 3. And when we take LCM here what we get, so we get here 14 minus 7 plus 2 into t to the power 7 by 3. So we get 3 into, 14 plus 2 is 16 minus 7, so 9 by 14. So into 9 by 14, t to the power 7 by 3 which is 27 by 14, t to the power 7 by 3.

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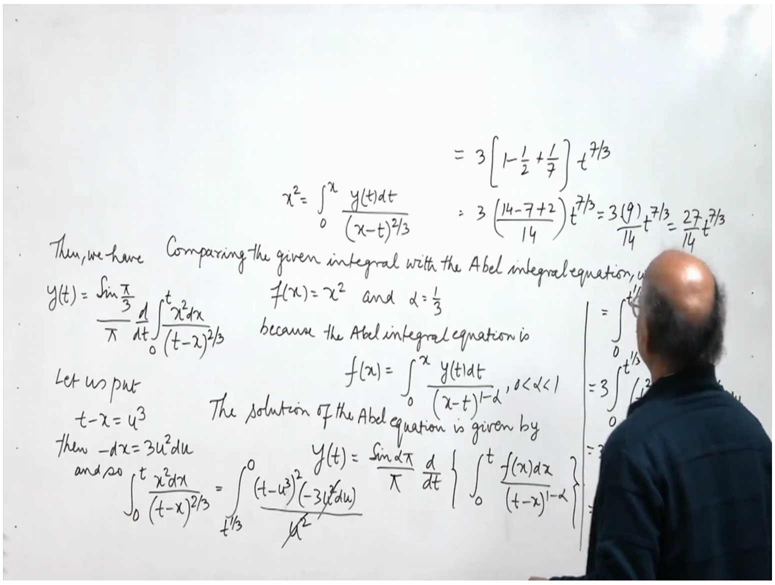

So we have calculated the value of the integral. Now let us put it here. So thus y t is equal to, sin pie by 3. Sin 6 t is root 3 by 2. So root 3 by 2 pie, d over d t of 27 by 14, t to the power 7 by 3. And this is equal to root 3 by 2 pie, 27 by 14 into 7 by 3, t to the power 4 by 3. So this you can cancel and this you can cancel.

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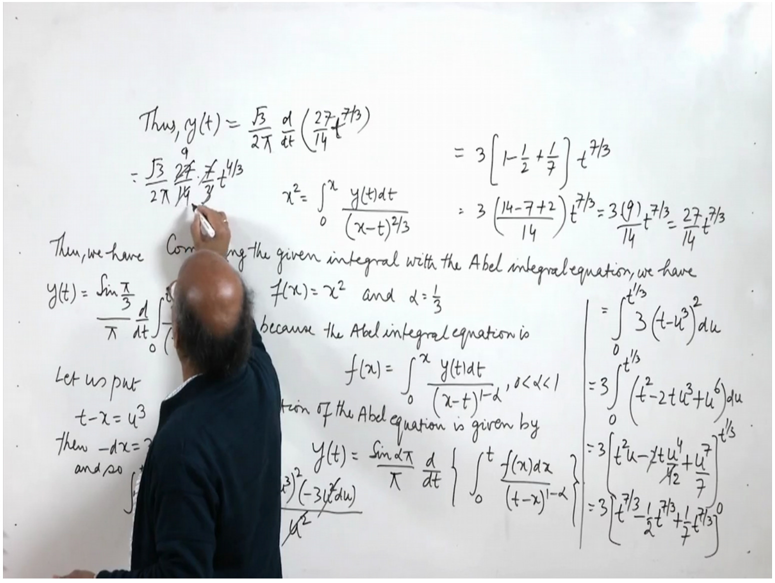

So what we get is this is 9 root 3 by 4 pie, t to the power 4 by 3. And thus we have the solution of the given integral equation as y t equal to 9 root 3 divided 4 pie, t to the power 4 by 3.

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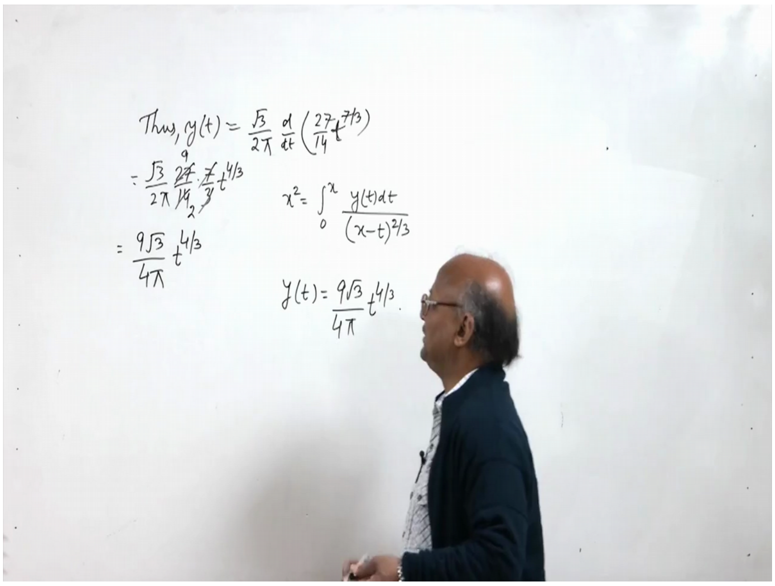

So with this I would like to conclude my lecture. In the next lecture we will discuss the general form of the Cauchy integral equation and how we can solve the general form of a Cauchy integral equation. Thank you very much for your attention.