Integral Equations, Calculus of Variations and their Applications
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Lecture 02
Conversion of IVP into integral equations

Hello Friends I welcome you all to my lecture on Conversion of Initial Value Problem which we write in short as IVP into integral equations.

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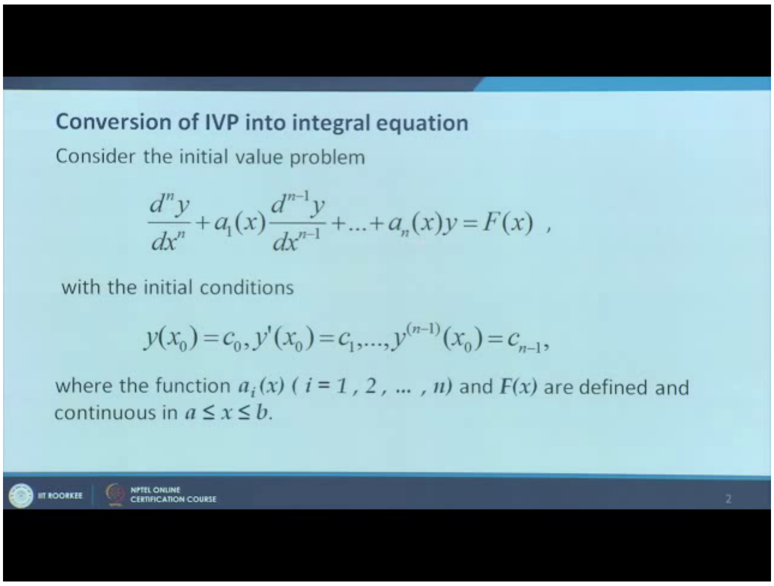

Let us consider an initial value problem where we have a n th order differential equation ordinary differential equation d n y by d x n plus a 1(x) plus d n minus 1 y by d x n minus 1 and so on a n x into y equal to F(x). And the initial conditions are y (x 0) is let us say c 0 y dash (x 0) is C 1 and y n minus 1 at derivative of y(x) at x 0 is equal to C n minus 1.

And let us assume that the coefficient functions A I x here in the differential equation are continuous in the close interval [a, b]. And the function f (x) on the right side is also continuous in the closed interval [a, b]. Ok, so we will be converting this differential equation this initial value problem to an integral equation.

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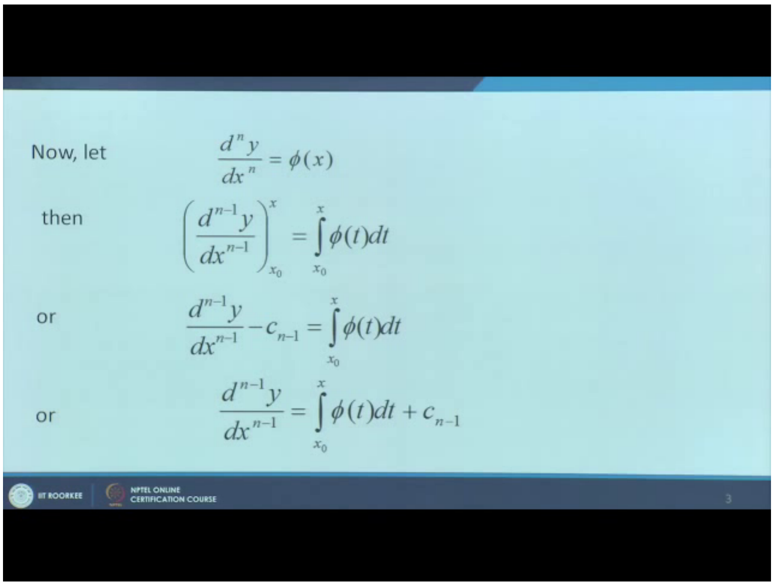

So let us see we will we start with let us define d n y over d x n equal to Phi(x). Then we integrate this with respect to x over the interval (x 0 to x).

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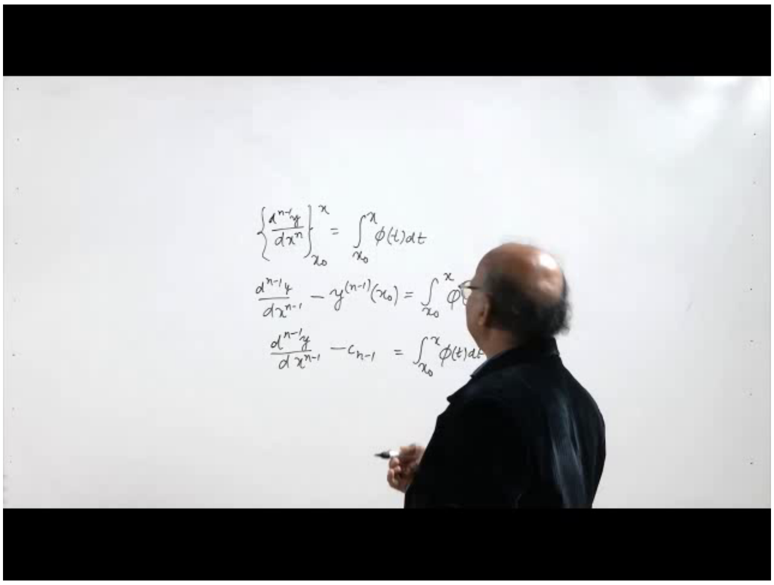

So we shall have {d n minus 1 y over d x n minus 1} (x 0 to x) equal to integral (x 0 to x) Phi t dt. Now this is n minus derivative d n minus 1 y over d x n minus 1 minus n minus 1 of the derivative at the point x 0 is equal to integral x 0 to x Phi t dt. Since we are assuming y n minus one of the derivative y(x 0) to be 0 sorry to be equal to c n minus 1 we get.

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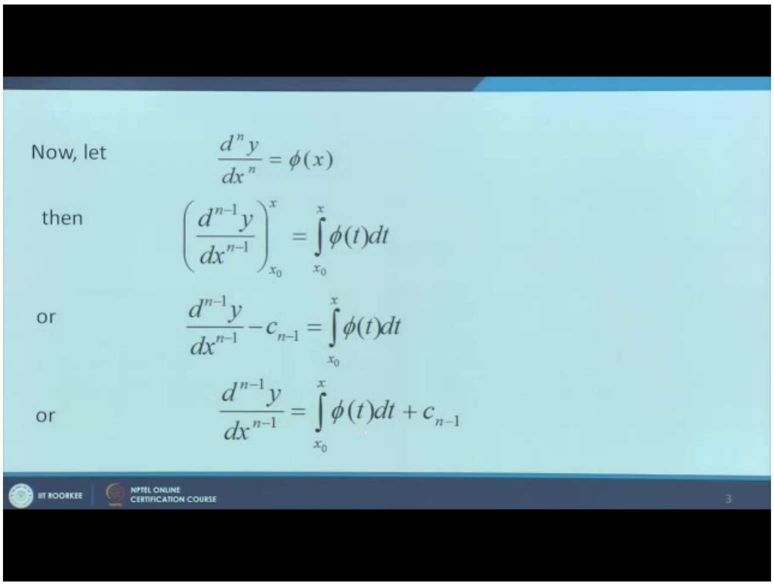

or we can say d n minus 1 y over d x n minus 1 is equal to integral x 0 to x phi t dt plus c n minus 1.

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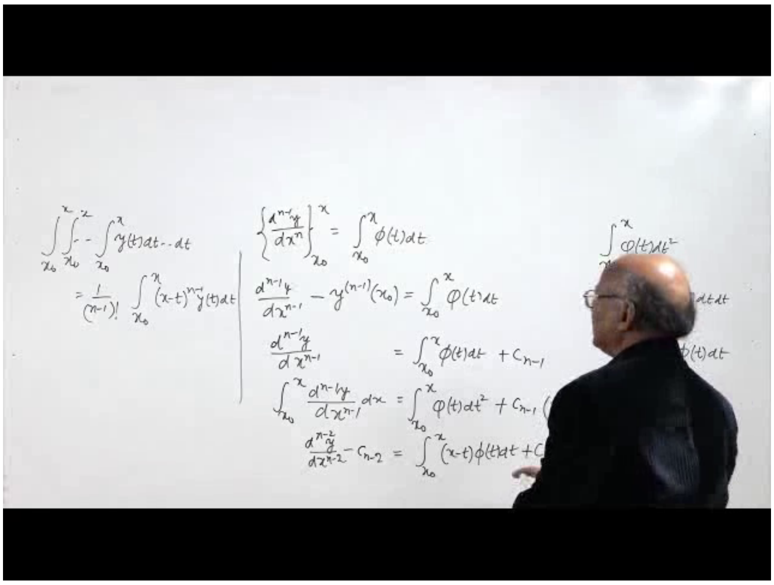

Let us again integrate both sides, so we can write it as let us again integrate it with respect to x over the interval x 0 to x, so x 0 to x we will have integral x 0 to x Phi t dt square this is nothing but double integral of phi t and the two times integral of Phi t with respect to t . So this is plus c n minus 1 x, x 0 to x.

So when we integrated with respect to x over the interval x 0 to x what we get is this and this is nothing but d n minus 2 y over d x n minus 2 minus the value of d n minus 2 y over d x n minus 2 at the point x 0 which we write c n minus 2. Now this integral x 0 to x phi t dt square this is nothing but we are writing this is x 0 to x phi t dt dt, ok.

Now this is nothing but integral x 0 to x x minus t into phi t dt because we have already seen that the n integrals over x 0 to x y t dt is equal to 1 over n minus 1 factorial integral x 0 to x x minus t raise to the power n minus 1 into y t dt. So using this formula for n equal to 2 we get this as x 0 to x x minus t into Phi t dt plus c n minus 1 x minus x 0 I can write like this.

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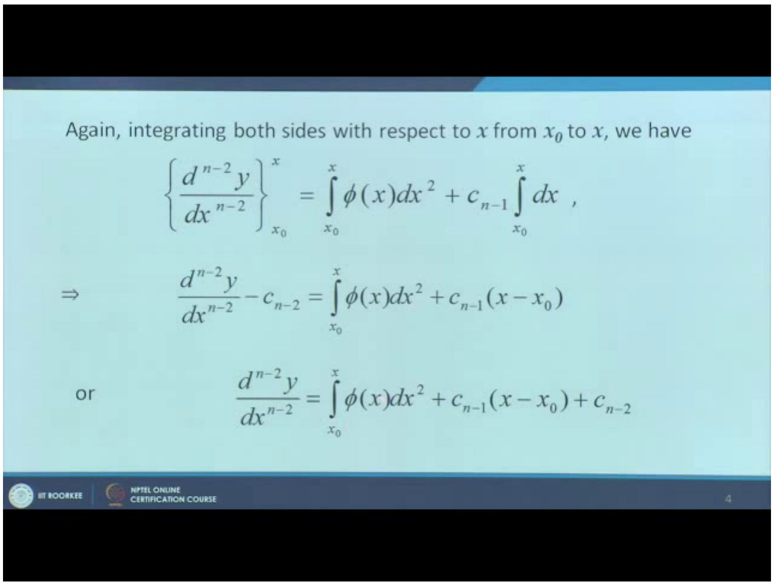

So what we get is y n minus 2, n minus 2 are the derivative of y is equal to integral x 0 to x phi t dt square which is this is integral x 0 to x, x minus t into phi t dt plus c n minus 1 x minus x 0 plus n minus 2. Thus we get d n minus 2 y over d x n minus 2 equal to integral x 0 to x x minus t phi t dt plus c n minus 1 x minus x 0 plus c n minus 2.

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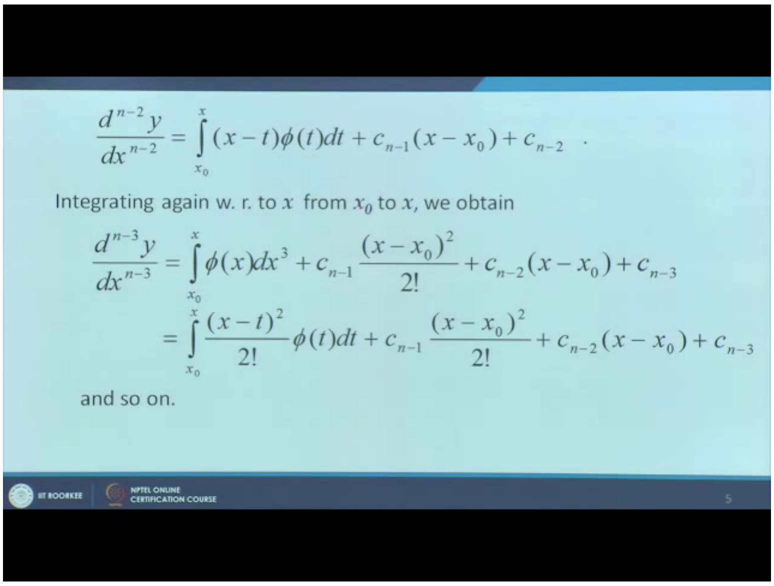

Now let us integrate it again with respect to x from x 0 to x then we shall get the n minus third derivative of y with respect to x as there is three times integral of phi (x) with respect to x. So during this formula we will be getting integral (x 0 to x), (x minus t) whole square over 2 factorial phi t dt and then (c n minus 1) (x minus x 0) whole square by 2 factorial (c n minus 2) (x minus x 0) plus (c n minus 3) and so on.

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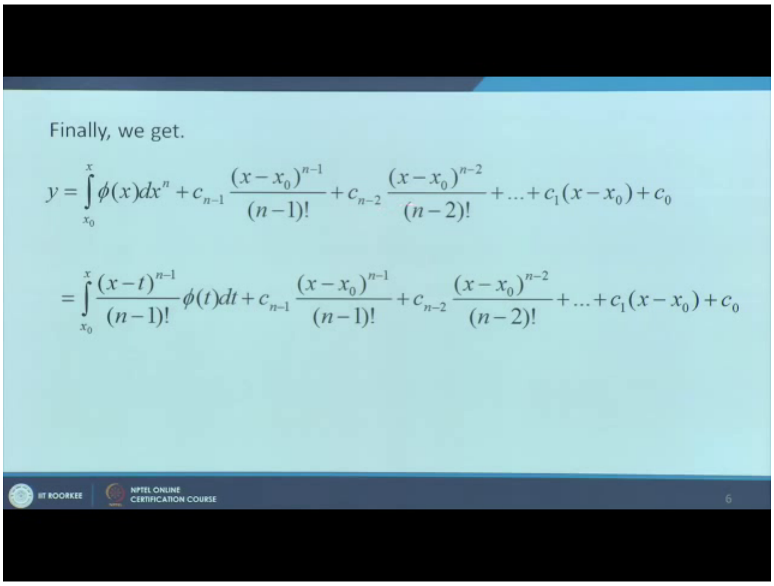

So after n integrals n times integrals taking integration n times we will get y equal to x 0 to x phi x d x n this means that n th derivative n integrals of phi(x) with respect to x over the interval x 0 to x plus c n minus 1 x minus x 0 to the power n minus 1 over n minus factorial and so on c 0 is equal to (x 0 to x),(x minus t) to the power (n minus 1) over (n minus 1) factorial phi t dt plus (c n minus 1) (x minus x 0) to the power (n minus 1) over (n minus 1) factorial and so on c 0.

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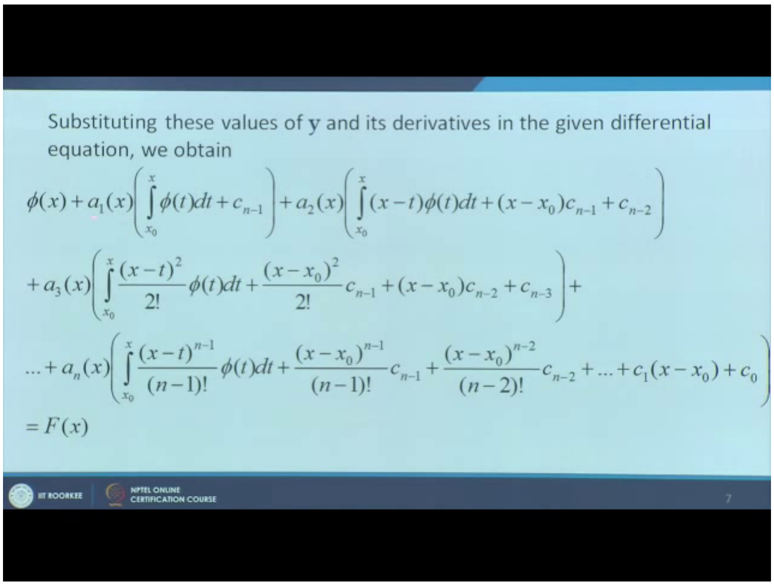

Now let us put these values y and its derivatives in the given initial value problem in the given differential equation we shall have phi (x) which is the expression for n th derivative of y with respect to x.

Then a 1 times integral x 0 to x phi t dt plus c n minus 1 this is the expression for n minus 1) th derivative of y with respect to x and a 2 (x) into this, this is the expression for (n minus 2) th derivative of y with respect to x and then a 3(x) times this a n(x) times this is y. The expression for y equal to f(x)

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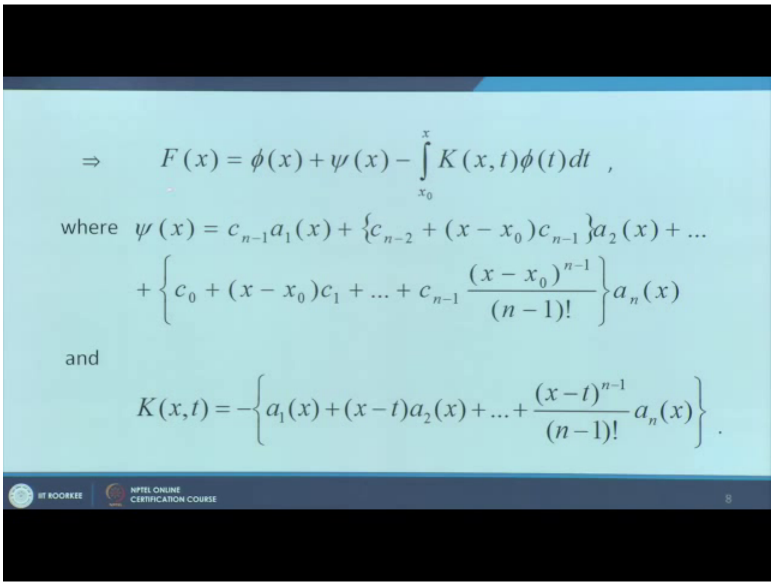

And then collecting coefficients we can then write phi(x) we can write this equation the previous equation as f(x) equal to phi(x) plus psi(x) minus integral x 0 to x k(x,t) into phi t dt where psi(x) is nothing but c n minus 1 a 1(x) plus c n minus 2 into x minus x 0 c n minus 1 a 2(x) and so on.

This expression into a n(x) so this is psi(x) then k(x,t) is the function of x and t which is minus a 1(x) plus x minus t into x and so on x minus) to the power (n minus 1) over (n minus 1) factorial a n(x). So we can write this equation in a simplified form like this after collecting the coefficients. So f(x) is equal to phi(x) plus psi(x) minus integral over x 0 to x k(x, t) phi t dt.

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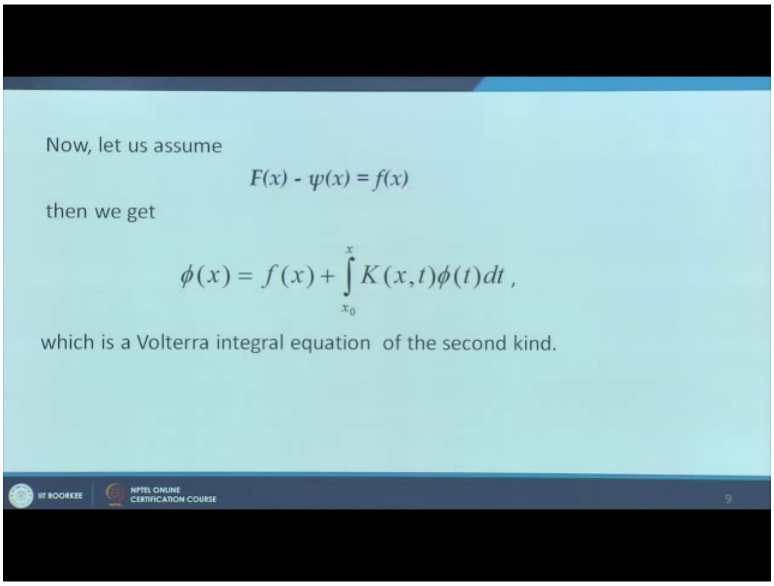

Now what we do is let us assume f(x) minus psi(x) equal to small f(x) then we shall get if you consider f(x) minus psi(x) to be small f(x) then f(x) equal to phi(x) minus integral x 0 to x k(x,t) phi t dt or we can say phi(x) equal to f(x) plus integral x 0 to x k(x,t) into phi t dt which is a Volterra Integral equation of the second kind where f(x) is known to us k(x,t) is also known to us and phi(x) is the unknown function.

So then how we convert an initial value problem to Volterra Integral equation of the second kind we can solve this Volterra Integral equation of the second kind for the unknown function phi to obtain the solution of the initial value problem.

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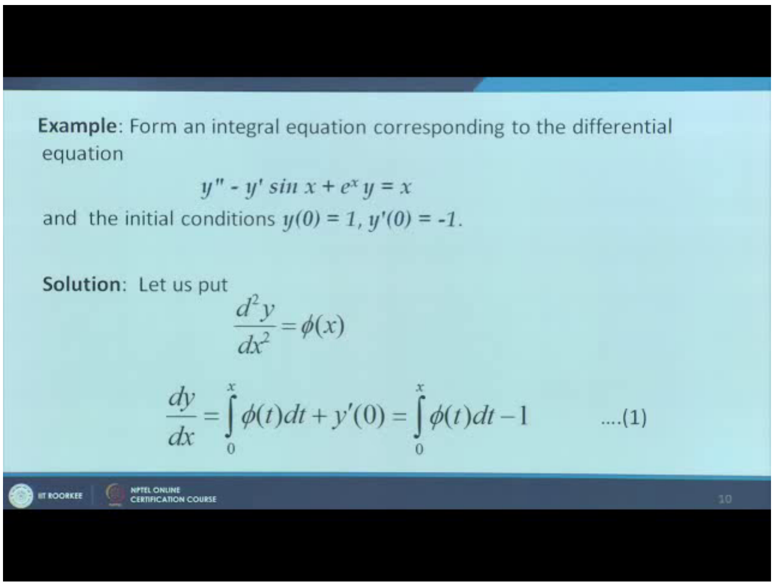

Now let us discuss an example of an initial value problem and see how we can convert it to Volterra Integral equation of the second kind? So this is second order differential equation y double dash minus y dash sin x plus e to the power x into y equal to x the initial conditions are y(0) equal to 1 and y dash(0) equal to minus 1.

So here x0 point is x 0 equal to 0 and we are given the two conditions for the second order differential equations. So let us put d square by the highest derivative of y with respect to x as equal to phi(x). So d square y over dx square equal to phi(x) we put then we integrate it with respect to x so we get dy by dx at x and dy by dx at 0.So d square y over d x square equal phi(x),

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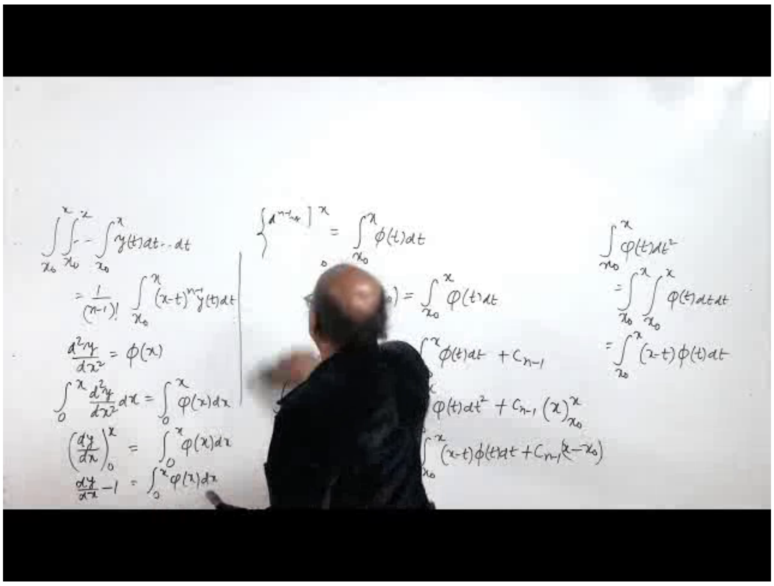

let us integrate over the interval x o to x x 0 is 0 here so we integrate it over the interval 0 to x with respect to x. And what we get is dy over dx. Now dy by dx at 0 you have been given as minus 1 so we will get here dy by dx minus 1 equal to this integral. And therefore what we get is dy by dx equal to integral 0 to x phi t dt plus y dash 0, y dash 0 is minus 1 so we get integral 0 to x phi t dt minus 1.

Again we integrate this equation with respect to x when we integrate it again with respect to x what we will get is?

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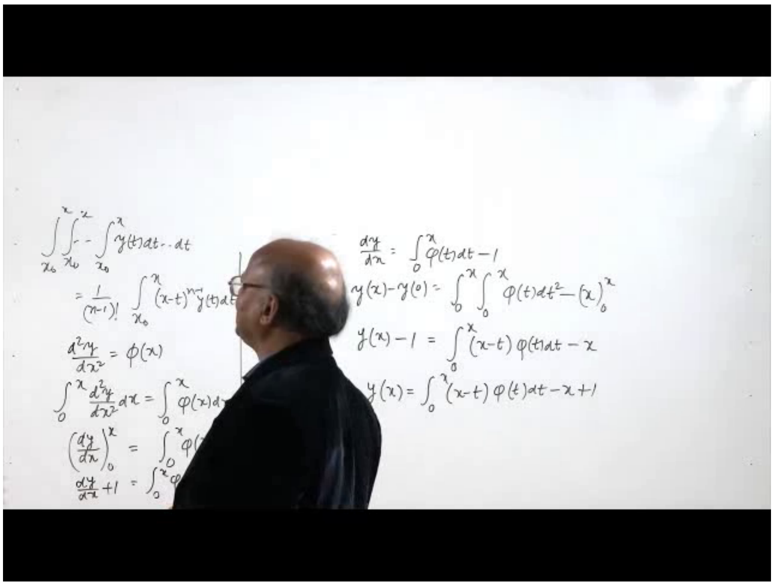

See we have dy by dx equal to integral 0 to x we can write phi t dt plus 1. Now we integrate it again with respect to x over the interval 0 to x. So y at x minus y at 0 we have minus we get this, ok. Now y at 0 is going to be equal to 1 so we have y (x minus 1) plus this is integral 0 to x (x minus t) into phi t dt plus x.

So what we get is y(x) equal to 0 to x (x minus t) phi t dt minus x y dash 0 is given to be equal to minus 1 y dash 0 is given to be minus 1 so we get here this is actually plus 1 here, so what we get is minus here so what we get is y(z) equal to this. So using the value of y(x 0) as y we get the expression for y.

Now substituting the value of y and dy by dx and d square y by dx square in the given differential equation we will have phi (x) equal to x minus sin x plus x minus 1 into e to the power x plus this which is the Volterra Integral equation of the second kind. Let us see how do we get this.

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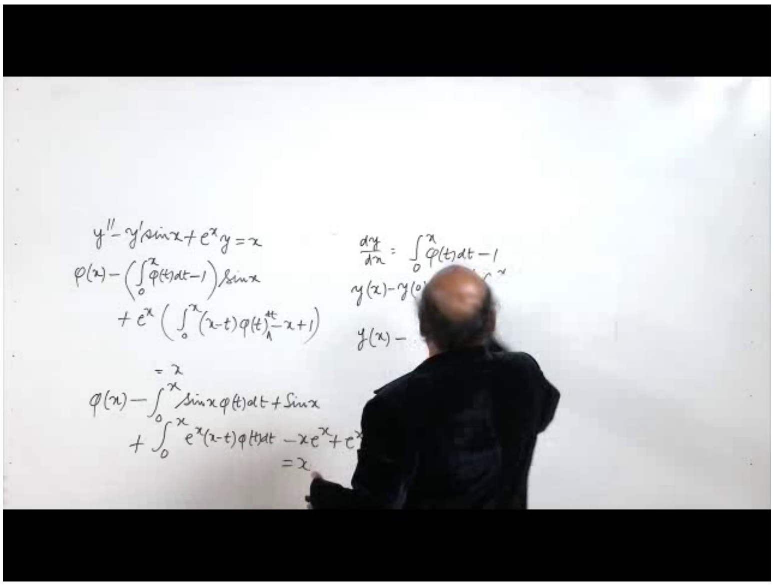

So the given equation is y double dash minus y dash sin(x) plus e to the power x into y equal to x y double dash is equal to phi x minus y dash is equal to integral 0 to x phi t dt minus 1 into sin x plus e to the power x into y and y is equal to integral 0 to x, x minus t into phi t dt minus x plus 1 this is the value of y equal to x.

So what we get is phi (x) minus integral 0 to x sin (x) into phi t dt plus sin (x) plus integral 0 to x e to the power x into (x minus t) into phi t dt. Here I have dt here minus x e to the power x plus e to the power x equal to x this is what we have. Now we can rearrange terms here and what we will get is from here we will get the following.

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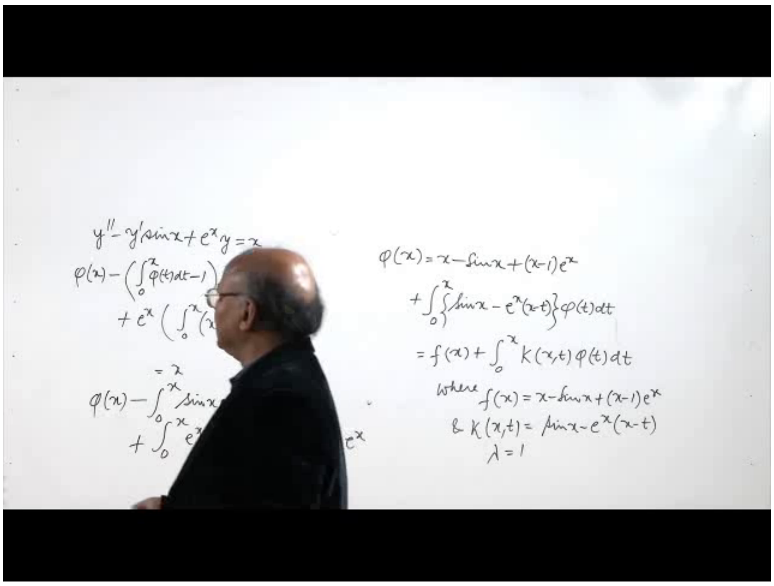

Phi(x) equal to x minus sin (x) and then these terms come to the right side as (x minus 1) into e x plus integral 0 to x, sin (x) minus e to the power x into x minus t into phi t dt. So this is f(x) which is equal to f(x) plus integral 0 to x k(x,t) into phi t dt where f(x) is x minus sin x plus x minus 1 e to the power x and k (x,t) is equal to sin (x) minus e to the power x into x minus t and lambda is equal to 1.

So we get f(x) equal to phi (x) equal to f(x) plus 0 to x k(x, t) phi t dt so which is the Volterra Integral equation of the second kind.

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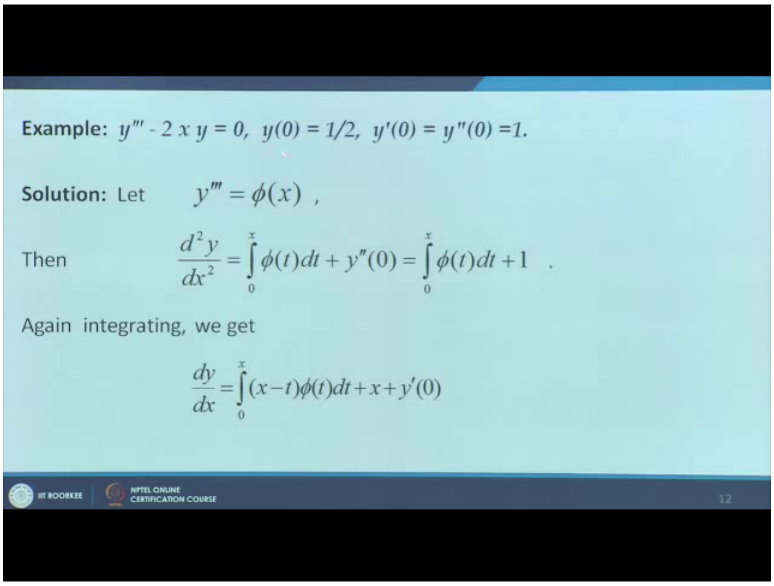

So we can convert the initial value problem in this manner. Another example we can consider say y triple dash minus 2 x y equal to 0 where we are given three conditions y(0) is equal to half y dash 0 equal to 2 y double dash 0 equal to 1.

So as usual let us start with the highest derivative of y with respect to x that is y triple dash we assume it equal to phi x integrated once with respect to x. Since x 0 is given here 0 so we integrate over the interval 0 to x. So d square y over dx square over the interval 0 to x will be d square y over d x square and its value at x equal to 0. Its value x equal to 0 is given to be equal to 1.

So we will get d square y over d x square equal to 0 to x phi t dt plus y double dash 0 and y double dash is equal to 1 so we will get integral 0 to x phi t dt plus 1. Now let us integrate it again with respect to x we will get dy by dx evaluated at x and 0. At 0 its value is 0 to 1 so we will get integral 0 to x (x minus t) phi t dt and integral here of the right side.

When you integrate the right side we will have double integral here so double integral of phi t with respect to t so we will get that can be converted into a single integral 0 to x x minus t phi t dt integral of 1 is x and over interval 0 to x it is x and then we add here y dash 0. So y dash 0 is equal to 1 so we get here dy by dx equal to integral 0 to x (x minus t) phi t dt plus x plus 1.

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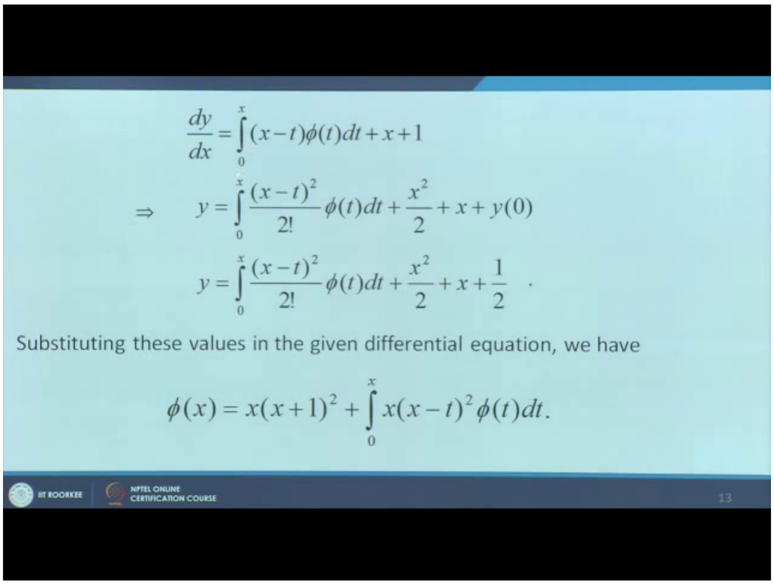

Again we integrate with respect to x to get y equal to integral 0 to x (x minus t) whole square 2 factorial by phi t dt and then here we get x square by 2 plus x plus the value of y x 0 and value of y x 0 is one to be equal to half. So we put that value and we get y equal to integral 0 to x (x minus t) whole square by 2 factorial phi t dt plus f square by 2 plus x plus half .

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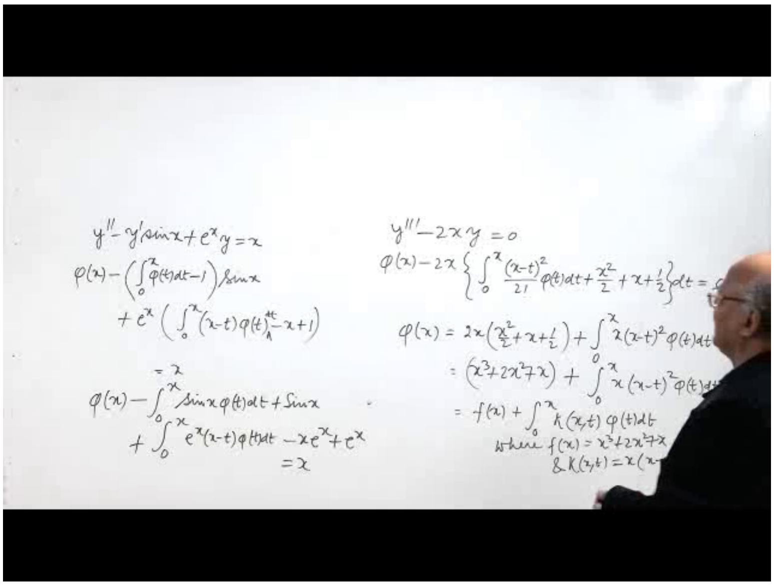

Oh substitute the let us substitute these values in the given differential equation. So the given differential equation is y triple dash minus 2 x by equal to 0, y triple dash is equal to phi here minus 2x times let us put the value of y, the value of y is integral 0 to x (x minus t) whole square by 2 factorial and phi t dt plus x square by 2 and we have x plus half equal to 0.

So we can write it as phi x equal to when we multiply by 2 x times this quantity so we have 2x into x square by 2 x plus half and then we have integral 0 to x x into (x minus t) whole square into phi t dt so this is x cube plus 2 x square plus x integral 0 to x. So this is your f (x) and lambda is 1 here k(x,t) is x into x minus t whole square. We can write like this,

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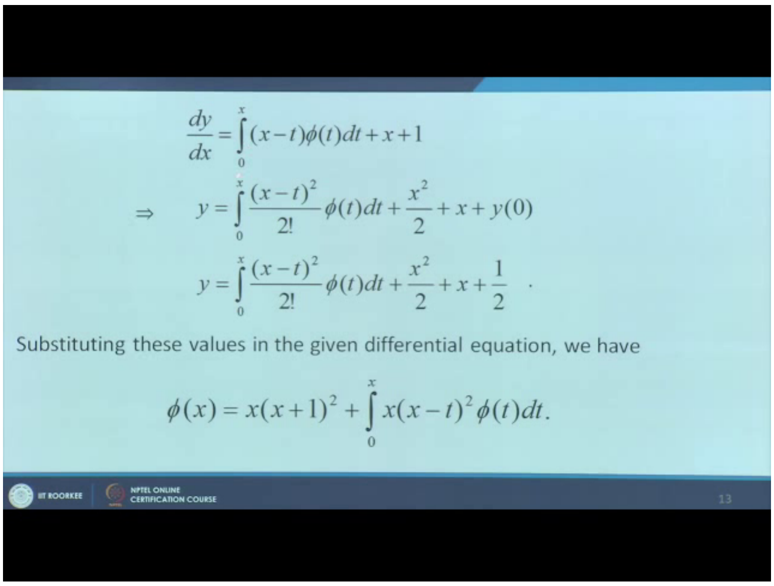

So we get Volterra Integral equation of the second kind. This x cube plus 2 x square plus x can be written as x times x square plus 2x plus 1 which is x times (x plus 1) whole square. So we get Volterra Integral equation of the second kind.

So this is how we convert an initial value problem to a Volterra Integral equation by here what we do is the highest derivative of y with respect to x we assume it as a function phi (x) and then go on integrating to get y and then substitute y and (())(24:19) derivatives in the given differential equation and arrange it into the form of an integral equation .

This is what I have to say in this lecture Thank you very much for your attention.