Course on Integral Equations, Calculus of Variations and their Applications
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Lecture 20
Fredholm method of solutions

So hello friends, welcome to the today's lecture of here we will discuss the classical Fredholm theory for solving Fredholm integral equation of second kind.

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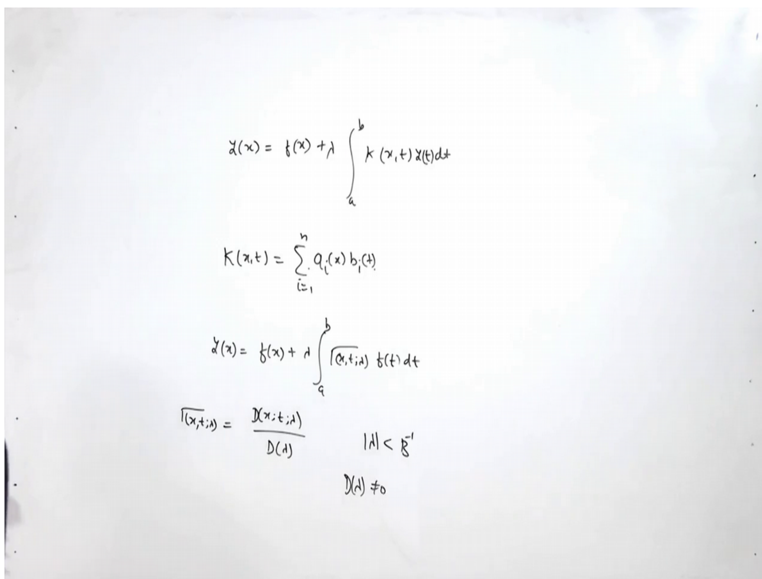

So basically we are looking at this kind of solution that suppose we have this Fredholm integral equation on this kind yx equal to f of x plus lambda times a to b K of x, t yt and dt. So in previous lecture we have seen that if this K of x, t is of separable type that summation i equal to 1 to n a i x b i t then we can write down the solution as y of x equal to f of x plus lambda times a to b gamma xt lambda and f of s d of s here f of t dt here I am using t so it is f of t and dt.

So here f of t and dt and we have also have seen that we can write this gamma xt lambda the resolvent kernel as ratio of 2 (())(1:45) in terms of D x, t, lambda divided by D lambda and this expression valid in a reason say modulus of lambda less than or equal to B inverse and provided that D lambda is not equal to 0. So in this case the solution is unique and you can get by evacuating this resolvent kernel gamma x, t, lambda and this we have seen.

Now what we want to do in today's lecture is generalize this theory for any given kernel. So it means that here we are assuming that this f of x and K x, t are L 2 functions and then we try to find out then can we write the solution of this Fredholm integral equation of second kind as this kind of thing where your gamma x, t lambda can be written as ratio of two functions D x, t, lambda upon D lambda where D lambda and d x, t lambda we can find out later on.

So to start with let us look at this method which is proposed by Fredholm itself himself and he started he gave three theorems known as Fredholm first theorem, second theorem and third theorem which is which gives the solutions for all lambdas and it is as follows.

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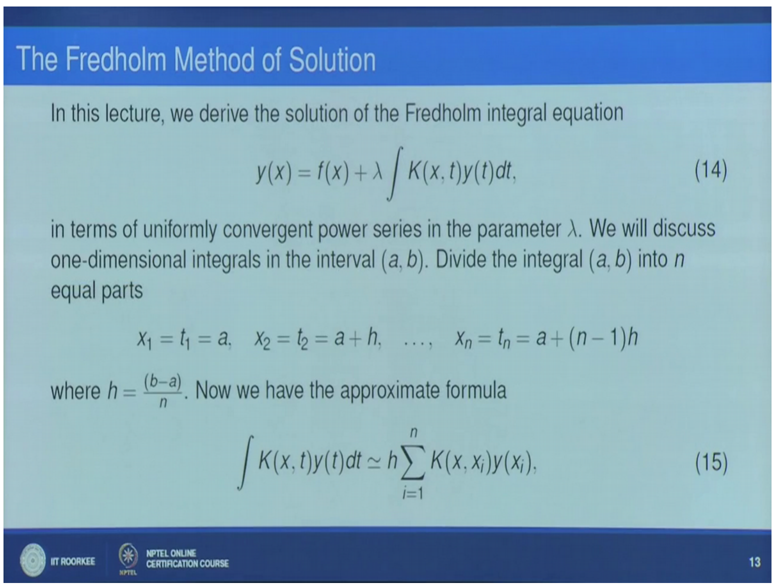

So just look at here what he did is to solve this particular equation what he try to do is he try to discretize this integral into finite summation for this what he did he simply divide this interval a to b into an equal number of parts.

So here let us assume that we have x 1 equal to t 1 as a and x 2 equal to t 2 equal to a plus h and so on or you can say you can take h as b minus a by n and using this I can simply say this K x, t yt dt can be written as this finite sum h times i equal to 1 to n K x x i y of x i.

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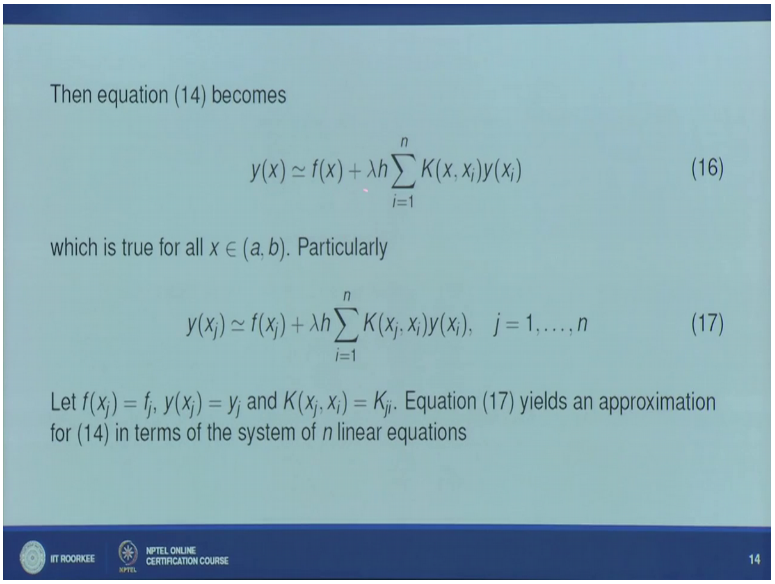

Now using this discretization using this writing integral into this finite sum we can write this as that equation 1 this y of x equal to f of x plus lambda integral here can be written as y of x is approximately equal to f of x plus lambda h i equal to 1 to n K x, xj y x j.

Now this expression is valid for all x in terms of a to b. So what to do here you let us consider this equation at x equal to your x i's.

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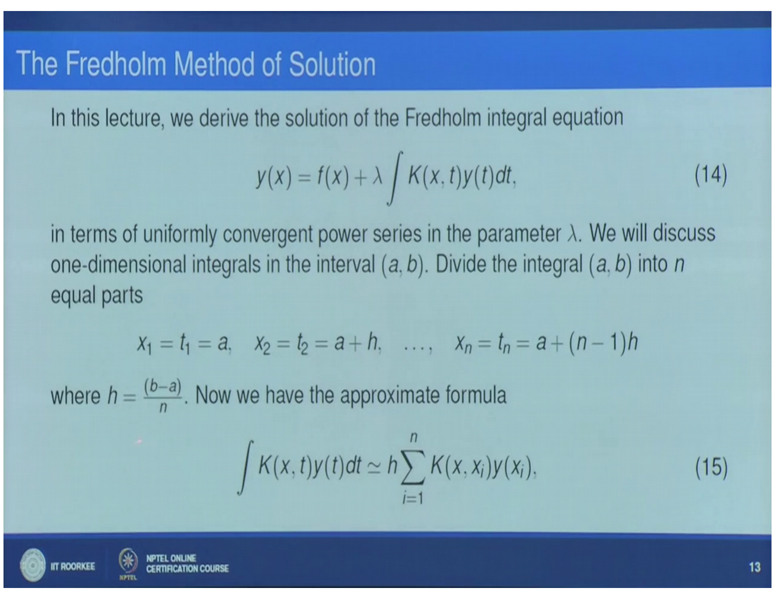

So what how we had define x i? x i is defined like this x 1 as a, x 2 as a plus h, x n as a plus n minus 1 h.

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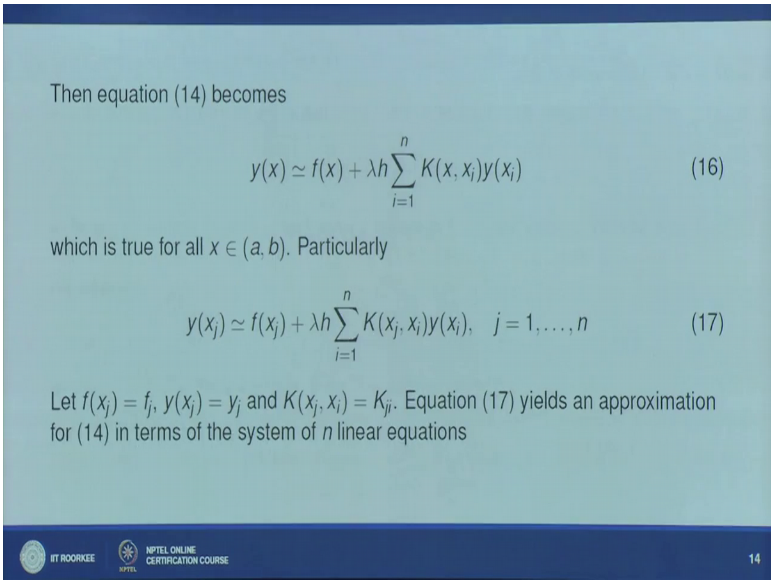

So looking at this equation at x equal to x i we can write down as y of x j is approximately equal to f of x j plus lambda h i equal to 1 to n K x j, x i y x i.

Now here this will give you this will give us the solution of this integral equation approximate solution at the point say x 1 to x n. So it is not giving all the solution for entire x but it is giving the solution or you can say it interpolate this solution at x 1 to x n. So it means that this solution obtain by 17 is the interpolation or it is the approximation of the solution at n points x 1 to x n, is it okay?

So let us try to find out the solution here so to solve this let us simplify the notation here let us write down yx j as y j fx j as f of j and K x j x i you write it K j i and using this simplified notation we can write this equation as algebraic equation, right?

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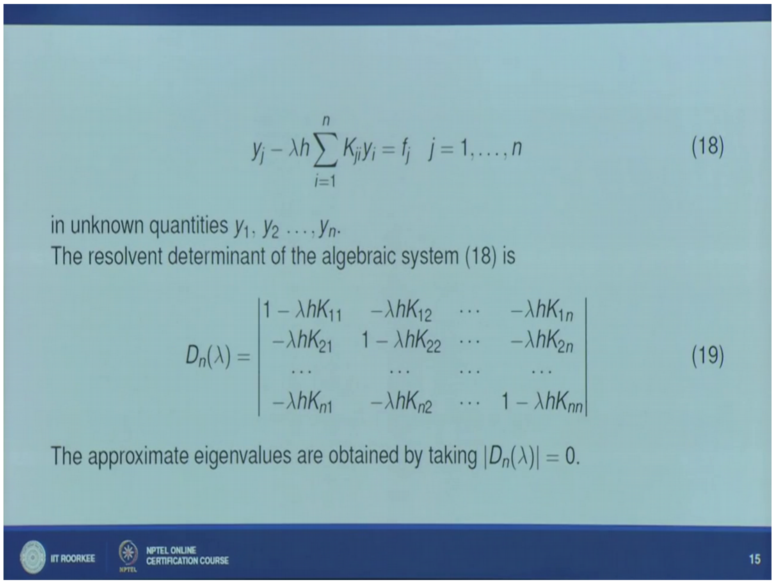

So this I can write as y j minus lambda h i equal to 1 to n K j i y i equal to f of j. So here if we can solve this we can get the solution y j here.

So it means that you can get the solution of the integral equation at particular points x 1 to x n. To get this, this you can solve in terms of you can write it this as d lambda matrix d lambda y i you can write it like this.

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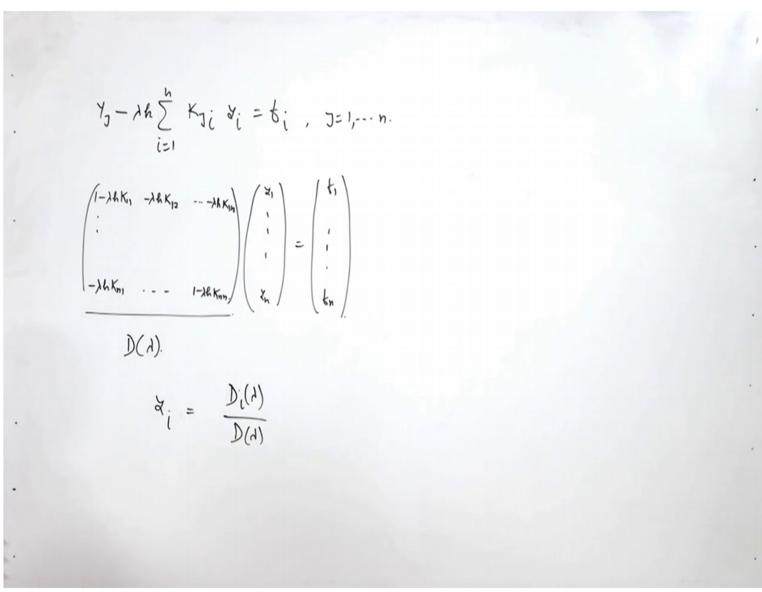

So here we have y j minus lambda h summation i equal to 1 to n K j i and y i equal to f of i and that is valid for j equal to 1 to say n. So if you write down you can write down this as like this, so y 1 to say y n and this is your f 1 to say f n and here you can write it 1 minus lambda h K 11 minus lambda h K 12 and so on minus lambda h K 1n and so on here you can write minus lambda h K n1 to 1 minus lambda h K nn.

So if you remember the similar kind of expression we have discuss for separable equation, now only thing is that there K 11 is replaced by a 11 and so on. So we can say that if we denote this as D of lambda determinant of this is denoted as D lambda then the solution y 1 to y n is depending on D lambda or you can say that in that case you can write down solution y i as say your D i lambda divided by D lambda I can write this as D yeah. So D i lambda is I can further write in terms of D lambda as D i j lambda divided by D lambda, okay you can write this as D lambda as this, okay.

So D i lambda is the ith column is replaced by f 1 to f n, okay so this we have seen.

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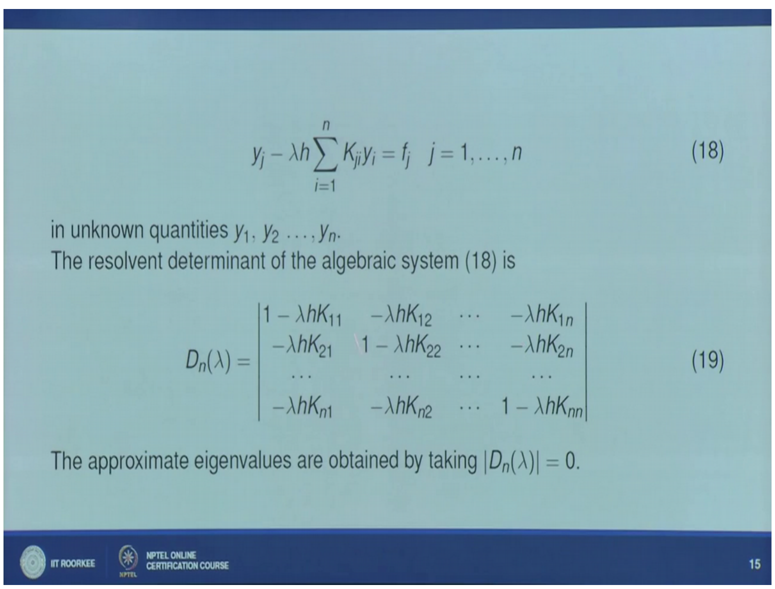

So we say that the this determinant if this determinant is non-zero then we can find out the unique solution y 1 to y n in terms of f 1 to f n and if this determinant is 0 then we can find out say approximate eigen value corresponding to this system 18. So by taking D n lambda equal to 0 we can get the eigen values and when D n lambda is non-equal to 0 in that case we can find out the solution y 1 to y n.

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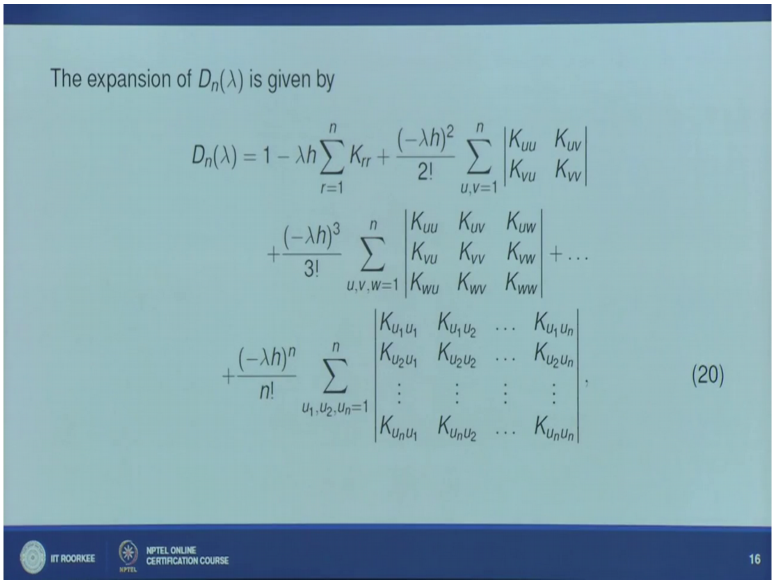

Now what we try to do here then let us try to find out the expression of this D lambda. So to find out this D lambda we try to express this in terms of expansion of in terms of minus lambda h in powers of minus lambda h. So for that we try to show that the D n lambda can be written as this finite series term, right?

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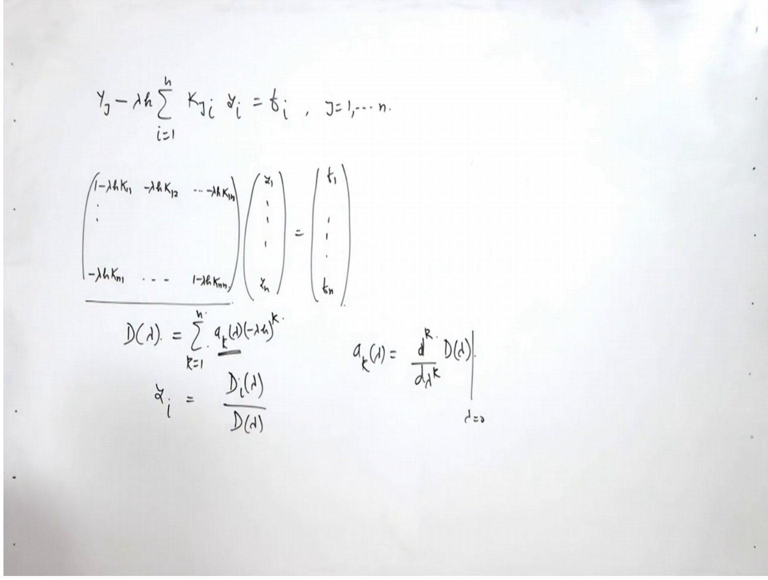

So D n lambda can be written like this so if you look at we want to write down D n lambda as this you can write this as summation you can write a k lambda minus lambda h power k.

So we need to find out what is this a k lambda k is from 1 to n. So if you remember we can do we can find out a k lambda as say d upon d lambda k d k by d lambda k of this capital d lambda and evaluated at lambda equal to 0. So if we can do this then you can get a k lambda but this is little bit evolving, so what we try to do here rather than writing this we try to collect the corresponding coefficient here.

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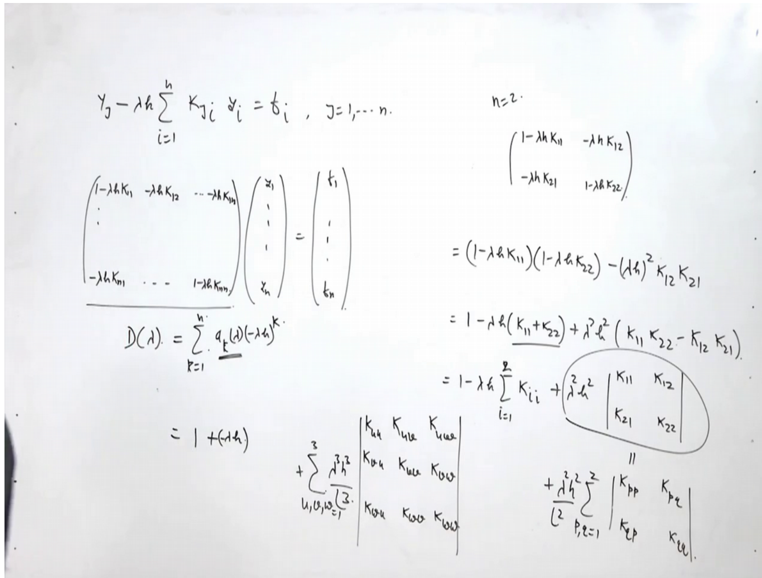

So what we try to do here if you look at the constant term is what if you look at the constant term is nothing but 1 which we can obtain by say multiplying this diagonal terms. So here you can say that the constant term is 1 and then we want to find out say the coefficient of minus lambda h here, so here let me write it what is the coefficient of minus lambda h. So if you look at we want say power one of this minus lambda h so for that we have to take all the determinant having only one column, right?

So for to look at this let us take very easy example let us take two pi two example, so here we simply let us take 1 minus lambda h k 11 minus lambda h k 12 and here we have minus lambda h k 21 and 1 minus lambda h k 22 and this is I am just showing that this example 20 you can this expression given in equation number 20 is the correct one. So for that let us take for n equal to 2 for n equal to 1 it is quite obvious for n equal to 1 it is nothing but 1 minus lambda h k 11 so that is quite obvious here.

So for n equal to 1 this is quite obvious, for n equal to 2 you can write it like this so when you expand this what you will get, 1 minus lambda h k 11 into 1 minus lambda h k 22 and minus lambda h whole square and k 12 k 21. So if you simplify this you can write this as 1 minus lambda h and it is k 11 plus k 22 plus lambda square h square and it is what it is k 11 k 22 and here we also have the same thing minus you can write down this lambda square h square common you can write this as k 12 and k 21.

So for n equal to 2 we try to show that 20 can be obtain by this. So here I can say that it is 1 minus lambda h and this is nothing but say you can write down this as summation i equal to 1 to 2 and you can write it k ii plus lambda square h square and this I can write as you have k 11, k 12, k 21, k 22 so this I can write as determinant of this, okay. So I am saying that here if I write down this this is nothing but you can say that it is k pp and k pq and k qp and k qq here I am just denoting this as something for which p is less than q, right?

So here I am writing this as this expression the value of this when p is less than q. So if I write down general one that summation determinant of this k pp, k pq, k qp, k qq and if we take the summation then they are two such permutation is possible. So I can write this as this second term I can write it as this summation plus lambda square h square summation here summation I am taking at pq from 1 to 2 and if you sting the two such combination then we have to divide by factorial 2.

So if we have two notation p and q then there are only two permutation is possible so you divide by this, okay. So it means that I can write this term as lambda square h square divided by number of the permutation here summation pq from 1 to 2 k pp, k pq, k qp, k qq. So for n equal to 2 you can see that equation number 20 or the expression given in terms of 20 is valid, you can also verify that for power in terms of this lambda cube h cube is given in terms of say k here I am using uu, k uv, k uw and k vu, k vv, k vw and k wu, k wv and k ww.

Now again since we have to take only one but here we are taking in this so if we want to take the summation here then summation over u, v, w and such kind of possibility is factorial 3, so we can write that from 1 to 3 here the coefficient of lambda cube h cube by factorial 3 can be given in terms of this.

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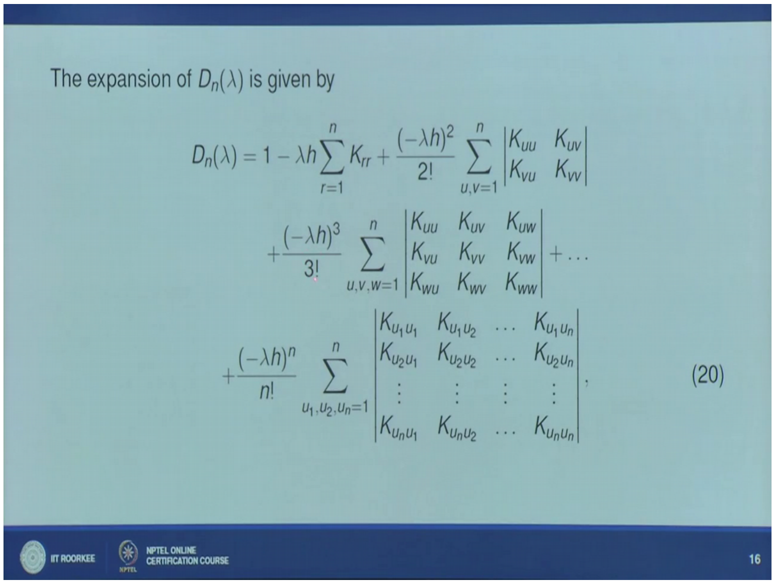

So this I am writing here as minus lambda h to power cube divided by number of the permutation available that is factorial 3 u, v, w from 1 to n and it is determinant like this. So in general you can write that the nth term can be given as minus lambda h to whole power n divided by factorial n which is the number of permutation available for v n to u n and k u1 u1 and this kind of determinant k u1 u1, k u1 u2 and so on.

So this is the expansion of D n lambda in terms of powers of minus lambda h.

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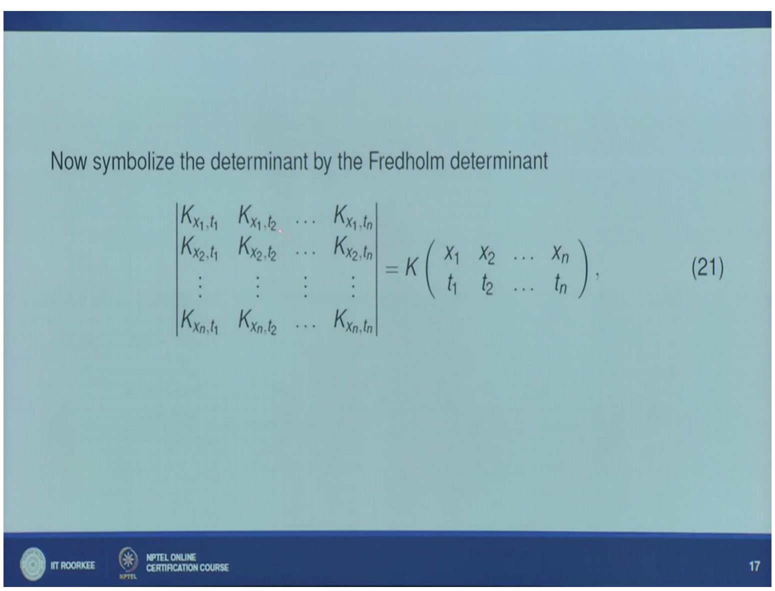

So if we take this and use this the short form for this so here we use this terminology here that if it is k x1 t 1 k x1 t 2 k x 1 tn so if you look at in the first row it is like the first variable is x 1 the second variable is vary from t 1 to t n and if you look at the column here your first (())(18:12) is t 1 is the second argument t 1 is a and first argument is from x 1 to x n.

So if you look at this, this can be summarized as this notation we are just denoting this determinant by k x 1 to x n and t 1 to t n.

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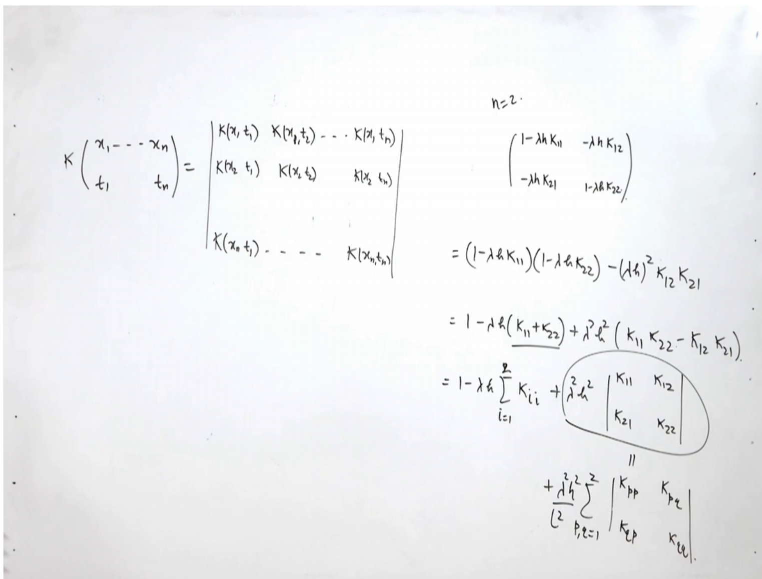

So here what we are trying to write it here, here we are writing this notation K and x 1 to say x n and t 1 to t n now we are denoting this notation as determinant here K x 1 now you take x 1 and then vary for t 1 to t n. So x 1 t 1 K x 2 t 2, sorry K x 1 t 2 and so on K x 1 and t n, right?

So first, now fix for 2, K x 2 and repeat the same procedure and similarly here we have K x n t 1 and K x n t n. So this is the notation we are using and if we use this notation then this determinant which is also known as Fredholm determinant then in this notation your D n lambda can be simplified as this.

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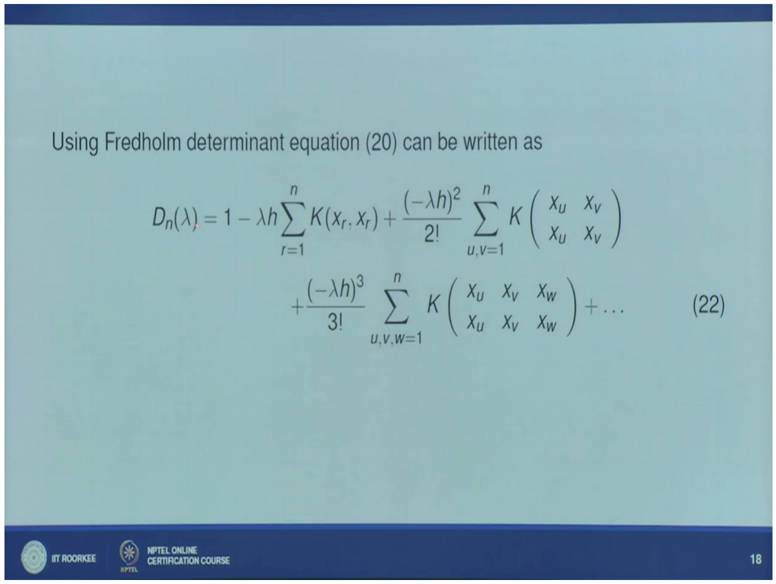

So here D n lambda is written as 1 minus lambda h from r equal to 1 to n K x r, x r plus minus lambda h to power 2 upon factorial 2 u v from 1 to n K x u, x v plus so on, right?

So D n lambda is given by this, now in terms of D n lambda we try to find out the solution of the Fredholm integral equation of the second kind.

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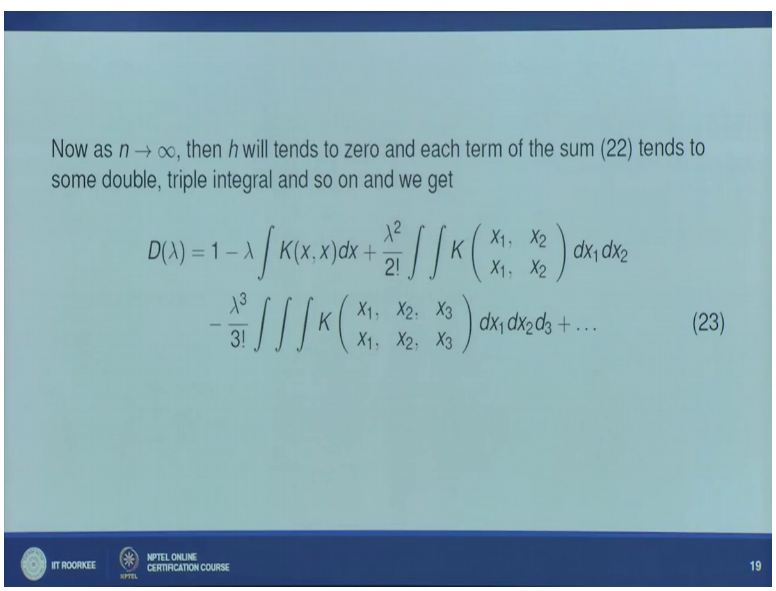

So here now if you remember with the help of this D n lambda you can find out only the values y 1 to y n where y 1 to y n is what y of x 1 and it is y of x n. So it means that with the help of D n lambda we are able to find out the solution only the points given here x 1 to x n but if you want to find out the solution at every given point x between a to b then what to do here we let us assume that this n is standing to infinity.

Now what is this n here, if you remember n is we have defined as we have truncated we have portioned this a b into an equal part and we write x 1 as say a and you can write x n as a plus n minus 1 times h which is given as b. So you can write your h as x minus b minus a by I think this u v have written as a plus n h, okay. So here you can say that let me write it here a plus n h as b, right?

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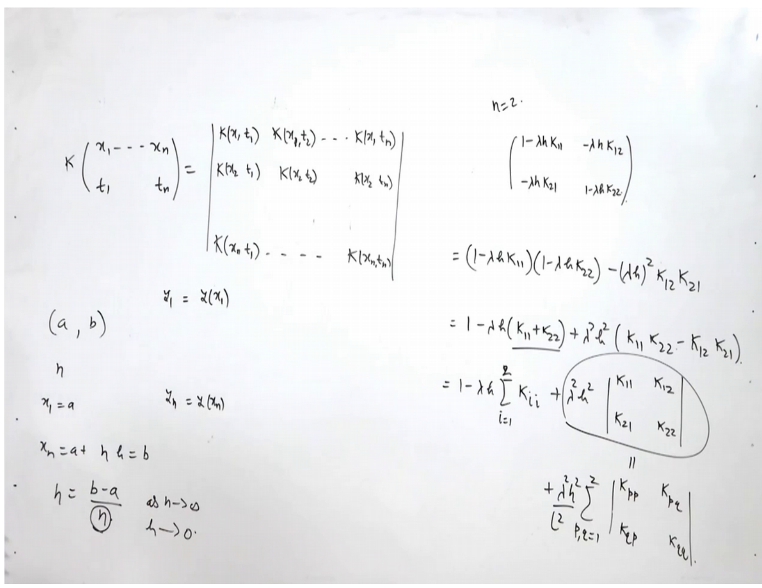

So here we can write this as h as n b minus a by n, okay.

So as n tanning to infinity so as n tanning to infinity your h is tanning to 0. So in that case we want to show that this D n lambda this h is standing to 0 and n tanning to infinity then this finite sum is going to go for infinite sum and we can say that this is going to converge to a limit first one integral and similarly this double summation is converted to your double integral and so on.

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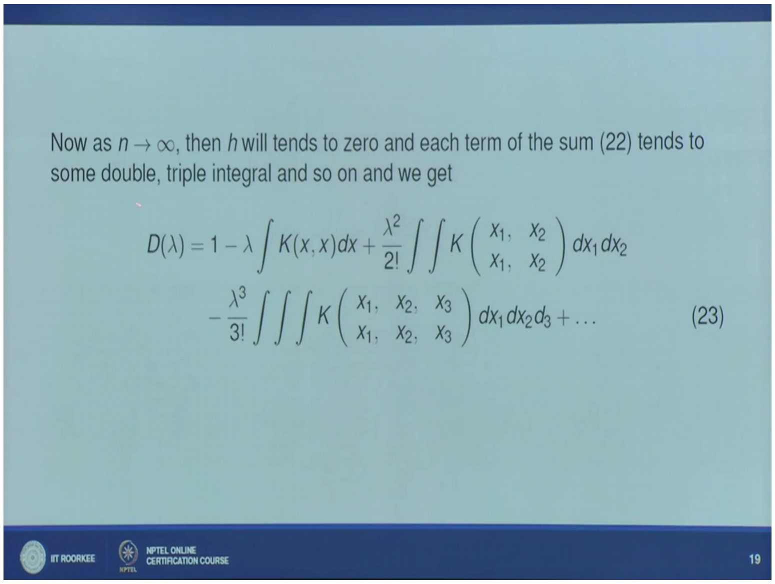

So as n tending to infinity then as we pointed out that h will tends to 0 and each term of the sum 22 means this is going to converted into double, triple integral and so. So if you look at this summation minus lambda h r equal to 1 to n K x r, x r this will go into single integral and your x r is going to x a and this I can write this as 1 minus lambda k x x d of x plus lambda square by 2 here and this this is reduced to here u is going to x and v is going to sum y so you can write this as double integral K x 1, x 1 x 2, x 2, d x 1, d x 2 and so on.

So similarly you can say that as n tending to infinity that this D n lambda is converted to this infinite series given in terms of this three. Now here whatever we have discussed that is just a formal discussion because we have not discussed the convergence part and so on because when I say that when you take n tending to infinity it means that this is going to be infinite series. So I can talk about convergence and all this thing provided I know the convergence part.

So it means that I should know that the series which I am writing here is convergent and then only I can write that the limit n tending to infinity D n lambda I can write only D lambda.

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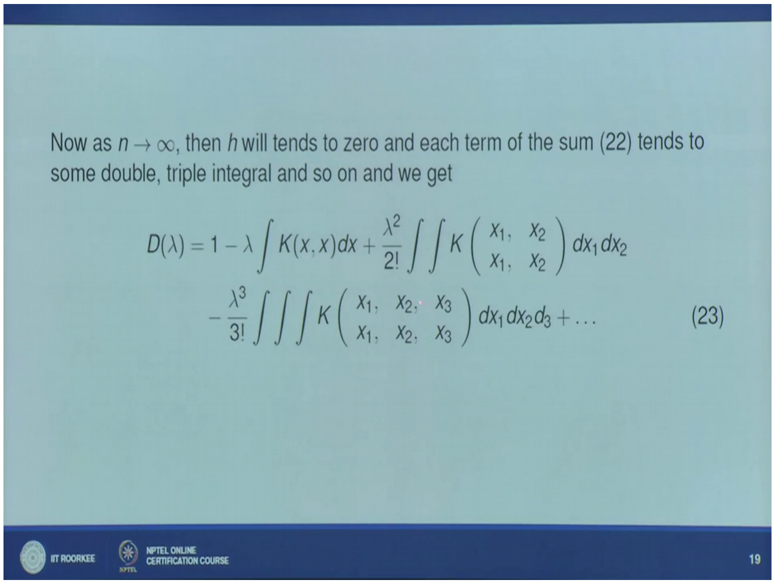

So here if it converges then only I can write D lambda as this. So here I am not giving the (()) (24:19) here so we have seen that as n tending to infinity then this h will tends to 0 and each term of the sum tends to some double, triple integral and so on and we can say that this one single summation will go to single integral double summation will go to double integral and so on because n tending to infinity and this h is tending to 0.

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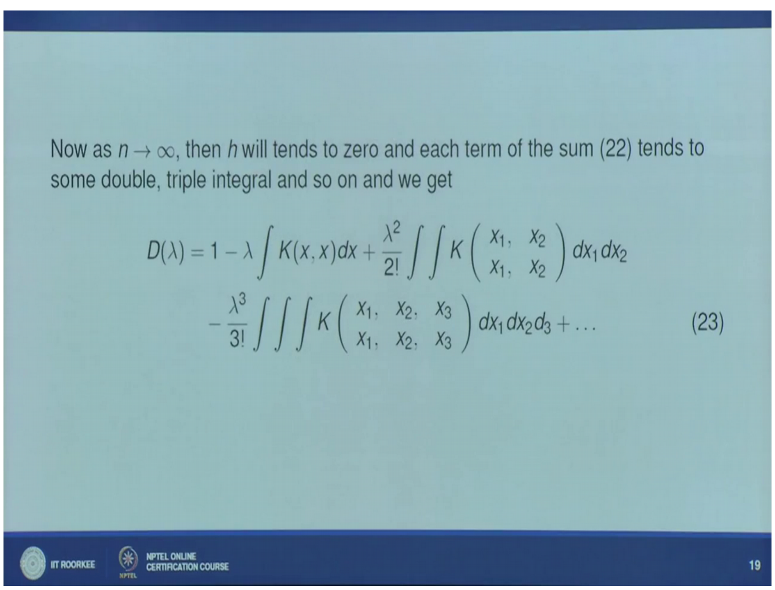

So here we can write that the limiting case of D n lambda is given by D lambda and it is given by 1 minus lambda K x x dx plus lambda square upon factorial 2 double summation K x 1 to x 2 given in this notation dx 1 to dx 2 and so on. Now here we assume that the convergence is given but this Fredholm discuss the convergence part provided that this kernel K x, y is bounded and (())(25:31).

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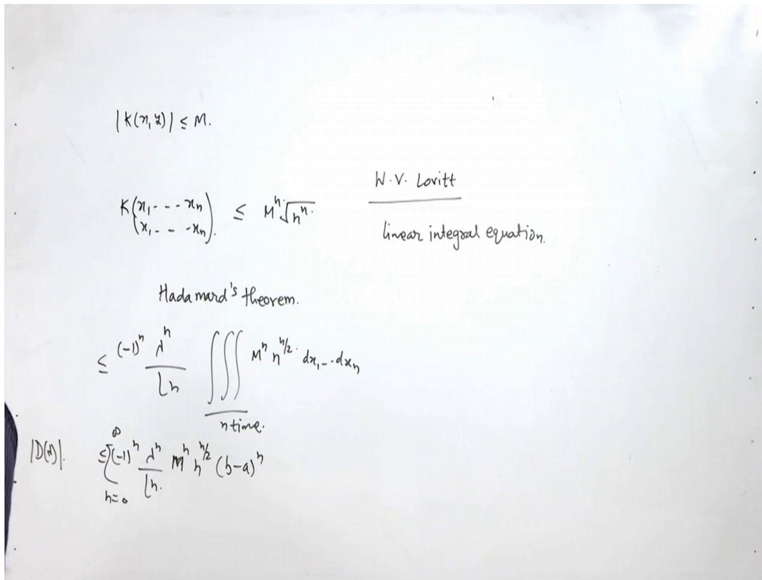

So he assumed that if modulus of K x, y is less than or equal to m then the expression this K x 1 to say x n x 1 to x n is bounded by M to power n factorial n to power n.

So this is given by a theorem known as Hadamard's theorem and by which we can show that the nth term is bounded by minus 1 to power n lambda to power n upon factorial n and this integral n times and this is further bounded by so this nth term is bounded by this and you can write this as M to power n n power n by 2 dx 1 to dx n and if you further simplify it is bounded by minus 1 to power n lambda to power n factorial n here and this is M to power n capital M to power n n to power n by 2 and this is nothing but b minus a to power n here.

So you can say that this series your d lambda modulus of d lambda is bounded by this summation n is from 0 to infinity and we can show that this series is convergence convergent then we can show that this series is also convergence by comparison test. So to show that it is converge you simply apply the ratio test and you can prove it. This procedure the including this Hadamard theorem which says that this is less than or equal to this is given by a book by W. V. Lovitt and title is linear integral equation.

So there you can find out that the convergence of d lambda is discussed and it is proved that this d lambda converges as an infinite series for all values of lambda so it is this converges is absolute and uniform for all values of lambda. So using this d lambda know we want to find out the solution y of x at any given point x between a to b for general kernel the only condition on kernel is that it is bounded and integrable.

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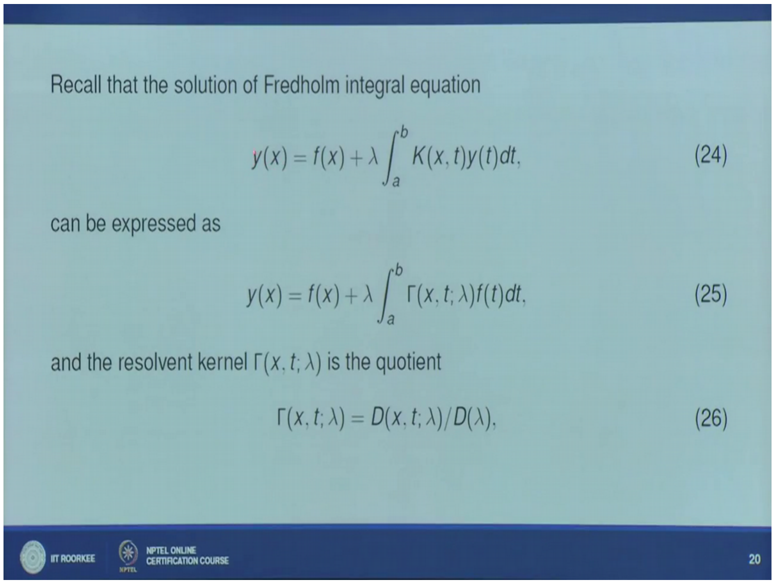

So if you remember we have discussed that in case of separable kernel and case of successive approximation that if we have this kind of equation yx equal to fx plus lambda a to b K x, t yt dt then the solution of this problem can be reduced to in the expression yx equal to f of x plus lambda a to b resolvent kernel gamma x t lambda ft dt where resolvent kernel gamma x t lambda is given by this ratio D x, t, lambda divided by D lambda provided that D lambda is non-zero.

So I can we try to find out in next lecture that how we can find out this resolvent kernel and what is the guarantee that it exists and how to find out this quantity D x, t, lambda. So this we have already obtained in the case of separable kernel in previous lecture in this next lecture we are going to say that in this case also for general kernel how we can find out this D x, t, lambda and how we can find out this gamma x, t, lambda so that we can find out the solution yx given as equation number 25.

So we will meet in next lecture to see how to find out this, so thank for listening us, thank you.