Integral Equations, Calculus of Variations and their Applications
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Lecture 27
Equations with Convolution Type Kernels I

Hello friends I welcome you to my lecture on equations with convolution type kernels. There will be two lectures on this topic. So this is first of these two lectures. Let us consider the linear integral equations with convolution type kernels. Let us take two functions phi 1 x and phi 2 x which are defined for x greater than or equal to 0 and or continuous then the convolution of these two functions phi 1 x and phi 2 x is defined by a function phi 3 x equal to integral 0 to x, phi 1 x minus t into phi 2 t d t.

It turns out that the function phi 3 x is also continuous for x greater than or equal to 0. Let us say that the Laplace transforms of the functions phi 1 x and phi 2 x be denoted by phi 1 s and phi 2 s respectively so that L phi 1 x is equal to phi 1 s and L phi 2 x equal to phi 2 s.

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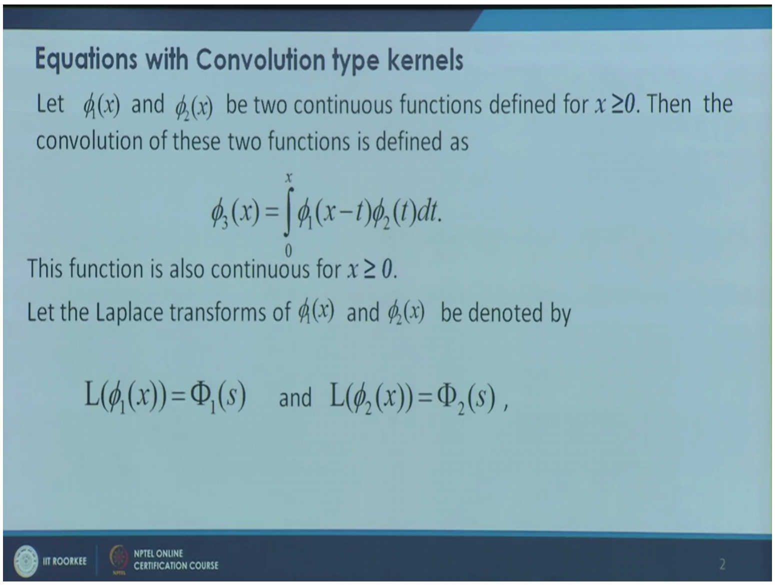

Then by the (convolu) convolution theorem for the Laplace transforms we have Laplace transform of phi 3 x. So Laplace transform of phi 3 x is the product of the Laplace transforms of phi 1 x and phi 2 x which is equal to phi 1 s into phi 2 s.

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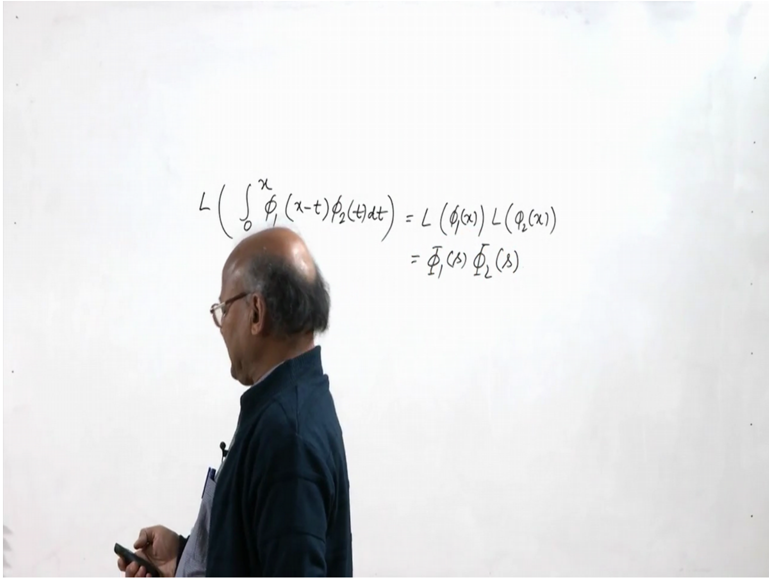

Now let us consider the Volterra type integral equation of the second kind, y x equal to f x plus integral 0 to x, K x minus t, y t d t. So here f x is a known function, K x minus t is also known, this kernel of the integral equation K is solely dependent on the difference x minus t of the arguments t and x and then you have y t, the unknown function.

So let us see how we can solve such an integral equation where the kernel K x t is solely dependent on the difference of the arguments x and t that is x minus t is called an integral equation with convolution type kernels.

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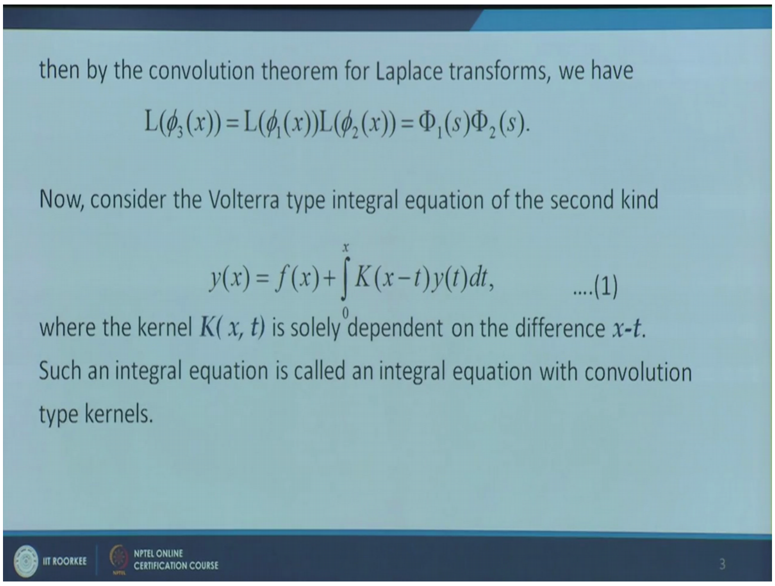

So what we will do is, okay. So let f x and K x t be sufficiently smooth functions such that as x goes to infinity, mod of f x is less than or equal to M 1 times e to the power alpha 1 x, mod of K x t is less than or equal to M 2 times e to the power alpha 2 x. That means f x and K x t are both of exponential order. Then using the method of successive approximations it turns out that mod of y x is less than or equal to M 3 times e to the power alpha 3 x.

Now consequently the Laplace transforms of f x, K x t and y x exist. Let us assume that Laplace transform of f x is F s and Laplace transform of K x t is K tilde s and Laplace transform of y x is equal to Y s. Here remember that K x t is kernel of the type x minus t, that means K x t is a function of x minus t.

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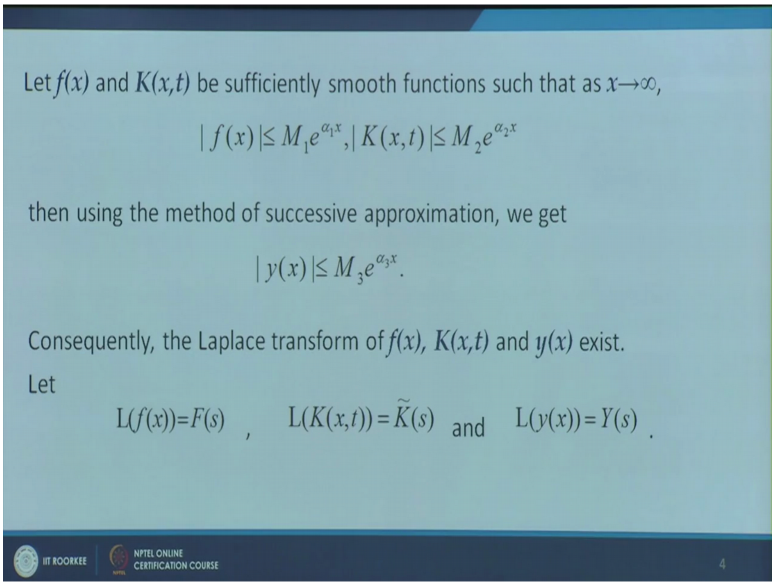

So what we will do is let us take the Laplace transform of both sides of the equation 1. When you take the Laplace transform of this equation 1, Laplace transform of y x will be Y s which will be equal to Laplace transform of f x which we have assumed as equal to F s. And then Laplace transform of integral 0 to x, K x minus t, y t d t which by making use of the convolution theorem for Laplace transforms we can write as the product of the Laplace transform of K x t and y t.

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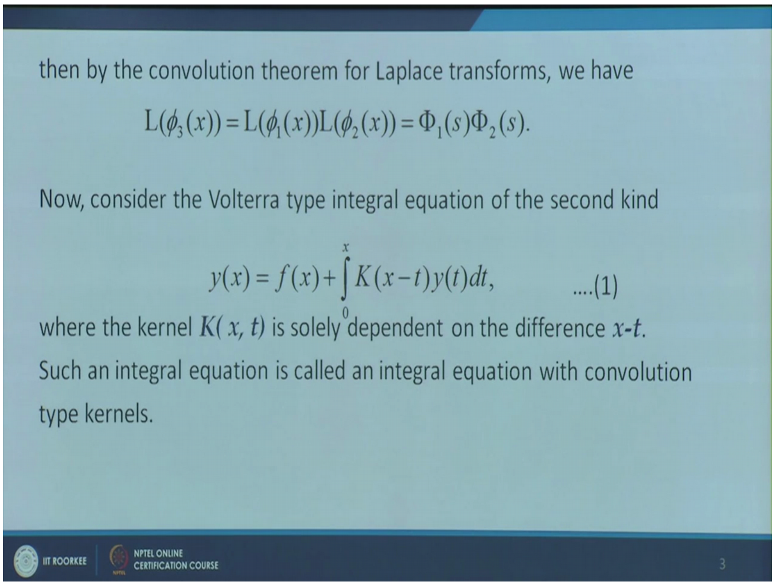

So this will be equal to Y s equal to F s plus Laplace transform of K x t which we have assumed as K tilde s into Y s. Then we can solve this equation for Y s and Y s will be equal to F s over 1 minus K tilde s where we are assuming that K tilde s is not equal to 1. So then the solution y x of the integral equation 1 is found by taking the inverse Laplace transform of Y s that is we get y x equal to L inverse Y s.

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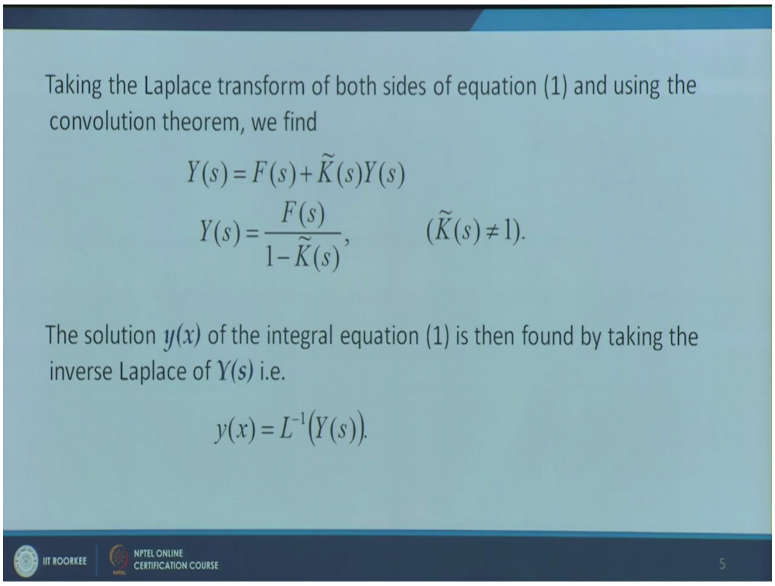

Now let us see how we apply this method to solve a integral equation of Volterra type of second kind. So let us consider the integral equation phi x equal to sin x plus 2 times integral 0 to x, cos x minus t, phi t d t. So let us take the Laplace transform of both sides. So then Laplace transform of both sides when we take we will get Laplace transform of phi x equal to

Laplace transform of sin x plus 2 times Laplace transform of this convolution of cos x and phi x, so 0 to x cos x minus t into phi t d t.

Or we can say phi s is equal to 1 over s square plus 1 plus 2 times Laplace transform of cos x into Laplace transform of phi x. Or we can say phi s equal to 1 by s square plus 1, 2 times s over s square plus 1 into phi s. Let us collect the coefficient of phi s, so phi s times 1 minus 2 s over s square plus 1 equal to 1 over s square plus 1. Or we can say phi s time s minus 1 whole square divided by s square plus 1 is equal to 1 over s square plus 1. Or we can say phi s equal to 1 over s minus 1 whole square.

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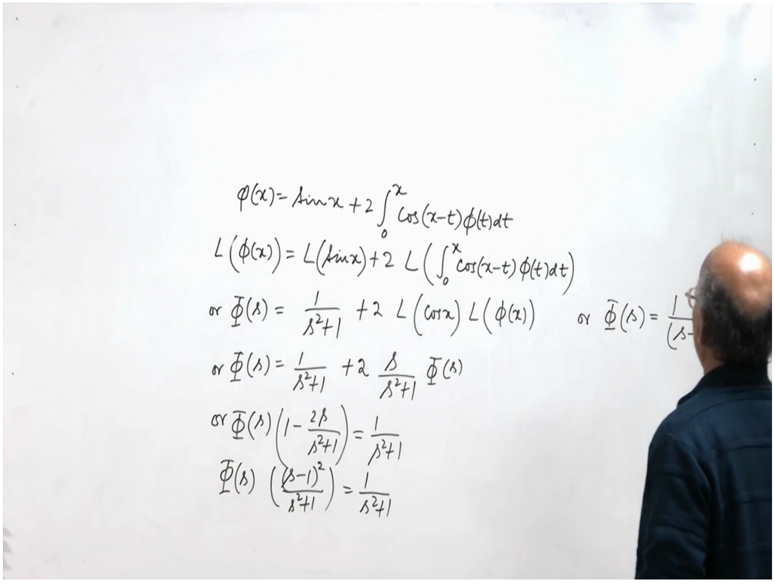

Now then the unknown function phi x is obtained by taking inverse Laplace transform of 1 over s minus 1 whole square. Now let us recall that Laplace transform of t is equal to 1 over s square. Laplace transform of t is 1 over s square and by the first shifting theorem when you multiply t by e to the power t, s is replaced by s minus 1.

So we get 1 over s minus 1 whole square. So thus we can say that hence L inverse of 1 over s minus 1 whole square is equal to t e to the power t. Now here phi is a function of x so I can write it as x times e to the power x.

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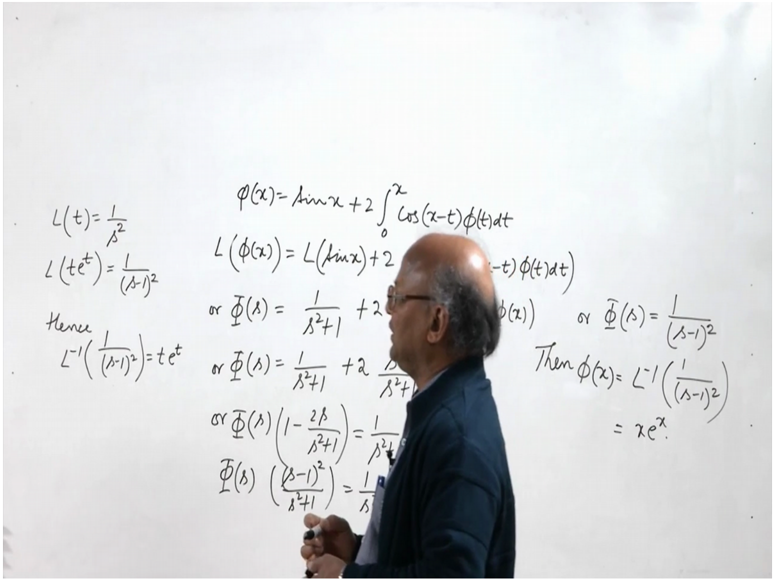

Thus the solution of the integral equation phi x equal to sin x plus 2 times integral 0 to x cos x minus t, phi t d t will be equal to x e to the power. Phi x equal to x e to the power x.

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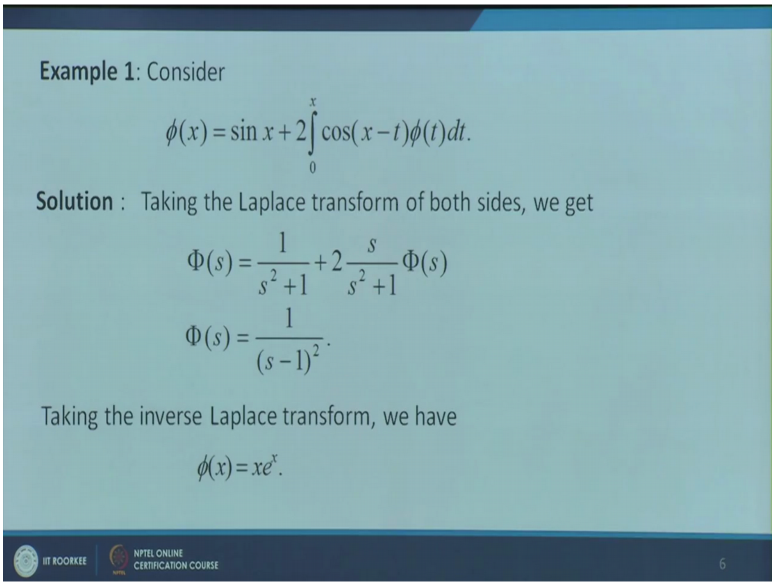

Now let us take one more example on this. Consider the Volterra integral equation of the second kind phi x equal to e to the power 2 x plus integral 0 to x, e to the power t minus x, phi t d t, okay. So we will be solving it by taking the Laplace transform of both sides but here let us (not) notice that when we compare it with the standard Volterra integral equation of second kind with convolution type kernel, here we should be having phi x equal to f x plus integral 0 to x, K x minus t, phi t d t.

So instead of t minus x we should be having x minus t.

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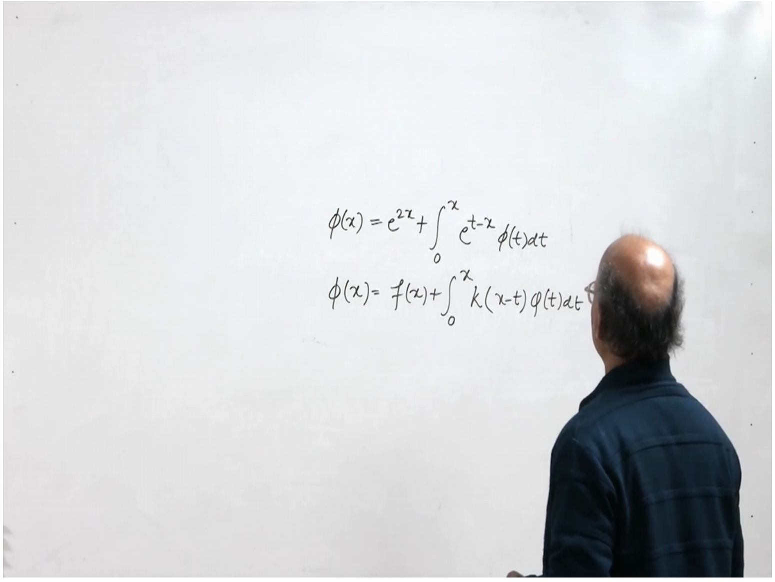

So what we write is we can write the given equation as phi x equal to e to the power 2 x plus integral 0 to x, e to the power minus x minus t into phi t d t. So K x t is e to the power minus x minus t. Now let us take the Laplace transform of both sides.

So taking Laplace transform of both sides of the above equation we have phi s is Laplace transform of phi x, then Laplace transform of e to the power 2 x is 1 over s minus 2 plus Laplace transform of the convolution 0 to x, e to the power minus x minus t, phi t d t. So this is Laplace transform of e to the power minus x into Laplace transform of phi x.

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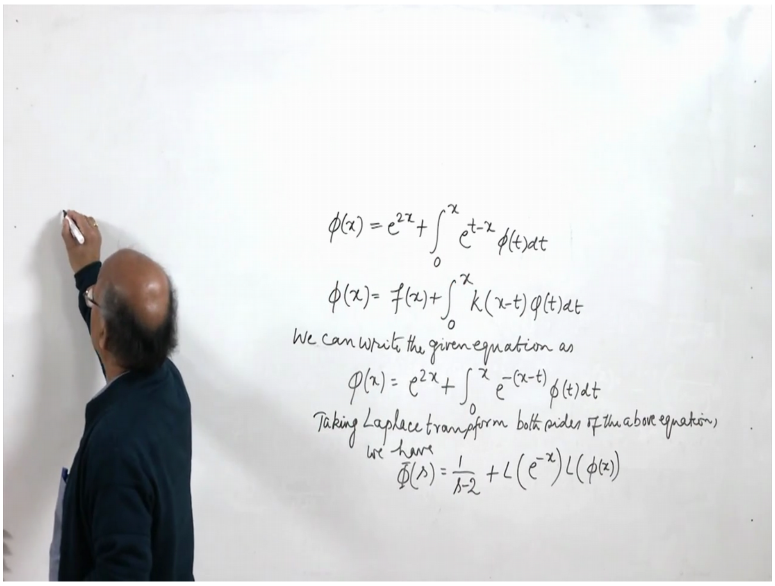

And so we can write it as phi s times 1 minus 1 over s plus 1 equal to 1 over s minus 2. So I will have s over s plus 1, 1 over s minus 2 or phi s equal to s plus 1 divided by s into s minus 2. We can break it into partial fractions. So A upon s plus B upon s minus 2. Then A is given by s plus 1 over s minus 2 at s equal to 0. So we have minus half. And B is equal to s plus 1 by s at s is equal to 2. So it is 3 by 2.

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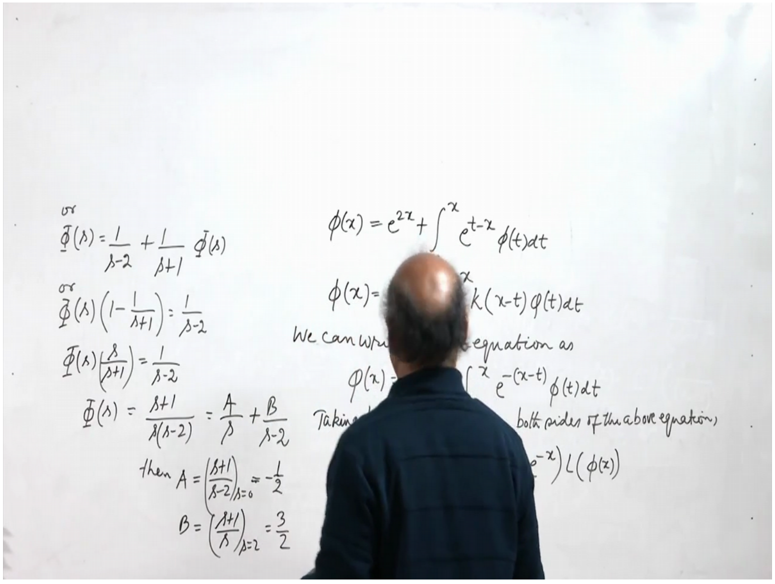

So thus we get phi s is equal to minus 1 by 2 into s plus 3 by 2, s minus 2. Now (Lap) taking inverse Laplace transform both sides, L inverse phi s equal to minus half, L inverse 1 by s plus 3 by 2, L inverse 1 over s minus 2 which gives us the solution of the given integral equation.

So phi x equal to minus 1 by 2, L inverse of 1 by s is equal to 1 so we get 1 plus 3 by 2, L inverse of 1 over s minus 2 is e to the power 2 x. So we get phi x equal to minus half plus 3 by 2, e to the power 2 x. This is the solution of the given integral equation.

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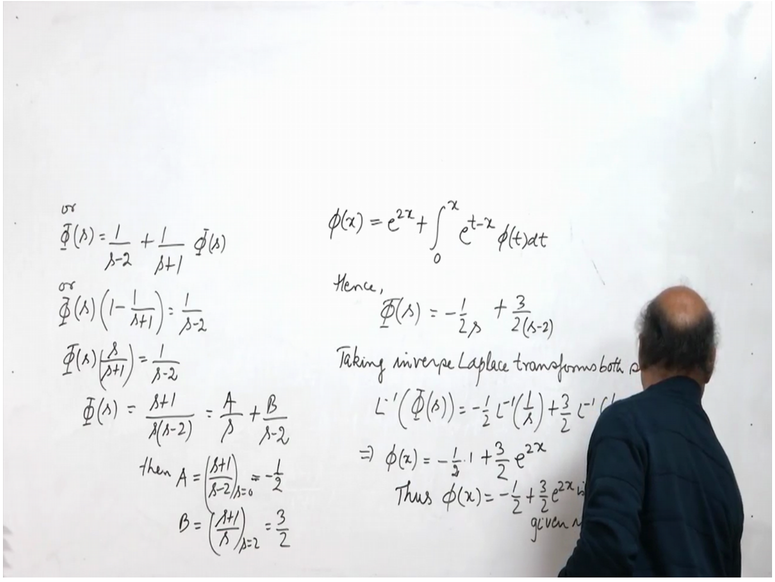

Okay now one more problem I have taken here. I can discuss it here, phi x equal to x minus 0 to x, sin hyperbolic x minus t, phi t d t. When we take Laplace transform of both sides Laplace transform of phi x is phi s, Laplace transform of x is 1 over s square minus Laplace transform of sin hyperbolic x. And Laplace transform of sin hyperbolic x is 1 over s square minus 1 into Laplace transform of phi x which is phi s.

So we can collect the coefficients of phi s which is 1 plus 1 over s square minus 1 and then divide by the coefficients of phi s. So we get phi s equal to 1 over s square minus 1 over s to the power 4 and then we can take the inverse Laplace transform of this equation. So L inverse of phi s gives us phi x and L inverse of 1 over s square is x, L inverse of 1 over s to the power 4 is x cube by 6. So the solution of the given Volterra integral equation of convolution type is phi x equal to x minus x cube by 6.

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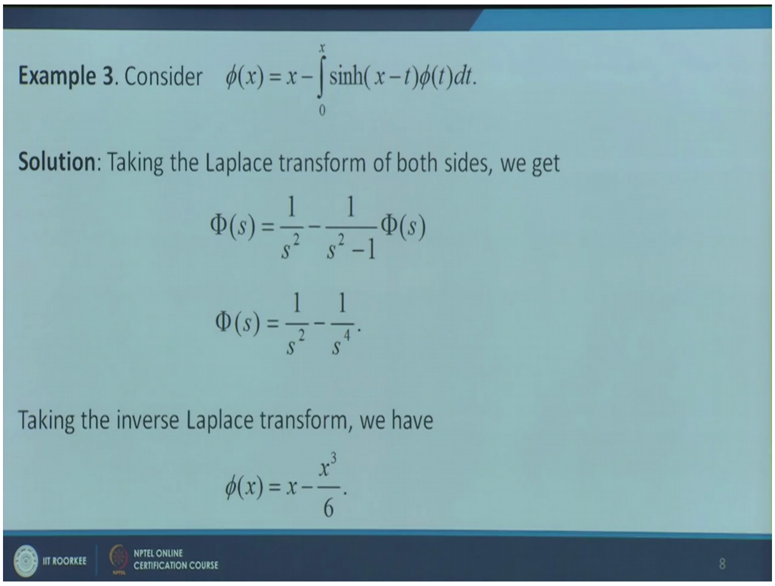

Now there is one more equation, phi x equal to 1 minus 2 x minus 4 x square plus integral. There is a problem which let us do here.

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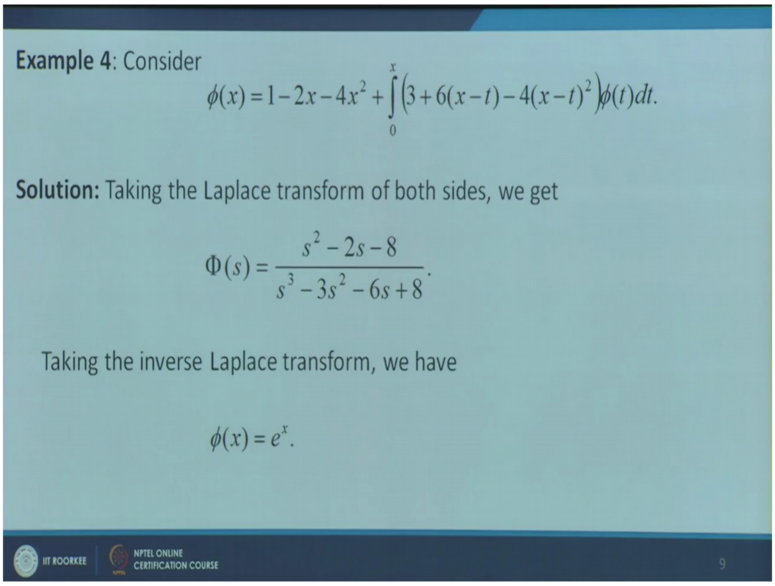

So phi x is equal to 1 minus 2 x minus 4 x square plus integral 0 to x, 3 plus 6 x minus t minus 4 x minus t whole square into phi t d t. In my lecture on Volterra integral equations of second kind where K x t is of some special forms there we have discussed this problem because the kernel here is a polynomial in t of degree 2.

So we can solve it by the method which we discussed there but you will see now that when we solve this by using the Laplace transform, the solution of this integral equation comes very easily. So let us take the Laplace transform of this.

So taking Laplace transform on both sides we get phi s, Laplace transform of 1 which is 1 by s minus 2 times Laplace transform of x is 1 over s square minus 4 times Laplace transform s square is Laplace transform of t to the power n is equal to n factorial divided by s to the power n plus 1. So Laplace transform of s square is 2 factorial that is 2 factorial divided by s to the power 3 plus Laplace transform of this convolution. So Laplace transform of 3 plus 6 x minus 4 x square into Laplace transform of phi x.

So what we get is phi s equal to 1 over s minus 2 over s square minus 8 over s cube plus Laplace transform of 3 plus 6 x minus 4 x square is 3 by s plus 6 by s square minus 4 into 2 factorial divided by s cube into phi s. We can then collect the coefficients of phi s, so what we get is 1 minus 3 by s minus 6 by s square plus 8 by s cube equal to, 1 by s minus 2 by s square minus 8 by s cube let us take LCM there.

So we get s cube and then we get s square minus 2 s minus 8. So simplifying we get phi s times s cube, we have s cube minus 3 s square minus 6 s plus 8, this is equal to s square minus 2 s minus 8 divided by s cube.

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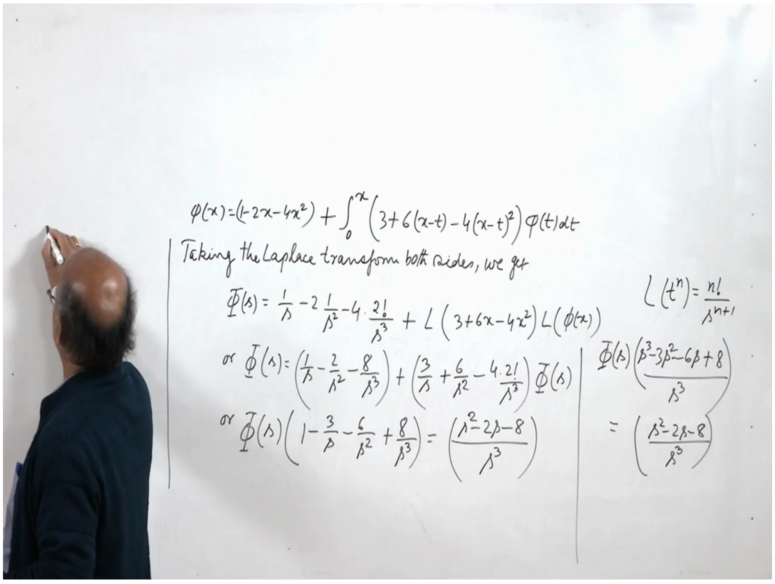

Now so phi s is equal to s square minus 2 s minus 8 divided by s cube minus 3 s square minus 6 s plus 8. Now we can see that the denominator is a polynomial of degree 3 in s and s is equal to 1 is a root of this polynomial because when you put s equal to 1 what we have is 1 minus 3 minus 6 plus 8. So equal to 1 is a root of this polynomial. So s minus 1 is a factor and when you divide it by s minus 1 what you get is, s square minus 2 minus 8. So s square minus 2 s minus 8 divided by s minus 1, the denominator can be written as s minus 1 into s square minus 2 s minus 8.

So we can cancel this and we get 1 over s minus 1. Now phi s is equal to 1 over s minus 1, so taking inverse Laplace transform, so L inverse of phi s is equal to L inverse of 1 over s minus 1 which implies phi x equal to e to the power x. So phi x is equal to e to the power x is the solution of the given Volterra integral equation of second kind.

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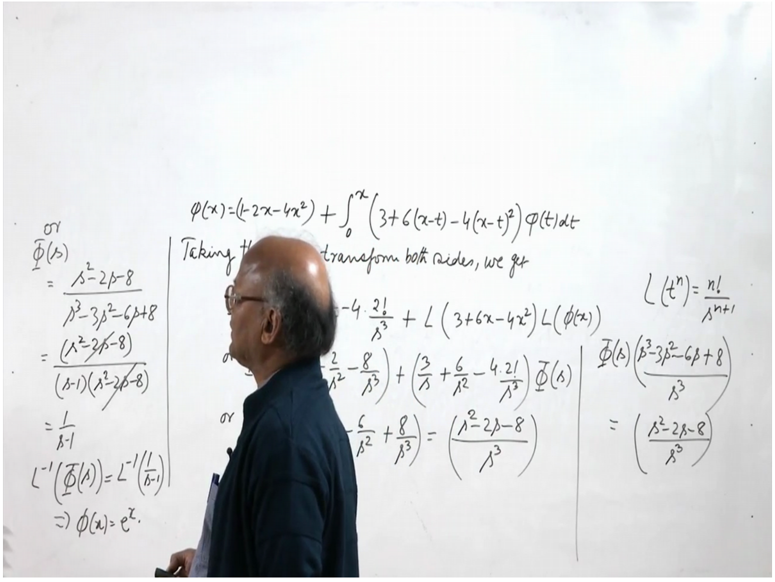

when we discussed this problem there when K x t is of some particular form, there the solution was quite lengthy but you can see here that by using the Laplace transform we can find the solution of this Volterra integral equation very easily. Now let us consider system of Volterra type integral equations.

A system of Volterra type integral equations can also be (salv) solved by making use of the Laplace transformation. So let us see how we solve the system of Volterra type integral equations of (seco) second kind.

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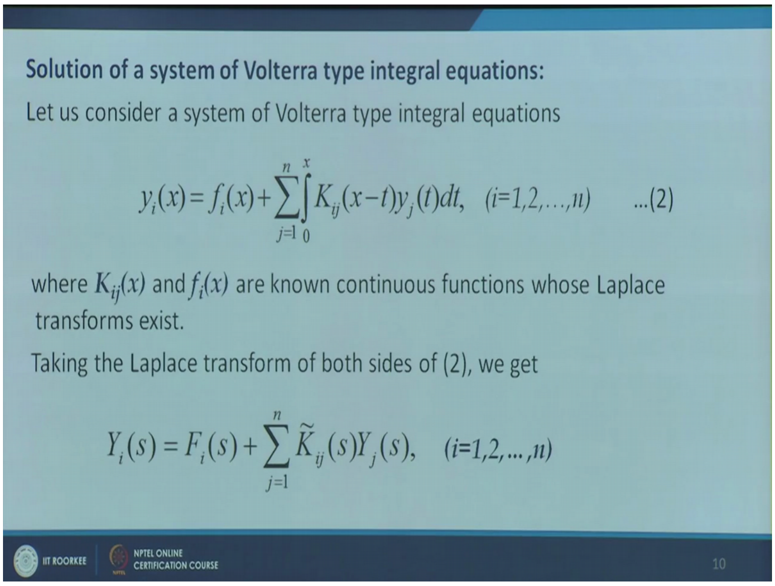

Let us consider a system of Volterra type integral equation, y i x equal to f i x plus sigma j equal to 1 to n, integral 0 to x, K i j x minus t, y j t d t. So there are n integral equations here and therefore there are n functions to be determined. The unknown functions are y i x where i runs from 1 to n. So there are n integral equations and therefore there n (indep) unknown functions y i x.

And the functions f i x for i equals to 1 to n and the functions K i j x are known continuous functions whose Laplace transforms exist. So we are assuming that the Laplace transforms of the kernel and the (func) function for f i x do exist. And then what we do is we take the Laplace transform of both sides of this equation.

When you take Laplace transform of both sides here what you get is (())(25:05) Laplace transform of y i x by capital Y i s and Laplace transform of f i x by capital F i s. And then we have sigma j equal to 1 to n. This is convolution of K i j x and y j x. So product of the Laplace transforms of K i j x and y j x. So Laplace transform of K i j x we are denoting by K i j tilde s and Laplace transform of y j x we are denoting by Y j s.

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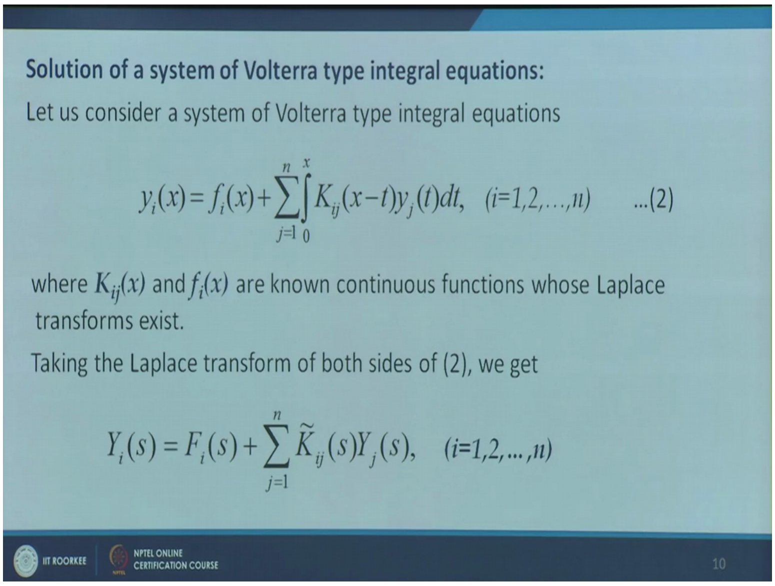

So here we are denoting Laplace transform of f i x by F i s, Laplace transform of y i x by Y i s and Laplace transform of K i j. So this is what we have and then we will have these n equations. So this is a system of linear algebraic equation in Y i s.

You can see here this a system of linear algebraic equations in Y i s and we shall solve these linear system and to determine the expressions for this function Y i s and then we will take the inverse Laplace transform of Y i s to get the unknown function y i x for i equal to 1, 2, 3 and so on up to n.

Let us say for example we consider this system of algebraic equations, phi 1 x equal to 1 minus 2 times integral 0 to x, e to the power 2 times x minus t phi 1 t d t (inte) plus integral 0 to x, phi 2 t d t. And the second equation is phi 2 x equal to 4 x minus integral 0 to x, phi 1 t d t plus 4 times integral 0 to x, x minus t, phi 2 t d t.

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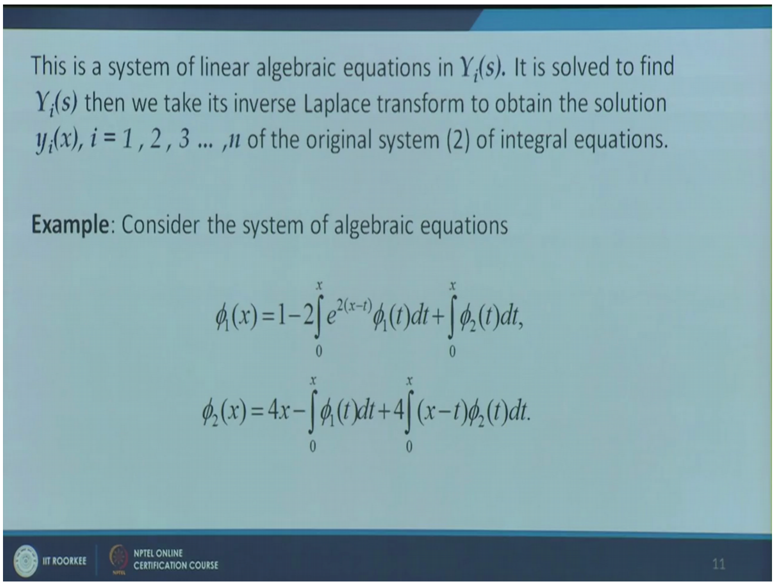

Then so we get a system of linear algebraic equations in Y i s. There will be n equations in the unknown function Y i s. We can determine the unknown functions Y i s from these n equations. And then in order to find the solution of the given system we will then take the inverse Laplace transform of Y i s. So taking inverse Laplace transform of Y i s we will determine the unknown function y i x for i equal to 1, 2, 3 and so on up to n which gives us the solution of the original system 2 of integral equations.

We have given an example, consider the system of algebraic equations, phi 1 x equal to 1 minus 2 times integral 0 to x, e to the power 2 x minus t, phi 1 t d t plus integral 0 to x, phi 2 t d t and phi 2 x equal to 4 x minus integral 0 to x, phi 1 t d t plus 4 times integral 0 to x, x minus t, phi 2 t d t.

So here there are two unknown functions phi 1 x and phi 2 x. We will be taking the Laplace transform of these two integral equations and then we will get the unknown functions phi 1 s and phi 2 s. The unknown phi 1 s and phi 2 s will be determined from the two algebraic equations in phi 1 s and phi 2 s and we shall be taking their inverse Laplace transform to determine the unknown functions phi 1 x and phi 2 x. With this I would like to conclude my lecture. Thank you very much for your attention.