Course on Integral Equations, Calculus of Variations and their Applications
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Lecture 19
Fredholm alternative Theorem - 2

Hello friends, let us continue our discussion for Fredholm integral equation of second kind when kernel is given as separable kernel.

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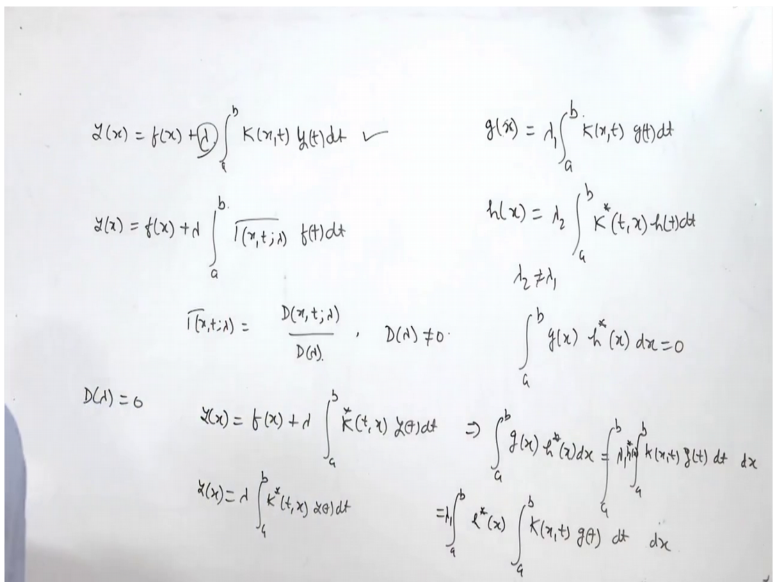

So there we have seen that if the determinant of the coefficient matrix denoted as d lambda is non-zero then the equation yx equal to f of x plus lambda a to b K x, t yt dt has a unique solution and it is given as y of x equal to f of x plus lambda a to b gamma xt lambda gamma xt lambda ft dt where gamma xt lambda is given as D x, t, lambda upon d lambda and d lambda is the coefficient matrix here.

And here we have assumed that d lambda is non-zero. Now the problem is that if d lambda so if this value of lambda happen to be the root of this coefficient matrix d lambda then this solution is not defined then we have to find out say conditions on f such that we may have solution. So in general we do not have any solution of this, right? But there are certain conditions when this f is having certain feature for which we have still have a solution of this particular problem.

So in case of d lambda equal to 0 to find out the solution we consider the transpose equation of the original equation the given equation, so how we define transpose equation? We define transpose equation like this y of x equal to f of x plus lambda a to b K star t, x y t dx here K star is the conjugate transpose of K x, t. So here this is the known as transpose equation or some time people call it adjoin equation of the first one.

So here we say that we try to find out the condition so here before starting with anything we try to discuss certain properties here, so first property we want to discuss is that the eigen function corresponding to distinct eigen values of transpose equation and the given equation are orthogonal to each other means what that suppose we have a eigen value like here let me write it here that use some other notation let me write it say here we write it say g of s equal to say lambda a to b and here we have K of x, t and let me use this K x g of x.

So here we have g of x equal to lambda 1 K x, t gt dt, so this is this implies what? That g is the eigen function of the equation of this Fredholm integral equation corresponding to lambda 1 and let us say that we have h of x which is lambda 2 a to b and we here we have a transpose of this K square t of x h of t dt. Now my claim is that this g and h which is eigen function corresponding to lambda 1 of the problem and h is the eigen function corresponding to lambda 2, here lambda 2 is non-equal to lambda 1 and eigen function of the transpose equation.

And we are claiming that this g of x and h of x is orthogonal to each other, what do you mean by this? It means that a to b g of x h star x d of x is equal to 0, so this we want to prove. So to prove this what we do here we simply multiply by h star x here and integrate and we do the same thing here. So what you will get here let me write it here so here we have a to b g of x h star x d of x is equal to what this is lambda 1 a to b and here we have K of x, t g of t dt and here I am writing a to b and this is what I am writing this, okay so here it is what multiplying this as h star x h star let me write h star x here and then d of x, right?

So now this I can write it here a to b and here we can write it here simplify this. So what we try to do here we can write this as lambda 1 I can take it out then h star x so h star x and here I can write it a to b K of x of t g of t dt and d of x. Now here we want to find out say expression of this so here we can find out h star x h star x is basically what you can take the transpose of this and we can put it here value and then we can simplify what should be the value here, okay. So here or you can say that we can change the order here, so when you change the order what you will get?

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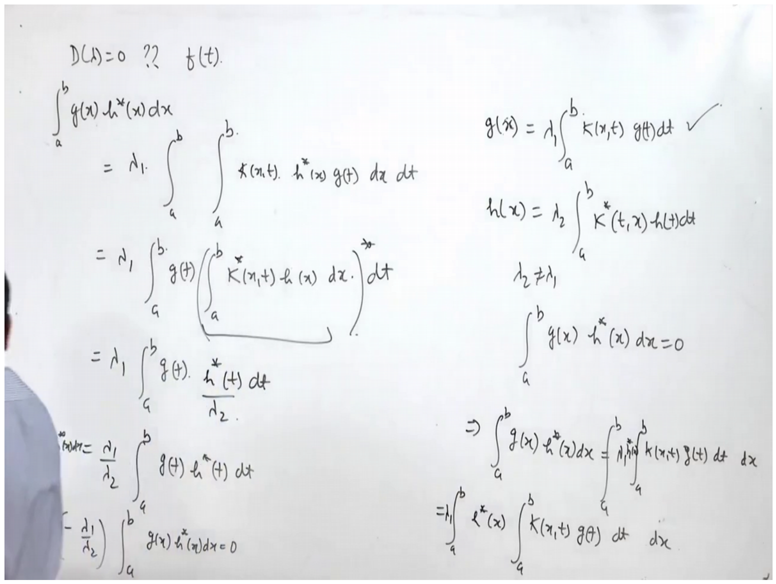

So here we have this is what? This is a to b g of x h star of x d of x is equal to lambda 1 now I am just changing the order here then we have a to b and this is what you will get K of x, t I am writing h star x and g of t and we have dx here and dt here then I can take gt out.

So we have lambda 1 a to b and then it is g of t a to b K x, t h star x and we have d of x. So this I can write lambda 1 a to b g of t, now here what I can write here this I can write this as what? So from here look at this how I can write it this, this I can write here I want to so here this is what I can write this transpose of this so dt I have missed. So I can take this as transpose here and then it is, okay so this I can write it here, okay so this I can write it here so transpose of this so it will go to this I can write it like this.

Now this is what? This is a to b K star x, t h x dx is basically what? It is giving you this, so here I can write gt and I can write this as h star t dt, is it okay? Divided by say lambda 2, right? So I am using this expression and we are writing the value of this so it is a to b K star t, x ht dt is nothing but hx divided by lambda 2 and conjugate transpose of this. So this is what lambda 1 divided by lambda 2 a to b g of t h star t dt here and this is nothing but a to b and g of x h star x d of x.

So t is just a dummy variable so I can replace it by x so this I can write 1 minus lambda 1 upon lambda 2, and here we have what? We have a to b g of x h star of x d of is equal to 0. Now lambda 1 is not equal to lambda 2 so this implies that gx is orthogonal to h star x. So here we can say that the eigen function corresponding to lambda 1 of this equation and the eigen function corresponding to distinct eigen value of the transpose equation are orthogonal to each other, right?

Now with the help of this we try to find out the condition that what happened if d lambda equal to 0 then what should be condition on your f of t so that we have a solution here.

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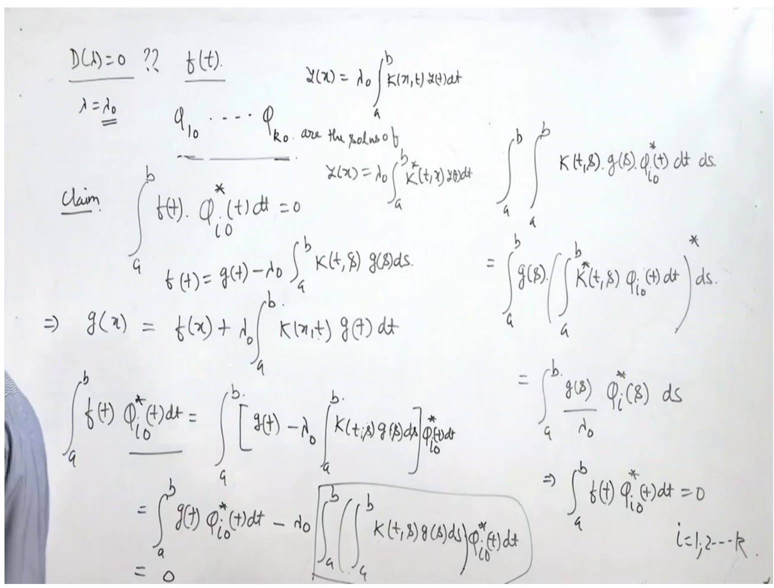

So for that so we want to know that when d lambda equal to 0 what should be the condition on f of t so that our equation has a solution. So here if you take that d lambda is equal to 0 let us say that lambda is equal to lambda not where lambda not is the solution of this root of this determinant d lambda, then we can call this as eigen value and let us say that the corresponding to this lambda not we have a an eigen function say let us say we have say phi 1 corresponding to phi 1 not say phi 1 not to say phi 1 say let us say phi k not.

So we let us say that corresponding to this lambda not we have these many eigen function of the equation this y of x equal to lambda not a to b K of x, t your yt dt. So it means that this phi 1 not to phi k not satisfy this homogeneous integral equation. Now we claim that we want to claim that if f of t is orthogonal to all phi i not t dt is equal to 0 then and then only we have say we have some solutions of this equation y of x equal to lambda not a to b K x, t yt dt.

So for that let us say that how we can write this so here let us say that let us say that we have a solution of the problem, so let us say that gx has a solution of what of the non-homogeneous condition that gx equal to f of x plus lambda a to b K of x, t g of t dt, right? So let g is one of the solution of this and then our claim is that if f satisfy this condition then we have a solution. So to show this let us say this so a to b f of t phi i not t dt and this I can write as, so this I can write here as a to b and I am writing the value of f of t.

So f of t is basically I can get is from this so here we have this g of t plus minus lambda minus lambda not a to b we are considering for lambda is equal to lambda not so let us say lambda not. So a to b and we have what K of x, t and g of t dt, right? So this is what we are writing here. So let us use some other symbol otherwise so here let us use some let us say s you are using so gx s d of s and this I am writing as so here it is t and dt here.

So what I am doing I am just writing the value of f of t so how we are getting the value of f of t from this I am getting value of f of t when I am writing f of t then I have to use some other dummy variable for this integration. So here I am using dummy variable as s and then I am writing here, so I am writing the value of f of t so how I can write so I can write f of t as g of t minus lambda not and a to b and here I am writing g of t and this is s, so integration variable is s g of s d of s, right?

So I am just using the value of ft so which I am writing here g of t minus lambda not a to b K t, s gs ds and then I am writing phi i not t dt, right? And then we try to get some kind of condition on ft. So this I can write it simplify a to b g of t and then when we have this then we have phi i not t dt minus lambda not a to b we have what this is lambda not a to b K t, s g of s d of s phi i not t dt, is it okay?

Now what we want to show is that ft is orthogonal to this means I have to take the conjugate of this, so please orthogonal t means this is the conjugate. So here please correct this and then we have this and then we have this and then we have this, is it okay? So here once we have this then what we do here we simply change the order of this integration. So when you change the order of this integration I am just looking at this particular part, so here I am writing here as a to b and a to b now if you change the order then t will come inside and ds will come out.

So it will be what K t, s and g of s and phi i not star t then I am taking dt inside and ds outside, is it okay? So this is what this implies that a to b and I am having g of s here and then what is inside is a to b K t, s and phi i not star t dt and then d of s. So this I can write as you can write it transpose here then inside it will be transpose here, right? So this I can write it like this. Now, we have what? This is phi i not t dt is a solution of what? It is eigen function corresponding to this lambda not, so lambda not of the transpose equation.

So here I am we have assumed here that corresponding to lambda equal to lambda not phi i not to phi k not are the solutions of transpose equation that is y of x equal to lambda not a to b K star t of x yt dt. So if we take the assumption on these then this is what? This I can write as a to b and this is nothing but what this I can write as the g of s as it is divided by lambda not and this is going to be phi i star and we have s and then d of s, is it okay? Now if you this is lambda time, yeah yx upon lambda. So this value is coming out to be this provided that phi i phi 1 0 to phi k 0 are the solutions of this the transpose equation or you can say that when lambda equal to lambda not if we have say eigen function corresponding to transpose equation corresponding to this lambda not then this value is equal to this and if you put this value then what we will have then this value is coming out to be 0, right?

So it means what, so this implies what? That a to b your f of t your phi i 0 t dt star is coming out to be 0 for since I have taken any i so I can write it that it is for i equal to 1, 2 and up to k. So it means that that we have a solution that here we have assumed that g is a solution of this particular problem. So if g is a solution then we must have that f t is orthogonal to the eigen function corresponding to the transpose equation, is it okay?

So that is the necessary condition, yeah there is a and we can prove that this is also a sufficient condition but I am not going into detail of that that is easily available that you can do you can prove it, okay or you can see the book of R.P Kanwal linear integral equation but we can show here that if g is the solution of the given problem that is Fredholm integral equation of second kind then the necessary condition that your ft has to be orthogonal to the eigen function corresponding to the transpose equation, is it okay?

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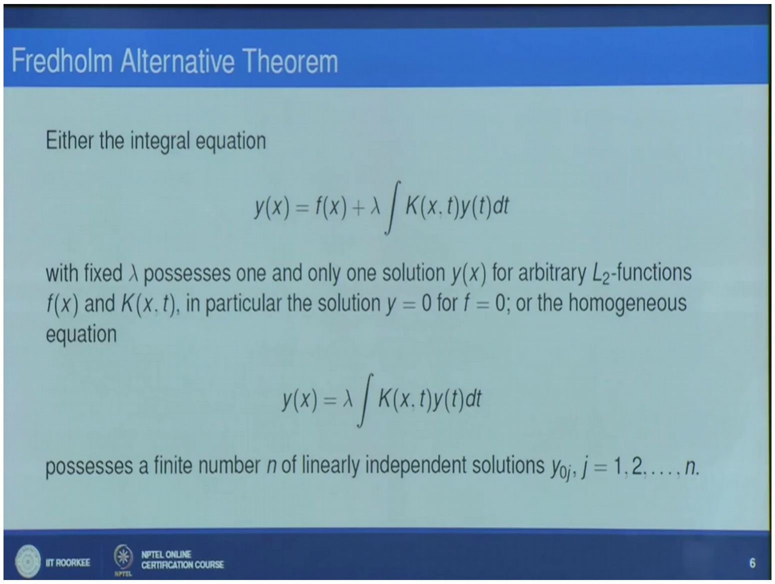

So if we summarize all this thing then we can write one theorem the theorem which is going to stated in next slide that is this Fredholm alternative theorem it says that either this integral equation this equation yx equal to f of x plus lambda K x, t yt dt with fixed lambda possess one and only one solution y of x for arbitrary L 2 functions f of x K x, t in particular the solution y equal to 0 for f equal 0.

So this portion this first portion is corresponding to when d lambda is equal to now d lambda is not equal to 0. So when d lambda is not equal to 0 you can find out your c unique way and you can get unique solution of this non-homogeneous first Fredholm integral equation of the first kind or you can say that if your f of x is equal to 0 then that case all those constant is simply vanished and in that case you have only trivial solution. So this first portion is corresponding to the fact that d lambda is not equal to 0 but if d lambda is equal to 0 then we have to look at the homogeneous equation yx equal to lambda a to b K x, t yt dt possess a finite number of linearly independent solution this we are one possibility corresponding to d lambda not equal to 0 other possibility is corresponding to d lambda equal to 0.

And this theorem is known as Fredholm alternative theorem, what we try to do here, here we have consider this particular theorem for kernel which is separable. Now what we try to do here we want to generalize the theory developed for separable kernel for kernel which is not separable or we can generalize this theory for a general kind of kernel.

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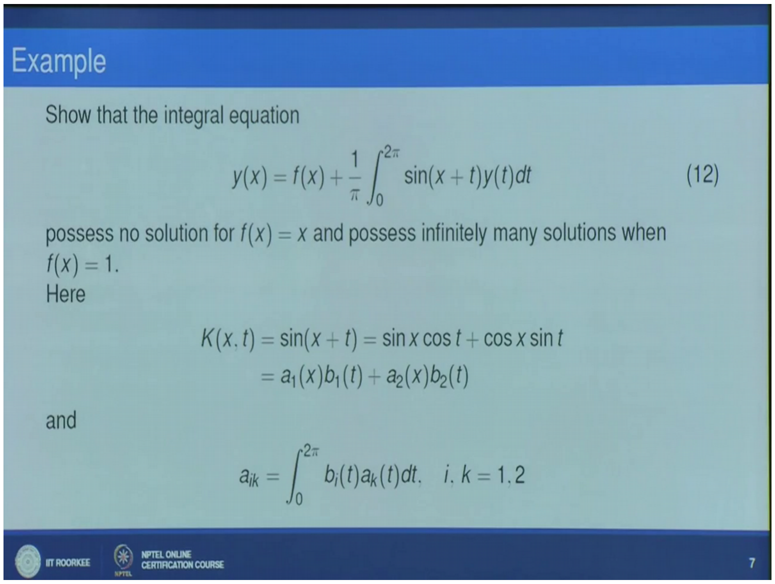

So that is before this we can consider a simple example here that is based on Fredholm alternative theorem and once we discussed this example later on then we will discuss how to generalize the discussion which we have presented in this lecture or in the previous lecture to the general kind of kernel. So one with just one quick look to this particular example, so here we have equation given here y of x equal to f of x plus 1 upon pi 0 to 2 pi sin x plus t yt dt. So here question is that if f of x is x then we do not have any solution but if f of x is equal to 1 then we have (())(21:59) solutions here, right?

So for that since look at here the kernel is sin of x plus t and if you simplify this is nothing but a kind of separable kernel. So when you write it separable kernel which is written as a 1 x b 1 t plus a 2 x b 2 t then we can write down the a i k as this.

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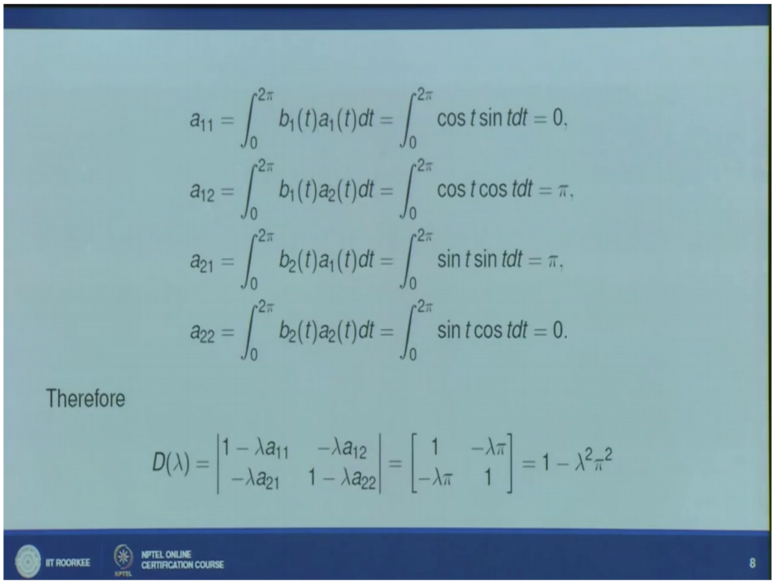

So here we can calculate all these a 1 a 1 a 11, a 12, a 21, a 22 and which by which we can form the matrix d lambda coefficient matrix d lambda and determinant of d lambda is given as 1 minus lambda square pi square.

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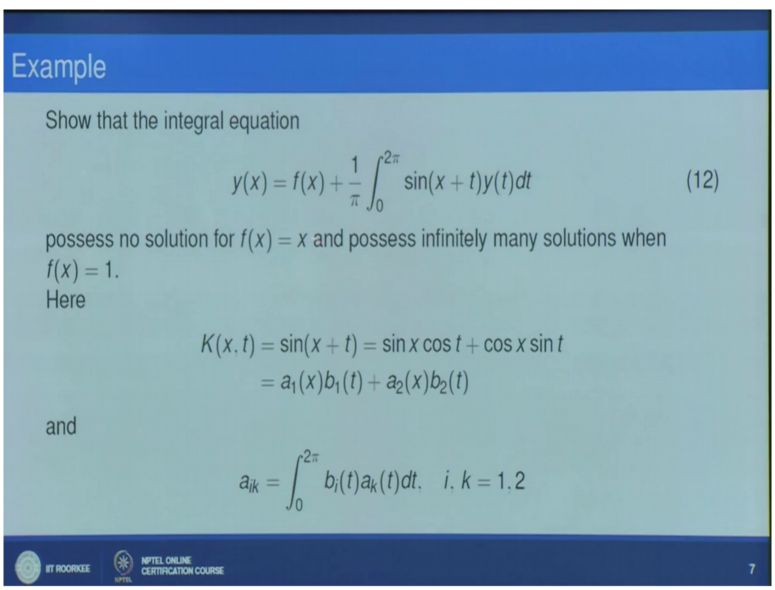

So we may we know that if d lambda is non-zero then we have a unique solution here, okay. And unique solution is given as y of x equal to f of x plus lambda times resolvent integral of resolvent kernel ft dt, right? The resolvent kernel means gamma x, t, lambda ft dt which is given as this equation number 11. So when d lambda is non-zero we have a solution like this but if you look at in this particular problem your lambda is 1 by pi and for which this d lambda is coming out to be 0.

So it means that we do not have a unique solution for this then we need to find out the situation when we have say no solution or more than one solution. So for that we have to look at the eigen values and eigen function of the transpose equation. By the way here your kernel is sin of x plus t so transpose kernel is nothing but the same as sin of x plus t.

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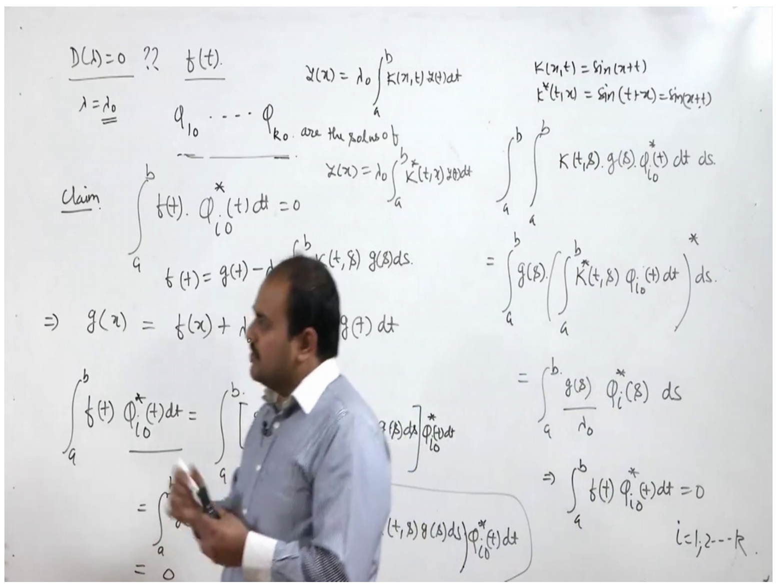

So here this K x, t is what? K x, t is sin of x plus t so this is real so K star t, x is going to be a sin of t plus x which is same as sin x plus t.

So here we have transpose problem is same as the same problem, right? So here when we want to discuss the eigen value eigen function of the transpose problem it is same as the considering the eigen value and eigen function of the same problem homogeneous problem.

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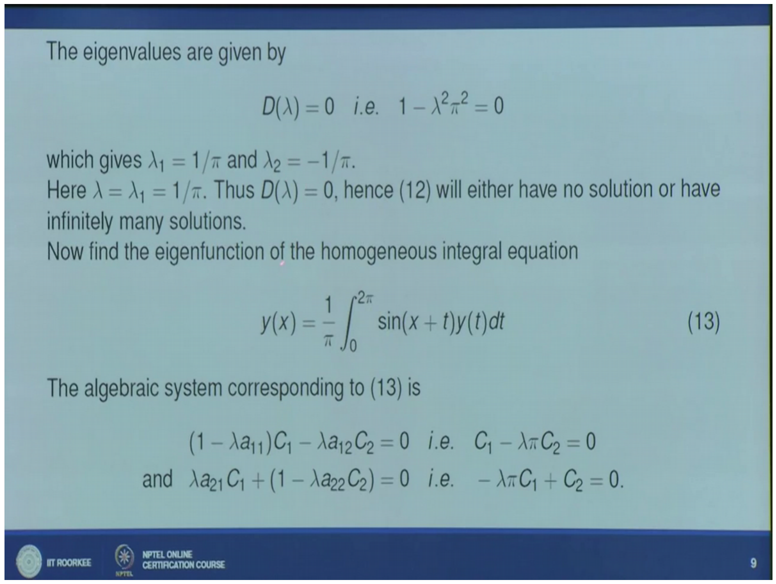

So here we are looking at that eigen function of the problem which is y of x equal to 1 upon pi 0 to 2 pi sin of x plus t yt dt and we want to find out say solution of this.

So when you find out the solution of this we have already discussed how to do it so here we simply expand it using the formula of sin of x plus t and assume that C 1 and C 2 where C 1 and C 2 is what you write sin of x cos of t yt dt, so you assume C 1 as 0 to 2 pi cos t yt dt and write as C 1. Similarly you can write C 2 as 0 to pi sin of t yt dt and using the expression of C 1 and C 2 we can simply solve our algebraic system like this.

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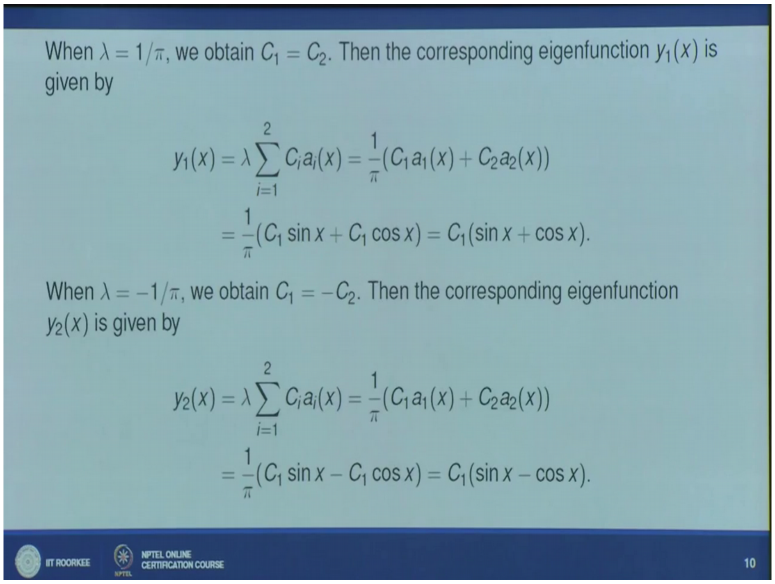

So here we have C 1 minus lambda pi C 2 equal to 0 and minus lambda pi C 1 plus C 2 equal to 0. So we want to find out say eigen value corresponding to lambda 1 equal to 1 by pi. So for this lambda is equal to 1 by pi means this is what this I can solve as C 1 equal to C 2, so when you put lambda as 1 by pi you can solve and you can get C 1 equal to C 2 and corresponding solution corresponding eigen function is given by y of x lambda is 1 by pi now and C 1 is equal to C 2 I can write this as 1 upon pi C 1 a 1 x plus C 2 a 2 x where C 2 is nothing but C 1.

So this I can write as C 1 times sin of x plus cos of x I am just writing 1 upon pi is merged with this so or you can take any constant now this is this you can write as C 1 telda you can say like this. So I can say that eigen function corresponding to lambda equal to 1 by pi is given as constant multiple of sin x plus cos of x. Similarly we can find out the eigen function corresponding to lambda equal to minus 1 upon pi so go back here and here we put lambda equal to minus 1 upon pi and we can solve that it is nothing but C 1 equal to minus C 2 and again we can simplify and your solution is constant multiple of sin x minus cos of x, right?

So we have y 1 x as sin x plus cos x into some constant multiple of this and y 2 x which is eigen function corresponding to lambda equal to minus 1 by pi and it is like this.

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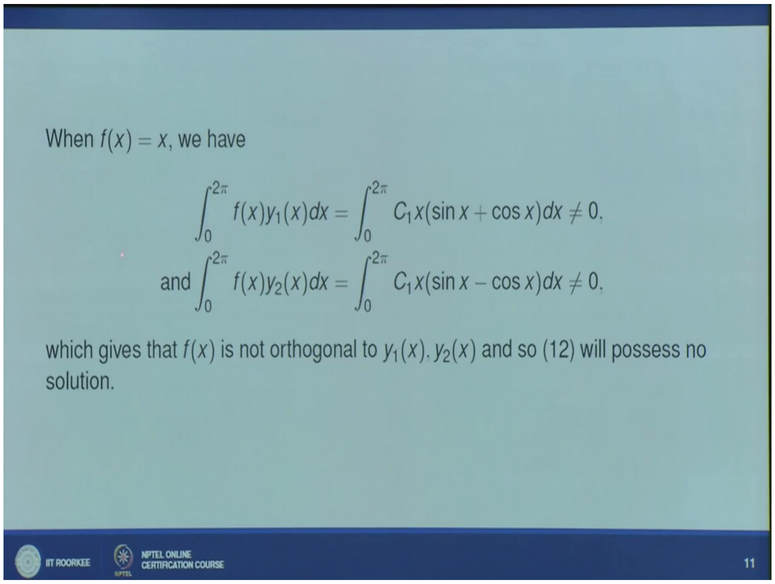

So here when we have f of x equal to x then we can calculate 0 to 2 pi f of x y 1 x dx and 0 to 2 pi fx y 2 x dx and if you put y 2 here then it is equal to this and when you calculate this you please calculate this and it is coming out to be 0.

So it means that it is not satisfying the condition stated here, right? So it means that we do not have so this implies that we cannot have a solution of the this non-homogeneous problem.

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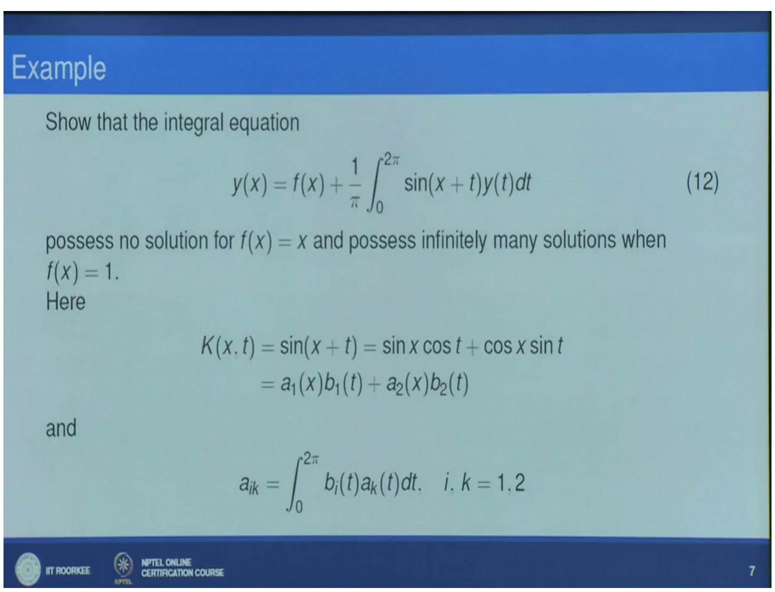

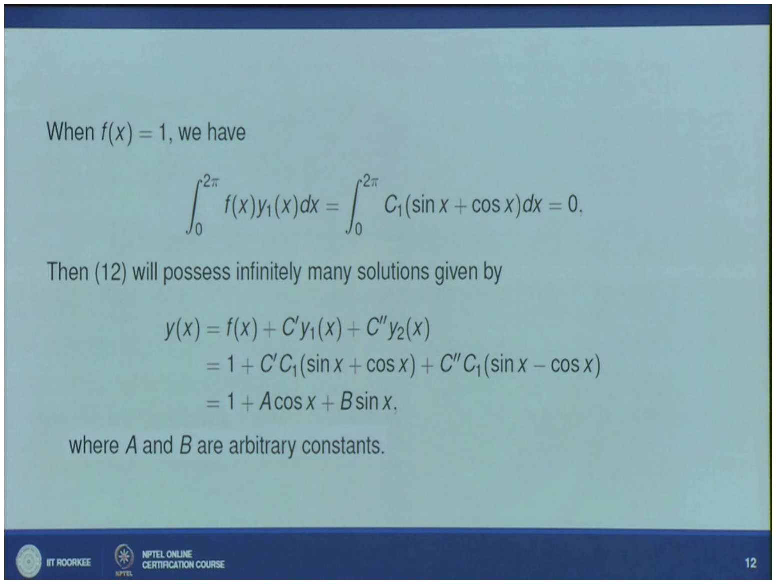

So it means that in this case our problem when fx equal to x our original problem y of x equal to x plus 1 upon pi 0 to 2 pi sin x plus t yt dt will not have any solution but if I look at f of x equal to 1 and find out the value 0 to 2 pi fx y 1 x dx equal to it is coming out to be 0 you can put the value of y 1 x and similarly you can say that it is 0 to 2 pi C 1 similarly you can check for 0 to 2 pi fx f of x y 2 x d of x.

And you can check that both are coming out to be 0 here we have done only for y 1 x you can check for y 2 x and it is also coming as 0. And in this case this condition is satisfied, right? And it means that your non-homogeneous problem will have a (())(28:14) solution so which we can write it here y of x equal to f of x plus say any constant multiple of by 1 x plus some constant multiple of y 2 x.

So when you write it then we can write that y of x is equal to x plus a cos of x plus b of sin of x. So in this case when f of x equal to 1 we have this condition that this 1 is orthogonal to eigen functions corresponding to transpose equation here transpose equation is the same equation. So here the this condition is satisfied and we have a solution y of x given as x plus a cos x plus b sin x or you can say that we have (())(28:55) solution.

So that we have discussed, now in next lecture we try to generalize the concept discuss in this lecture and the previous lecture to general kind of kernels, thank you very much.